17.3 Kinematics of rolling and sliding

Pure rolling in 2-D

In this section, we would like to add to the vocabulary of special motions by considering *pure rolling*. Most commonly, one discusses pure rolling of round objects on flat ground, like wheels and balls, and rolling of round things on other round things like gears and cams.

2-D rolling of a round wheel on level ground

The simplest case, the no-slip rolling of a round wheel, is an instructive starting point. First, we define the geometric and kinematic variables as shown in fig. 17.35. For convenience, we pick a point D which was at $x_D = 0$ at the start of rolling, when $x_C = 0$. The key to the kinematics is that:

> *The arc length traversed on the wheel is the distance traveled by the wheel center.*

That is,

$$
\begin{aligned}
x_C &= s_D \\
&= R\phi \\
\Rightarrow v_C = \dot{x}_C &= R\dot{\phi} \\
\Rightarrow a_C = \dot{v}_C = \ddot{x}_C &= R\ddot{\phi}
\end{aligned}
$$

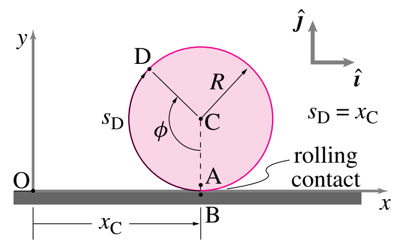


Figure 17.35: Pure rolling of a round wheel on a level support.

So the rolling condition amounts to the following set of restrictions on the position of C, $\vec{r}_C$, and the rotations of the wheel ϕ:

$$\vec{r}_C = R\phi\hat{\imath} + R\hat{\jmath}, \quad \vec{v}_C = R\dot{\phi}\hat{\imath}, \quad \vec{a}_C = R\ddot{\phi}\hat{\imath}, \quad \vec{\omega} = -\dot{\phi}\hat{k}, \text{ and } \quad \vec{\alpha} = \dot{\vec{\omega}} = -\ddot{\phi}\hat{k}.$$

If we want to track the motion of a particular point, say D, we could do so by using the following parametric formula:

$$
\begin{aligned}
\vec{r}_D &= \vec{r}_C + \vec{r}_{D/C} \\
&= R(\phi\hat{\imath} + \hat{\jmath}) + R(-\sin\phi\,\hat{\imath} - \cos\phi\,\hat{\jmath}) \\
&= R\left[(\phi - \sin\phi)\hat{\imath} + (1 - \cos\phi)\hat{\jmath})\right] \\
\Rightarrow \vec{v}_D &= R\left[(\dot{\phi}(1 - \cos\phi)\hat{\imath} + \dot{\phi}\sin\phi\,\hat{\jmath})\right] \qquad (17.32) \\
\Rightarrow \vec{a}_D &= R\dot{\phi}^2(\sin\phi\,\hat{\imath} + \cos\phi\,\hat{\jmath}).
\end{aligned}
$$

assuming $\dot{\phi}$= constant

Note that if $\phi = 0$ or 2π or 4π, etc., then the point D is on the ground and eqn. (17.32) correctly gives that

$$\vec{v}_D = R\left[\dot{\phi}\underbrace{(1 - \overbrace{\cos(2n\pi)}^{1})}_{0}\right]\hat{\imath} + \dot{\phi}\underbrace{\sin(2n\pi)}_{0}\hat{\jmath} = \vec{\mathbf{0}}.$$

Instantaneous kinematics

Instead of tracking the wheel from its start, we could analyze the kinematics at the instant of interest. Here, we make the observation that the wheel rolls without slip. Therefore, the point on the wheel touching the ground has no velocity relative to the ground.

$$\overbrace{\vec{\boldsymbol{v}}_A}^{\text{Velocity of point on the wheel touching the ground}} = \overbrace{\vec{\boldsymbol{v}}_B}^{\text{Velocity of ground} = \vec{\mathbf{0}}} \tag{17.33}$$

Now, we know how to calculate the velocity of points on a rigid body. So,

$$\vec{\boldsymbol{v}}_A = \vec{\boldsymbol{v}}_C + \vec{\boldsymbol{v}}_{A/C},$$

where, since A and C are on the same rigid body (fig. 17.35), we have from eqn. (16.22) that

$$\vec{\boldsymbol{v}}_{A/C} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{A/C}.$$

Putting this equation together with eqn. (17.33), we get

$$\begin{aligned} \vec{\boldsymbol{v}}_A &= \vec{\boldsymbol{v}}_B \\ \Rightarrow \underbrace{\vec{\boldsymbol{v}}_C}_{v_C\hat{\boldsymbol{\imath}}} + \underbrace{\vec{\boldsymbol{\omega}}}_{\omega\hat{\boldsymbol{k}}} \times \underbrace{\vec{\boldsymbol{r}}_{A/C}}_{-R\hat{\boldsymbol{\jmath}}} &= \vec{\mathbf{0}} \\ \Rightarrow v_C\hat{\boldsymbol{\imath}} + \omega R\hat{\boldsymbol{\imath}} &= \vec{\mathbf{0}} \end{aligned}$$

$$\Rightarrow \qquad v_C = -\omega R. \tag{17.34}$$

We use $\vec{\boldsymbol{v}}_C = v\hat{\boldsymbol{\imath}}$ since the center of the wheel goes neither up nor down. Note that if you measure the angle by ϕ, like we did before, then $\vec{\boldsymbol{\omega}} = -\dot{\phi}\hat{\boldsymbol{k}}$ so that positive rotation rate is in the counter-clockwise direction. Thus, $v_C = -\omega R = -(-\dot{\phi})R = \dot{\phi}R$.

Since there is always *some* point of the wheel touching the ground, we know that $v_C = -\omega R$ for all time. Therefore,

$$\vec{\boldsymbol{a}}_C = \dot{v}_C\hat{\boldsymbol{\imath}} = -\dot{\omega}R\hat{\boldsymbol{\imath}}.$$

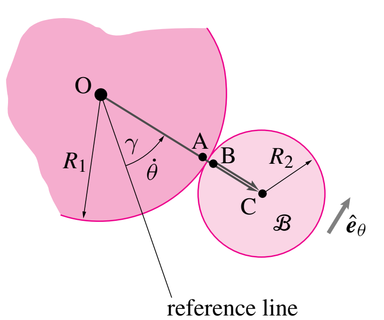


Figure 17.36: One round object rolling on another round object.

