17.3 Kinematics of rolling and sliding

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, and rolling of round things on other round things like gears and cams.

2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in fig. 17.35. For convenience, we pick a point D which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

> *The arc length traversed on the wheel is the distance traveled by the wheel center.*

That is,

$$\begin{aligned} x_C &= s_D \\ &= R\phi \\ \Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\ \Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi} \end{aligned}$$

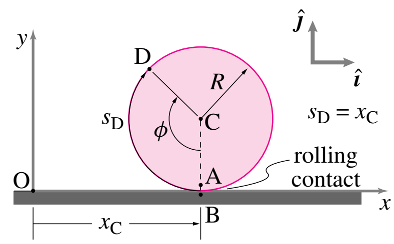


Figure 17.35: Pure rolling of a round wheel on a level support.

So the rolling condition amounts to the following set of restrictions on the position of C, $\vec{r}_C$, and the rotations of the wheel ϕ:

$$\vec{r}_C = R\phi\hat{\imath} + R\hat{\jmath}, \quad \vec{v}_C = R\dot{\phi}\hat{\imath}, \quad \vec{a}_C = R\ddot{\phi}\hat{\imath}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \quad \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say D, we could do so by using the following parametric formula:

$$\begin{aligned} \vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\ &= R(\phi\hat{\imath} + \hat{\jmath}) + R(-\sin\phi\hat{\imath} - \cos\phi\hat{\jmath}) \\ &= R\left[(\phi - \sin\phi)\hat{\imath} + (1 - \cos\phi)\hat{\jmath})\right] \\ \Rightarrow \vec{v}_D &= R\left[(\dot{\phi}(1 - \cos\phi)\hat{\imath} + \dot{\phi}\sin\phi\hat{\jmath})\right] \qquad (17.32) \\ \Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\hat{\imath} + \cos\phi\hat{\jmath}). \end{aligned}$$

assuming $\dot{\phi}$= constant

Note that if $\phi = 0$ or 2π or 4π, etc., then the point D is on the ground and eqn. (17.32) correctly gives that

$$\vec{v}_D = R\left[\dot{\phi}\underbrace{(1 - \overbrace{\cos(2n\pi)}^{1})}_{0}\right]\hat{\imath} + \dot{\phi}\underbrace{\sin(2n\pi)}_{0}\hat{\jmath} = \vec{\mathbf{0}}.$$

Instantaneous kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

$$\overbrace{\vec{\boldsymbol{v}}_A}^{\text{Velocity of point on the wheel touching the ground}} = \overbrace{\vec{\boldsymbol{v}}_B}^{\text{Velocity of ground} = \vec{\mathbf{0}}} \qquad (17.33)$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{\boldsymbol{v}}_A = \vec{\boldsymbol{v}}_C + \vec{\boldsymbol{v}}_{A/C},$$

where, since A and C are on the same rigid body (fig. 17.35), we have from eqn. (16.22) that

$$\vec{\boldsymbol{v}}_{A/C} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{A/C}.$$

Putting this equation together with eqn. (17.33), we get

$$\begin{aligned} \vec{\boldsymbol{v}}_A &= \vec{\boldsymbol{v}}_B \\ \Rightarrow \underbrace{\vec{\boldsymbol{v}}_C}_{v_C\hat{\boldsymbol{\imath}}} + \underbrace{\vec{\boldsymbol{\omega}}}_{\omega\hat{\boldsymbol{k}}} \times \underbrace{\vec{\boldsymbol{r}}_{A/C}}_{-R\hat{\boldsymbol{\jmath}}} &= \vec{\mathbf{0}} \\ \Rightarrow v_C\hat{\boldsymbol{\imath}} + \omega R\hat{\boldsymbol{\imath}} &= \vec{\mathbf{0}} \end{aligned}$$

$$\Rightarrow \qquad v_C = -\omega R. \qquad (17.34)$$

We use $\vec{\boldsymbol{v}}_C = v\hat{\boldsymbol{\imath}}$ since the center of the wheel goes neither up nor down. Note that if you measure the angle by ϕ, like we did before, then $\vec{\boldsymbol{\omega}} = -\dot{\phi}\hat{\boldsymbol{k}}$ so that positive rotation rate is in the counter-clockwise direction. Thus, $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$.