Rolling of round objects on round surfaces

For round objects rolling on or in another round object, the analysis is similar to that for rolling on a flat surface. Common applications are the so-called epicyclic, hypo-cyclic, or planetary gears (See Box 17.5 on planetary gears on page 4). Referring to fig. 17.36, we can calculate the velocity of C with respect to a fixed frame two ways and compare:

$$
\begin{aligned}
\vec{\boldsymbol{v}}_C &= \vec{\boldsymbol{v}}_B + \vec{\boldsymbol{v}}_{C/B} \\
\vec{\boldsymbol{v}}_C &= \overbrace{\underbrace{\vec{\boldsymbol{v}}_A}_{\vec{\mathbf{0}}} + \underbrace{\vec{\boldsymbol{v}}_{B/A}}_{\vec{\mathbf{0}}}}^{\vec{\boldsymbol{v}}_B} + \vec{\boldsymbol{v}}_{C/B}. \\
\dot{\theta}(R_1 + R_2)\hat{\boldsymbol{e}}_\theta &= \omega_{\mathcal{B}} R_2 \hat{\boldsymbol{e}}_\theta \\
\Rightarrow \omega_{\mathcal{B}} &= \frac{\dot{\theta}(R_1 + R_2)}{R_2} = \dot{\theta}(1 + \frac{R_1}{R_2}).
\end{aligned}
$$

Example: Two quarters.

The formula above can be tested in the case of $R_1 = R_2$ by using two quarters or two dimes on a table. Roll one quarter, call it $\mathcal{B}$, around another quarter pressed fast to the table. You will see that as the rolling quarter $\mathcal{B}$ travels around the stationary quarter one time, it makes two full revolutions. That is, the orientation of $\mathcal{B}$ changes twice as fast as the angle of the line from the center of the stationary quarter to its center. Or, in the language of the calculation above, $\omega_{\mathcal{B}} = 2\dot{\theta}$.

Sliding

Although wheels and balls are known for rolling, they do sometimes slide such as when a car screeches at fast acceleration or sudden braking or when a bowling ball is released on the lane.

The *sliding velocity* is the velocity of the material point on the wheel (or ball) relative to its contacting substrate. In the case of pure rolling, the sliding velocity is zero. In the case of a ball or wheel moving against a stationary support surface, whether flat or curved, the sliding velocity is

$$\vec{\boldsymbol{v}}_{\text{sliding}} = \vec{\boldsymbol{v}}_{\text{circle center}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{contact/center}} \tag{17.35}$$

Example: Bowling ball

The velocity of that point on the bowling ball which is instantaneously in contact with the alley (ground) is $\vec{\boldsymbol{v}}_{\mathrm{C}} = v_{\mathrm{G}}\hat{\boldsymbol{\imath}} + \omega\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{C/G}} = (v_{\mathrm{G}} + \omega R)\hat{\boldsymbol{\imath}}$. So unless $\omega = -v_{\mathrm{G}}/R$ the ball is sliding.

Note that, if sliding, the friction force on the ball opposes the slip of the ball and tends to accelerate the ball's rotation towards rolling. That is, for example, if the ball is not rotating the sliding velocity is $v_G\hat{\boldsymbol{\imath}}$, the friction force is in the $-\hat{\boldsymbol{\imath}}$ direction and angular momentum balance about the center-of-mass implies $\dot{\omega} < 0$ and a clockwise rotational acceleration. No matter what the initial velocity or rotational rate the ball will eventually roll.

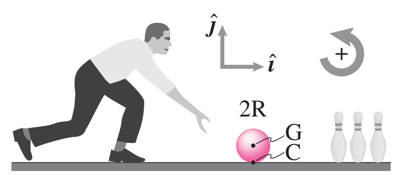
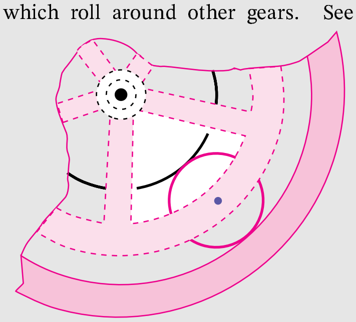


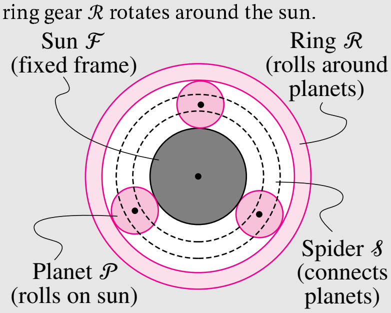
Figure 17.37: The bowling ball is sliding so long as $v_G \neq -\omega R$

17.5 The Sturmey-Archer hub

In 1903, the year the Wright Brothers first flew powered airplanes, the Sturmey-Archer company patented the internal-hub three-speed bicycle transmission. This marvel of engineering was sold on the best bikes until finicky but fast racing bicycles using derailleurs started to push them out of the market in the 1960's. Now, a hundred years later, internal bicycle hubs (now made by Shimano and Sachs) are having something of a revival, particularly in Europe. These internal-hub transmissions utilize a system called *planetary gears*, gears which roll around other gears. See the figure below.



In order to understand this gear system, we need to understand its kinematics—the motion of its parts. The central 'sun' gear $\mathcal{F}$ is stationary, at least we treat it as stationary in this discussion since it is fixed to the bike frame, so it is fixed in body $\mathcal{F}$. The 'planet' gears roll around the sun gear. Let's call one of these planets $\mathcal{P}$. The spider $\mathcal{S}$ connects the centers of the rolling planets. Finally, the ring gear $\mathcal{R}$ rotates around the sun.



The gear transmission steps up the angular velocity when the spider $\mathcal{S}$ is driven and ring $\mathcal{R}$, which moves faster, is connected to the wheel. The transmission steps down the angular velocity when the ring gear is driven and the slower spider is connected to the wheel. The third 'speed' in the three-speed gear transmission is direct drive (the wheel is driven directly).

What are the 'gear ratios' in the planetary gear system? The 'trick' is to recognize that for rolling contact that the contacting points have the same velocity, $\vec{v}_A = \vec{v}_B$ and $\vec{v}_D = \vec{v}_E$. Let's define some terms.

$$\vec{\omega}_{\mathcal{S}} = \omega_{\mathcal{S}}\hat{\boldsymbol{k}} \qquad \text{angular velocity of the spider}$$
$$\vec{\omega}_{\mathcal{P}} = \omega_{\mathcal{P}}\hat{\boldsymbol{k}} \qquad \text{angular velocity of the planet}$$
$$\vec{\omega}_{\mathcal{R}} = \omega_{\mathcal{R}}\hat{\boldsymbol{k}} \qquad \text{angular velocity of the ring}$$

Now, we can find the relation of these angular velocities as follows. Look at the velocity of point C in two ways. First,

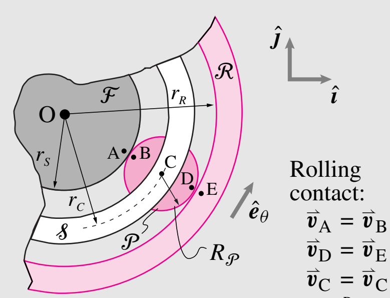
$$\begin{aligned}
\overbrace{\vec{v}_C}^{\text{A point on the spider}} &= \overbrace{\vec{v}_C}^{\text{A point on the planetary gear}} \\
\Rightarrow \vec{\omega}_{\mathcal{S}} \times \vec{r}_C &= \underbrace{\vec{v}_B}_{\vec{\mathbf{0}}} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{C/B} \\
\Rightarrow \omega_{\mathcal{S}} r_C &= \omega_{\mathcal{P}} R_P \\
\Rightarrow \omega_{\mathcal{P}} &= \frac{r_C}{R_P}\omega_{\mathcal{S}}
\end{aligned} \qquad (17.36)$$

Next, let's look at point D and E:

$$\begin{aligned}
\vec{v}_D &= \vec{v}_E \\
\vec{v}_A + \vec{v}_{D/A} &= \vec{\omega}_{\mathcal{R}} \times \vec{r}_R \\
\vec{\mathbf{0}} + \vec{\omega}_{\mathcal{P}} \times \vec{r}_{D/A} &= \omega_{\mathcal{R}}\hat{\boldsymbol{k}} \times \vec{r}_R \\
\omega_{\mathcal{P}}(2R_P)\hat{\boldsymbol{e}}_\theta &= \omega_{\mathcal{R}} r_R \hat{\boldsymbol{e}}_\theta \\
\Rightarrow \omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\underbrace{\omega_{\mathcal{P}}}_{\boxed{\omega_{\mathcal{P}} = \frac{r_C}{R_P}\omega_{\mathcal{S}}}}
\end{aligned}$$

$$\begin{aligned}
\omega_{\mathcal{R}} &= \frac{2R_P}{r_R}\,\frac{\overbrace{r_C}^{\boxed{r_C = r_S + R_P}}}{R_P}\omega_{\mathcal{S}} \\
&= \frac{2(r_S + R_P)}{\underbrace{r_R}_{\boxed{r_R = r_S + 2R_P}}}\omega_{\mathcal{S}}
\end{aligned}$$

$$\Rightarrow \frac{\omega_{\mathcal{R}}}{\omega_{\mathcal{S}}} = 2\frac{1 + \frac{R_P}{r_S}}{1 + \frac{2R_P}{r_S}} = \text{angular velocity step-up.}$$



Typically, the gears have radius ratio of $\frac{R_P}{r_S} = \frac{3}{2}$ which gives a gear ratio of $\frac{5}{4}$. Thus, the ratio of the highest gear to the lowest gear on a Sturmey-Archer hub is $\frac{5}{4}/\frac{4}{5} = \frac{25}{16} = 1.5625$. You might compare this ratio to that of a modern mountain bike, with eighteen or twenty-one gears, where the ratio of the highest gear to the lowest is about 4:1.

SAMPLE 17.10 **Falling ladder:** The ends of a ladder of length $L = 3\,\mathrm{m}$ slip along the frictionless wall and floor shown in Figure 17.38. At the instant shown, when $\theta = 60^\circ$, the angular speed $\dot\theta = 1.15\,\mathrm{rad/s}$ and the angular acceleration $\ddot\theta = 2.5\,\mathrm{rad/s^2}$. Find the absolute velocity and acceleration of end B of the ladder.

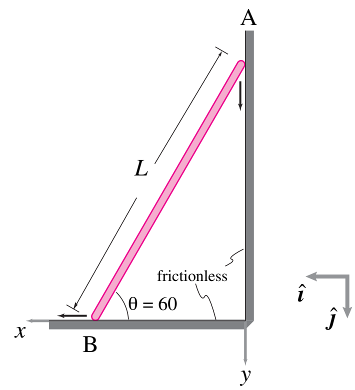


Figure 17.38

Solution Since the ladder is falling, it is rotating clockwise. From the given information:

$$\begin{aligned}\vec\omega &= \dot\theta\hat{\boldsymbol k} = -1.15\,\mathrm{rad/s}\hat{\boldsymbol k}\\ \dot{\vec\omega} &= \ddot\theta\hat{\boldsymbol k} = -2.5\,\mathrm{rad/s^2}\hat{\boldsymbol k}.\end{aligned}$$

We need to find $\vec{\boldsymbol v}_B$, the absolute velocity of end B, and $\vec{\boldsymbol a}_B$, the absolute acceleration of end B.

Since the end A slides along the wall and end B slides along the floor, we know the directions of $\vec{\boldsymbol v}_A$, $\vec{\boldsymbol v}_B$, $\vec{\boldsymbol a}_A$ and $\vec{\boldsymbol a}_B$.