Since there is always *some* point of the wheel touching the ground, we know that $v_C = -\omega R$ for all time. Therefore,

$$\vec{\boldsymbol{a}}_C = \dot{v}_C\hat{\boldsymbol{\imath}} = -\dot{\omega}R\hat{\boldsymbol{\imath}}.$$

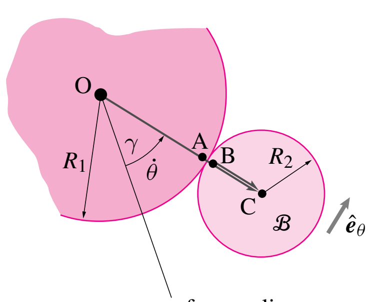


Figure 17.36: One round object rolling on another round object.

Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. Common applications are the so-called epicyclic, hypo-cyclic, or planetary gears (See Box 17.5 on planetary gears on page 4). Referring to fig. 17.36, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

$$\begin{aligned}
\vec{\boldsymbol{v}}_C &= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{v}}_{C/B}\\
\vec{\boldsymbol{v}}_C &= \overbrace{\underbrace{\vec{\boldsymbol{v}}_A}_{\vec{\mathbf{0}}} + \underbrace{\vec{\boldsymbol{v}}_{B/A}}_{\vec{\mathbf{0}}}}^{\vec{\boldsymbol{v}}_B} + \vec{\boldsymbol{v}}_{C/B}.\\
\dot{\theta}(R_1+R_2)\hat{\boldsymbol{e}}_\theta &= \omega_{\mathcal{B}} R_2 \hat{\boldsymbol{e}}_\theta\\
\Rightarrow \omega_{\mathcal{B}} &= \frac{\dot{\theta}(R_1+R_2)}{R_2} = \dot{\theta}(1+\frac{R_1}{R_2}).
\end{aligned}$$

Example: Two quarters.

The formula above can be tested in the case of $R_1 = R_2$ by using two quarters or two dimes on a table. Roll one quarter, call it $\mathcal{B}$, around another quarter pressed fast to the table. You will see that as the rolling quarter $\mathcal{B}$ travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of $\mathcal{B}$ changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above, $\omega_{\mathcal{B}} = 2\dot{\theta}$.

Sliding

Although wheels and balls are known for rolling, they do sometimes slide such as when a car screeches at fast acceleration or sudden braking or when a bowling ball is released on the lane.

The *sliding velocity* is the velocity of the material point on the wheel (or ball) relative to its contacting substrate. In the case of pure rolling, the sliding velocity is zero. In the case of a ball or wheel moving against a stationary support surface, whether flat or curved, the sliding velocity is

$$\vec{\boldsymbol{v}}_{\text{sliding}} = \vec{\boldsymbol{v}}_{\text{circle center}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{contact/center}} \tag{17.35}$$

Example: Bowling ball

The velocity of that point on the bowling ball which is instantaneously in contact with the alley (ground) is $\vec{\boldsymbol{v}}_{\text{C}} = v_{\text{G}}\hat{\boldsymbol{\imath}} + \omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\text{C/G}} = (v_{\text{G}} + \omega R)\hat{\boldsymbol{\imath}}$. So unless $\omega = -v_{\text{G}}/R$ the ball is sliding.

Note that, if sliding, the friction force on the ball opposes the slip of the ball and tends to accelerate the ball's rotation towards rolling. That is, for example, if the ball is not rotating the sliding velocity is $v_G\hat{\boldsymbol{\imath}}$, the friction force is in the $-\hat{\boldsymbol{\imath}}$ direction and angular momentum balance about the center-of-mass implies $\dot{\omega} < 0$ and a clockwise rotational acceleration. No matter what the initial velocity or rotational rate the ball will eventually roll.

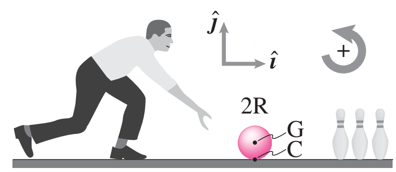
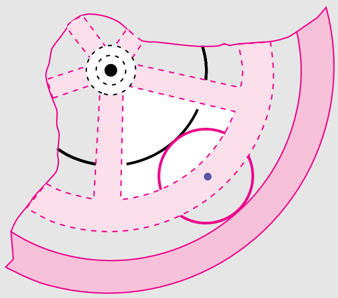


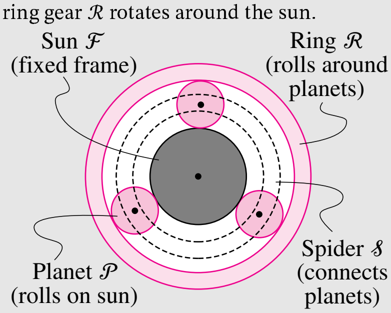
Figure 17.37: The bowling ball is sliding so long as $v_G \neq -\omega R$

17.5 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub three-speed bicycle transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. Now, a hundred years later, internal bicycle hubs (now made by Shimano and Sachs) are having something of a revival, particularly in Europe. These internal-hub transmissions utilize a system called *planetary gears*, gears which roll around other gears. See the figure below.