Let $\vec{\boldsymbol v}_A = v_A\hat{\boldsymbol\jmath}$, $\vec{\boldsymbol a}_A = a_A\hat{\boldsymbol\jmath}$, $\vec{\boldsymbol v}_B = v_B\hat{\boldsymbol\imath}$ and $\vec{\boldsymbol a}_B = a_B\hat{\boldsymbol\imath}$ where the scalar quantities v_A, a_A, v_B and a_B are unknown.

$$\begin{aligned}\text{Now, }\ \vec{\boldsymbol v}_A &= \vec{\boldsymbol v}_B + \vec{\boldsymbol v}_{A/B} = \vec{\boldsymbol v}_B + \vec\omega\times\vec{\boldsymbol r}_{A/B}\\ \text{or }\ v_A\hat{\boldsymbol\jmath} &= v_B\hat{\boldsymbol\imath} + \dot\theta\hat{\boldsymbol k}\times\underbrace{L(-\cos\theta\hat{\boldsymbol\imath} - \sin\theta\hat{\boldsymbol\jmath})}_{\vec{\boldsymbol r}_{A/B}}\\ &= (v_B + \dot\theta L\sin\theta)\hat{\boldsymbol\imath} - \dot\theta L\cos\theta\hat{\boldsymbol\jmath}.\end{aligned}$$

Dotting both sides of the equation with $\hat{\boldsymbol\imath}$, we get:

$$\begin{aligned}v_A\underbrace{\hat{\boldsymbol\jmath}\cdot\hat{\boldsymbol\imath}}_{0} &= (v_B + \dot\theta L\sin\theta)\underbrace{\hat{\boldsymbol\imath}\cdot\hat{\boldsymbol\imath}}_{1} + \dot\theta L\cos\theta\underbrace{\hat{\boldsymbol\jmath}\cdot\hat{\boldsymbol\imath}}_{0}\\ \Rightarrow\quad 0 &= v_B + \dot\theta L\sin\theta\\ \Rightarrow\quad v_B &= -\dot\theta L\sin\theta = -(-1.15\,\mathrm{rad/s})\cdot 3\,\mathrm{m}\cdot\frac{\sqrt3}{2}\\ &= 2.99\,\mathrm{m/s}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol v}_B = 2.9\,\mathrm{m/s}\hat{\boldsymbol\imath}}$$

Similarly,

$$\begin{aligned}\vec{\boldsymbol a}_A &= \vec{\boldsymbol a}_B + \dot{\vec\omega}\times\vec{\boldsymbol r}_{A/B} + \overbrace{\vec\omega\times(\vec\omega\times\vec{\boldsymbol r}_{A/B})}^{-\omega^2\vec{\boldsymbol r}_{A/B}}\\ a_A\hat{\boldsymbol\jmath} &= a_B\hat{\boldsymbol\imath} + \ddot\theta\hat{\boldsymbol k}\times L(-\cos\theta\hat{\boldsymbol\imath} - \sin\theta\hat{\boldsymbol\jmath}) - \dot\theta^2 L(-\cos\theta\hat{\boldsymbol\imath} - \sin\theta\hat{\boldsymbol\jmath})\\ &= (a_B + \ddot\theta L\sin\theta + \dot\theta^2 L\cos\theta)\hat{\boldsymbol\imath} + (-\ddot\theta L\cos\theta + \dot\theta^2 L\sin\theta)\hat{\boldsymbol\jmath}.\end{aligned}$$

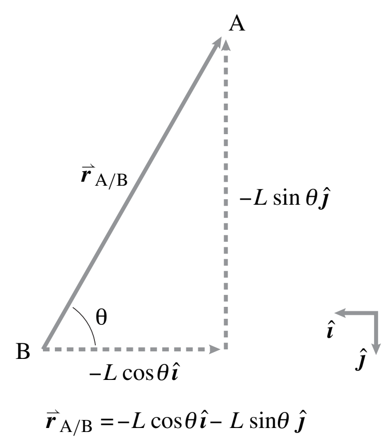


Figure 17.39

Dotting both sides of this equation with $\hat{\boldsymbol\imath}$ (as we did for velocity) we get:

$$\begin{aligned}0 &= a_B + \ddot\theta L\sin\theta + \dot\theta^2 L\cos\theta\\ \Rightarrow\quad a_B &= -\ddot\theta L\sin\theta - \dot\theta^2 L\cos\theta\\ &= -(-2.5\,\mathrm{rad/s^2}\cdot 3\,\mathrm{m}\cdot\frac{\sqrt3}{2}) - (-1.15\,\mathrm{rad/s})^2\cdot 3\,\mathrm{m}\cdot\frac12\\ &= 4.51\,\mathrm{m/s^2}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol a}_B = 4.51\,\mathrm{m/s^2}}$$

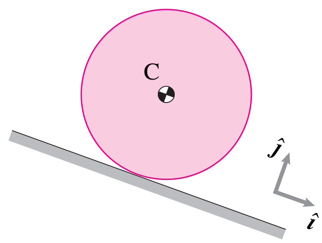


Figure 17.40

SAMPLE 17.11 A cylinder of diameter 500 mm rolls down an inclined plane with uniform acceleration (of the center-of-mass) $a = 0.1\,\mathrm{m/s^2}$. At an instant t_0, the mass-center has speed $v_0 = 0.5\,\mathrm{m/s}$.

1. Find the angular speed ω and the angular acceleration $\dot{\omega}$ at t_0.
2. How many revolutions does the cylinder make in the next 2 seconds?
3. What is the distance travelled by the center-of-mass in those 2 seconds?

Solution This problem is about simple kinematic calculations. We are given the velocity, $\dot{x}$, and the acceleration, $\ddot{x}$, of the center-of-mass. We are supposed to find angular velocity ω, angular acceleration $\dot{\omega}$, angular displacement θ in 2 seconds, and the corresponding linear distance x along the incline. The radius of the cylinder $R = \text{diameter}/2 = 0.25\,\mathrm{m}$.

1. From the kinematics of pure rolling,

$$\begin{aligned} \omega &= \frac{\dot{x}}{R} = \frac{0.5\,\mathrm{m/s}}{0.25\,\mathrm{m}} = 2\,\mathrm{rad/s}, \\ \dot{\omega} &= \frac{\ddot{x}}{R} = \frac{0.1\,\mathrm{m/s^2}}{0.25\,\mathrm{m}} = 0.4\,\mathrm{rad/s^2}. \end{aligned}$$

$$\boxed{\omega = 2\,\mathrm{rad/s}, \quad \dot{\omega} = 0.4\,\mathrm{rad/s^2}}$$

2. We can find the number of revolutions the cylinder makes in 2 seconds by solving for the angular displacement θ in this time period. Since,

$$\ddot{\theta} \equiv \dot{\omega} = \text{constant},$$

we integrate this equation twice and substitute the initial conditions, $\dot{\theta}(t = 0) = \omega = 2\,\mathrm{rad/s}$ and $\theta(t = 0) = 0$, to get

$$\begin{aligned} \theta(t) &= \omega t + \frac{1}{2}\dot{\omega}t^2 \\ \Rightarrow \quad \theta(t = 2\,\mathrm{s}) &= (2\,\mathrm{rad/s}) \cdot (2\,\mathrm{s}) + \frac{1}{2}(0.4\,\mathrm{rad/s}) \cdot (4\,\mathrm{s^2}) \\ &= 4.8\,\mathrm{rad} = \frac{4.8}{2\pi}\,\mathrm{rev} = 0.76\,\mathrm{rev}. \end{aligned}$$

$$\boxed{\theta = 0.76\,\mathrm{rev}}$$

3. Now that we know the angular displacement θ, the distance travelled by the mass-center is the arc-length corresponding to θ, *i.e.*,

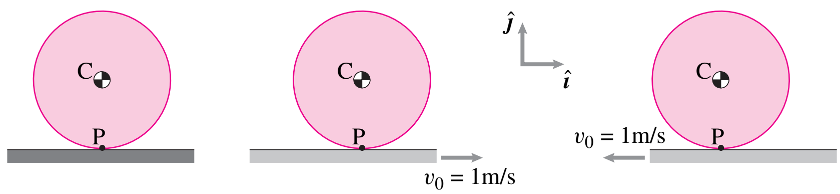
$$x = R\theta = (0.25\,\mathrm{m}) \cdot (4.8) = 1.2\,\mathrm{m}.$$

$$\boxed{x = 1.2\,\mathrm{m}}$$

Note that we could have found the distance travelled by the mass-center by integrating the equation $\ddot{x} = 0.1\,\mathrm{m/s^2}$ twice.

SAMPLE 17.12 Condition of pure rolling. A cylinder of radius $R = 20\,\mathrm{cm}$ rolls on a flat surface with absolute angular speed $\omega = 12\,\mathrm{rad/s}$ under the conditions shown in the figure (In cases (ii) and (iii), you may think of the 'flat surface' as a conveyor belt). In each case,

1. Write the condition for pure rolling.
2. Find the velocity of the center C of the cylinder.



(i) Fixed base (ii) Base moves to the right (iii) Base moves to the left

Figure 17.41:

Solution At any instant during rolling, the cylinder makes a point-contact with the flat surface. Let the point of instantaneous contact on the cylinder be P, and let the corresponding point on the flat surface be Q. The condition of pure rolling, in each case, is $\vec{v}_P = \vec{v}_Q$, that is, there is no relative motion between the two contacting points (a relative motion will imply slip). Now, we analyze each case.

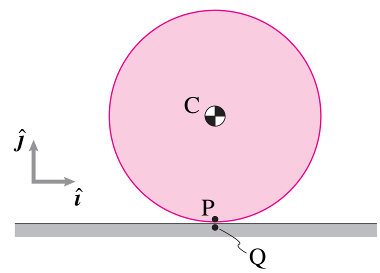


Figure 17.42: The cylinder rolls on the flat surface. Instantaneously, point P on the cylinder is in contact with point Q on the flat surface. For pure rolling, points P and Q must have the same velocity.

Case(i) In this case, the bottom surface is fixed. Therefore,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = \vec{0}$.
2. Velocity of the center:

$$\begin{aligned} \vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = \vec{0} + (-\omega \hat{k}) \times R\hat{\jmath} \\ &= \omega R \hat{\imath} = (12\,\mathrm{rad/s}) \cdot (0.2\,\mathrm{m})\hat{\imath} = 2.4\,\mathrm{m/s}\hat{\imath}. \end{aligned}$$

Case(ii) In this case, the bottom surface moves with velocity $\vec{v} = 1\,\mathrm{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = 1\,\mathrm{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = 1\,\mathrm{m/s}\hat{\imath}$.
2. Velocity of the center:

$$\begin{aligned} \vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = v_0 \hat{\imath} + \omega R \hat{\imath} \\ &= 1\,\mathrm{m/s}\hat{\imath} + 2.4\,\mathrm{m/s}\hat{\imath} = 3.4\,\mathrm{m/s}\hat{\imath}. \end{aligned}$$

Case(iii) In this case, the bottom surface moves with velocity $\vec{v} = -1\,\mathrm{m/s}\hat{\imath}$. Therefore, $\vec{v}_Q = -1\,\mathrm{m/s}\hat{\imath}$. Thus,

1. The condition of pure rolling is: $\vec{v}_P = \vec{v}_Q = -1\,\mathrm{m/s}\hat{\imath}$.
2. Velocity of the center:

$$\begin{aligned} \vec{v}_C &= \vec{v}_P + \vec{\omega} \times \vec{r}_{C/P} = -v_0 \hat{\imath} + \omega R \hat{\imath} \\ &= -1\,\mathrm{m/s}\hat{\imath} + 2.4\,\mathrm{m/s}\hat{\imath} = 1.4\,\mathrm{m/s}\hat{\imath}. \end{aligned}$$

$$\begin{array}{llll} \text{(a)}: & \text{(i)}\vec{v}_P = \vec{0}, & \text{(ii)}\vec{v}_P = 1\,\mathrm{m/s}\hat{\imath}, & \text{(iii)}\vec{v}_P = -1\,\mathrm{m/s}\hat{\imath}, \\ \text{(b)}: & \text{(i)}\vec{v}_C = 2.4\,\mathrm{m/s}\hat{\imath}, & \text{(ii)}\vec{v}_C = 3.4\,\mathrm{m/s}\hat{\imath}, & \text{(iii)}\vec{v}_C = 1.4\,\mathrm{m/s}\hat{\imath} \end{array}$$

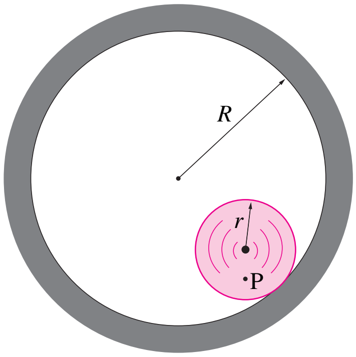


Figure 17.43: A uniform disk of radius r rolls without slipping inside a fixed cylinder.