In order to understand this gear system, we need to understand its kinematics—the motion of its parts. The central 'sun' gear $\mathcal{F}$ is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body $\mathcal{F}$. The 'planet' gears roll around the sun gear. Let's call one of these planets $\mathcal{P}$. The spider $\mathcal{S}$ connects the centers of the rolling planets. Finally, the ring gear $\mathcal{R}$ rotates around the sun.



The gear transmission steps up the angular velocity when the spider $\mathcal{S}$ is driven and ring $\mathcal{R}$, which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third 'speed' in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the 'gear ratios' in the planetary gear system? The 'trick' is to recognize that for rolling contact that the contacting points have the same velocity, $\vec{v}_A = \vec{v}_B$ and $\vec{v}_D = \vec{v}_E$. Let's define some terms.

$$\vec{\omega}_{\mathcal{S}} = \omega_{\mathcal{S}}\hat{k} \qquad \text{angular velocity of the spider}$$
$$\vec{\omega}_{\mathcal{P}} = \omega_{\mathcal{P}}\hat{k} \qquad \text{angular velocity of the planet}$$
$$\vec{\omega}_{\mathcal{R}} = \omega_{\mathcal{R}}\hat{k} \qquad \text{angular velocity of the ring}$$

Now, we can find the relation of these angular velocities as follows. Look at the velocity of point C in two ways. First,

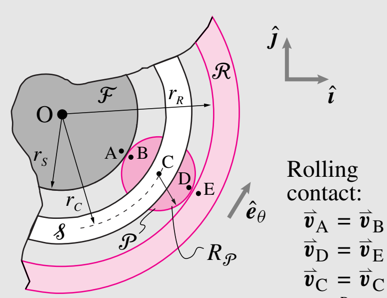
$$\overbrace{\vec{v}_C}^{\boxed{\text{A point on the spider}}} = \overbrace{\vec{v}_C}^{\boxed{\text{A point on the planetary gear}}}$$
$$\Rightarrow \vec{\omega}_{\mathcal{S}}\times\vec{r}_C = \underbrace{\vec{v}_B}_{\vec{0}} + \vec{\omega}_{\mathcal{P}}\times\vec{r}_{C/B}$$
$$\Rightarrow \omega_{\mathcal{S}} r_C = \omega_{\mathcal{P}} R_P$$
$$\Rightarrow \omega_{\mathcal{P}} = \frac{r_C}{R_P}\omega_{\mathcal{S}} \qquad (17.36)$$

Next, let's look at point D and E:

$$\vec{v}_D = \vec{v}_E$$
$$\vec{v}_A + \vec{v}_{D/A} = \vec{\omega}_{\mathcal{R}}\times\vec{r}_R$$
$$\vec{0} + \vec{\omega}_{\mathcal{P}}\times\vec{r}_{D/A} = \omega_{\mathcal{R}}\hat{k}\times\vec{r}_R$$
$$\omega_{\mathcal{P}}(2R_P)\hat{e}_\theta = \omega_{\mathcal{R}} r_R\hat{e}_\theta$$
$$\Rightarrow \omega_{\mathcal{R}} = \frac{2R_P}{r_R}\underbrace{\omega_{\mathcal{P}}}_{\boxed{\omega_{\mathcal{P}} = \frac{r_C}{R_P}\omega_{\mathcal{S}}}}$$

$$\omega_{\mathcal{R}} = \frac{2R_P}{r_R}\frac{\overbrace{r_C}^{\boxed{r_C = r_S + R_P}}}{R_P}\omega_{\mathcal{S}}$$
$$= \frac{2(r_S + R_P)}{\underbrace{r_R}_{\boxed{r_R = r_S + 2R_P}}}\omega_{\mathcal{S}}$$

$$\Rightarrow \frac{\omega_{\mathcal{R}}}{\omega_{\mathcal{S}}} = 2\frac{1+\frac{R_P}{r_S}}{1+\frac{2R_P}{r_S}} = \text{angular velocity step-up.}$$



Typically, the gears have radius ratio of $\frac{R_P}{r_S} = \frac{3}{2}$ which gives a gear ratio of $\frac{5}{4}$. Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is $\frac{5}{4}/\frac{4}{5} = \frac{25}{16} = 1.5625$. You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.