SAMPLE 17.13 Motion of a point on a disk rolling inside a cylinder. A uniform disk of radius r rolls without slipping with constant angular speed ω inside a fixed cylinder of radius R. A point P is marked on the disk at a distance ℓ ($\ell < r$) from the center of the disk. At a general time t during rolling, find

1. the position of point P,
2. the velocity of point P, and
3. the acceleration of point P

Solution Let the disk be vertically below the center of the cylinder at $t = 0\,\text{s}$ such that point P is vertically above the center of the disk (Fig. 17.44). At this instant, Q is the point of contact between the disk and the cylinder. Let the disk roll for time t such that at instant t the line joining the two centers (line OC) makes an angle ϕ with its vertical position at $t = 0\,\text{s}$. Since the disk has rolled for time t at a constant angular speed ω, point P has rotated counter-clockwise by an angle $\theta = \omega t$ from its original vertical position P'.

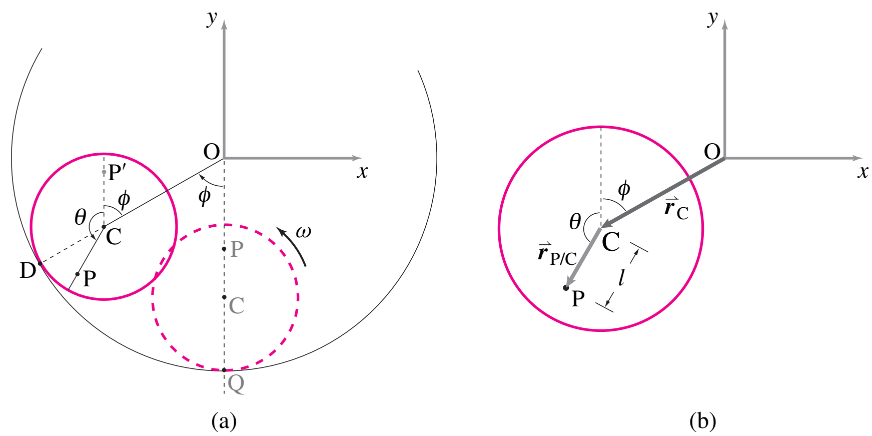


Figure 17.44: Geometry of motion: keeping track of point P while the disk rolls for time t, rotating by angle $\theta = \omega t$ inside the cylinder.

1. **Position of point P:** From Fig. 17.44(b) we can write

$$\vec{\boldsymbol{r}}_P = \vec{\boldsymbol{r}}_C + \vec{\boldsymbol{r}}_{P/C} = (R-r)\hat{\boldsymbol{\lambda}}_{OC} + \ell\hat{\boldsymbol{\lambda}}_{CP}$$

where

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OC} &= \text{a unit vector along OC } = -\sin\phi\,\hat{\boldsymbol{\imath}} - \cos\phi\,\hat{\boldsymbol{\jmath}},\\ \hat{\boldsymbol{\lambda}}_{CP} &= \text{a unit vector along CP } = -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}.\end{aligned}$$

Thus,

$$\vec{\boldsymbol{r}}_P = [-(R-r)\sin\phi - \ell\sin\theta]\hat{\boldsymbol{\imath}} + [-(R-r)\cos\phi + \ell\cos\theta]\hat{\boldsymbol{\jmath}}.$$

We have thus obtained an expression for the position vector of point P as a function of ϕ and θ. Since we also want to find velocity and acceleration of point P, it will be nice to express $\vec{\boldsymbol{r}}_P$ as a function of t. As noted above, $\theta = \omega t$; but how do we find ϕ as a function of t? Note that the center of the disk C is going around point O in circles with angular velocity $-\dot{\phi}\hat{\boldsymbol{k}}$. The disk, however, is rotating with angular velocity $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ about the instantaneous center of rotation, point D. Therefore, we can calculate the velocity of point C in two ways:

$$\begin{aligned}
& \vec{\boldsymbol{v}}_C &=& \vec{\boldsymbol{v}}_C \\
\text{or} \quad & \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{C/D} &=& -\dot{\phi}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{C/O} \\
\text{or} \quad & \omega\hat{\boldsymbol{k}} \times r(-\hat{\boldsymbol{\lambda}}_{OC}) &=& -\dot{\phi}\hat{\boldsymbol{k}} \times (R-r)\hat{\boldsymbol{\lambda}}_{OC} \\
\text{or} \quad & -\omega r(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_{OC}) &=& -\dot{\phi}(R-r)(\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}}_{OC}) \\
\Rightarrow \quad & \frac{r}{R-r}\omega &=& \dot{\phi}.
\end{aligned}$$

Integrating the last expression with respect to time, we obtain

$$\phi = \frac{r}{R-r}\omega t.$$

Let

$$q = \frac{r}{R-r},$$

then, the position vector of point P may now be written as

$$\vec{\boldsymbol{r}}_P = [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\imath}} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\jmath}}. \quad (17.37)$$

2. **Velocity of point P:** Differentiating Eqn. (17.37) once with respect to time we get

$$\vec{\boldsymbol{v}}_P = -\omega[(R-r)q\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\imath}} + \omega[(R-r)q\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\jmath}}.$$

Substituting $(R-r)q = r$ in $\vec{\boldsymbol{v}}_P$ we get

$$\vec{\boldsymbol{v}}_P = -\omega r[\{\cos(q\omega t) + \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\imath}} - \{\sin(q\omega t) - \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\jmath}}]. \quad (17.38)$$

3. **Acceleration of point P:** Differentiating Eqn. (17.38) once with respect to time we get

$$\vec{\boldsymbol{a}}_P = -\omega^2 r[-\{q\sin(q\omega t) + \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\imath}} - \{q\cos(q\omega t) - \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\jmath}}]. \quad (17.39)$$

$$\begin{aligned}
\vec{\boldsymbol{r}}_P &= [-(R-r)\sin(q\omega t) - \ell\sin(\omega t)]\hat{\boldsymbol{\imath}} + [-(R-r)\cos(q\omega t) + \ell\cos(\omega t)]\hat{\boldsymbol{\jmath}} \\
\vec{\boldsymbol{v}}_P &= -\omega r[\{\cos(q\omega t) + \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\imath}} - \{\sin(q\omega t) - \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\jmath}}] \\
\vec{\boldsymbol{a}}_P &= -\omega^2 r[-\{q\sin(q\omega t) + \frac{\ell}{r}\sin(\omega t)\}\hat{\boldsymbol{\imath}} - \{q\cos(q\omega t) - \frac{\ell}{r}\cos(\omega t)\}\hat{\boldsymbol{\jmath}}]
\end{aligned}$$

SAMPLE 17.14 The rolling disk: instantaneous kinematics. For the rolling disk in Sample 17.13, let $R = 4\,\text{ft}$, $r = 1\,\text{ft}$ and point P be on the rim of the disk. Assume that at $t = 0$, the center of the disk is vertically below the center of the cylinder and point P is on the vertical line joining the two centers. If the disk is rolling at a constant speed $\omega = \pi\,\text{rad/s}$, find

1. the position of point P and center C at $t = 1\,\text{s}$, $3\,\text{s}$, and $5.25\,\text{s}$,
2. the velocity of point P and center C at those instants, and
3. the acceleration of point P and center C at the same instants as above.

Draw the position of the disk at the three instants and show the velocities and accelerations found above.

Solution The general expressions for position, velocity, and acceleration of point P obtained in Sample 17.13 can be used to find the position, velocity, and acceleration of any point on the disk by substituting an appropriate value of ℓ in equations (17.37), (17.38), and (17.39). Since $R = 4r$,

$$q = \frac{r}{R-r} = \frac{1}{3}.$$

Now, point P is on the rim of the disk and point C is the center of the disk. Therefore,

$$\begin{aligned} \text{for point P:} \qquad \ell &= r, \\ \text{for point C:} \qquad \ell &= 0. \end{aligned}$$

Substituting these values for ℓ, and $q = 1/3$ in equations (17.37), (17.38), and (17.39) we get the following.

1. **Position:**

$$\begin{aligned} \vec{\boldsymbol{r}}_C &= -3r\left[\sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} + \cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{r}}_P &= \vec{\boldsymbol{r}}_C + r\left[-sin\left(\omega t\right)\hat{\boldsymbol{\imath}} + \cos\left(\omega t\right)\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

2. **Velocity:**

$$\begin{aligned} \vec{\boldsymbol{v}}_C &= -\omega r\left[\cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} - \sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{v}}_P &= -\omega r\left[\left\{\cos\left(\frac{\omega t}{3}\right) + \cos\left(\omega t\right)\right\}\hat{\boldsymbol{\imath}} - \left\{\sin\left(\frac{\omega t}{3}\right) - \sin\left(\omega t\right)\right\}\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

3. **Acceleration:**

$$\begin{aligned} \vec{\boldsymbol{a}}_C &= \frac{\omega^2 r}{3}\left[\sin\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\imath}} + \cos\left(\frac{\omega t}{3}\right)\hat{\boldsymbol{\jmath}}\right], \\ \vec{\boldsymbol{a}}_P &= \omega^2 r\left[\left\{\frac{1}{3}\sin\left(\frac{\omega t}{3}\right) + \sin\left(\omega t\right)\right\}\hat{\boldsymbol{\imath}} + \left\{\frac{1}{3}\cos\left(\frac{\omega t}{3}\right) - \cos\left(\omega t\right)\right\}\hat{\boldsymbol{\jmath}}\right]. \end{aligned}$$

We can now use these expressions to find the position, velocity, and acceleration of the two points at the instants of interest by substituting $r = 1\,\text{ft}$, $\omega = \pi\,\text{rad/s}$, and appropriate values of t. These values are shown in Table 17.1.

The velocity and acceleration of the two points are shown in Figures 17.45(a) and (b) respectively.

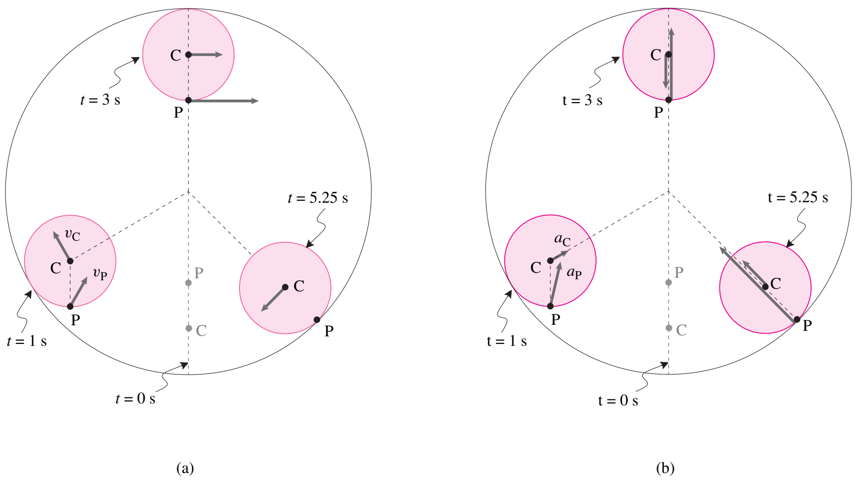


Figure 17.45: (a) Velocity and (b) Acceleration of points P and C at $t = 1$ s, 3 s, and 5.25 s.

It is worthwhile to check the directions of velocities and the accelerations by thinking about the velocity and acceleration of point P as a vector sum of the velocity (same for acceleration) of the center of the disk and the velocity (same for acceleration) of point P with respect to the center of the disk. Since the motions involved are circular motions at constant rate, a visual inspection of the velocities and the accelerations is not very difficult. Try it.

t	1 s	3 s	5.25 s
$\vec{r}_C$ (ft)	$3(-\frac{\sqrt{3}}{2}\hat{\imath} - \frac{1}{2}\hat{\jmath})$	$3\hat{\jmath}$	$3(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{r}_P$ (ft)	$\vec{r}_C - \hat{\jmath}$	$\vec{r}_C - \hat{\jmath}$	$4(\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_C$ (ft/s)	$\pi(-\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$\pi\hat{\imath}$	$\pi(-\frac{1}{\sqrt{2}}\hat{\imath} - \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s)	$\pi(\frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath})$	$2\pi\hat{\imath}$	$\vec{0}$
$\vec{a}_C$ (ft/s^2)	$\frac{\pi^2}{3}(\frac{\sqrt{3}}{2}\hat{\imath} + \frac{1}{2}\hat{\jmath})$	$-\frac{\pi^2}{3}\hat{\jmath}$	$\frac{\pi^2}{3}(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$
$\vec{v}_P$ (ft/s^2)	$11.86(.24\hat{\imath} + .97\hat{\jmath})$	$\frac{2\pi^2}{3}\hat{\jmath}$	$13.16(-\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath})$

Table 17.1: Position, velocity, and acceleration of point P and point C

SAMPLE 17.15 The rolling disk: path of a point on the disk. For the rolling disk in Sample 17.13, take $\omega = \pi$ rad/s. Draw the path of a point on the rim of the disk for one complete revolution of the center of the disk around the cylinder for the following conditions:

1. $R = 8r$,
2. $R = 4r$, and
3. $R = 2r$.

Solution In Sample 17.13, we obtained a general expression for the position of a point on the disk as a function of time. By computing the position of the point for various values of time t up to the time required to go around the cylinder for one complete cycle, we can draw the path of the point. For the various given conditions, the variable that changes in Eqn. (17.37) is q. We can write a computer program to generate the path of any point on the disk for a given set of R and r. Here is a pseudocode to generate the required path on a computer according to Eqn. (17.37).

A pseudocode to plot the path of a point on the disk:

```
(pseudo-code) program rollingdisk
%--------------------------------------------------------------
% This code plots the path of any point on a disk of radius
% 'r' rolling with speed 'w' inside a cylinder of radius 'R'.
% The point of interest is distance 'l' away from the center of
% the disk. The coordinates x and y of the specified point P are
% calculated according to the relation mentioned above.
%---------------------------------------------------------------
phi = pi/50*[1,2,3,...,100]   % make a vector phi from 0 to 2*pi
x = R*cos(phi)                % create points on the outer cylinder
y = R*sin(phi)
plot y vs x                   % plot the outer cylinder
hold this plot                % hold to overlay plots of paths
q = r/(R-r)                   % calculate q.
T = 2*pi/(q*w)                % calculate time T for going around-
                              % the cylinder once at speed 'w'.
t = T/100*[1,2,3, ..., 100]   % make a time vector t from 0 to T-
                              % taking 101 points.

rcx = -(R-r)*(sin(q*w*t))     % find the x coordinates of pt. C.
rcy = -(R-r)*(cos(q*w*t))     % find the y coordinates of pt. C.
rpx = rcx-l*sin(w*t)          % find the x coordinates of pt. P.
rpy = rcy + l*cos(q*t)        % find the y coordinates of pt. P.
plot rpy vs rpx               % plot the path of P and the path
plot rcy vs rcx               % of C. For path of C
```

Once coded, we can use this program to plot the paths of both the center and the point P on the rim of the disk for the three given situations. Note that for any point on the rim of the disk $l = r$ (see Fig 17.44).

1. Let $R = 4$ units. Then $r = 0.5$ for $R = 8r$. To plot the required path, we run our program `rollingdisk` with desired input,

   ```
   R = 4
   r = 0.5
   w = pi
   l = 0.5
   execute rollingdisk
   ```

   The plot generated is shown in Fig.17.46 with a few graphic elements added for illustrative purposes.

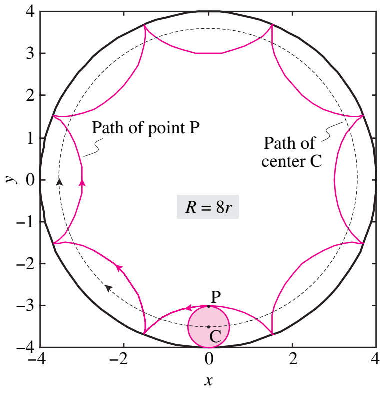


Figure 17.46: Path of point P and the center C of the disk for $R = 8r$.

2. Similarly, for $R = 4r$ we type:

   ```
   R = 4
   r = 1
   w = pi
   l = 1
   execute rollingdisk
   ```

   to plot the desired paths. The plot generated in this case is shown in Fig.17.47

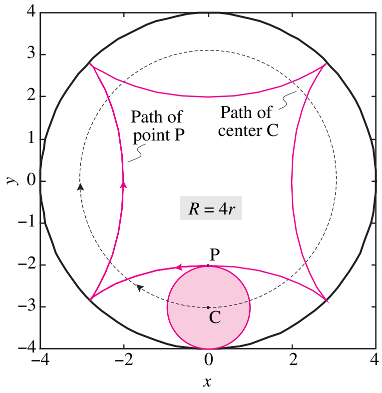


Figure 17.47: Path of point P and the center C of the disk for $R = 4r$.

3. The last one is the most interesting case. The plot obtained in this case by typing:

   ```
   R = 4
   r = 2
   w = pi
   l = 2
   execute rollingdisk
   ```

   is shown in Fig.17.48. Point P just travels on a straight line! In fact, every point on the rim of the disk goes back and forth on a straight line. Most people find this motion odd at first sight. You can roughly verify the result by cutting a hole twice the diameter of a coin (say a US quarter or dime) in a piece of cardboard and rolling the coin around inside while watching a marked point on the perimeter.

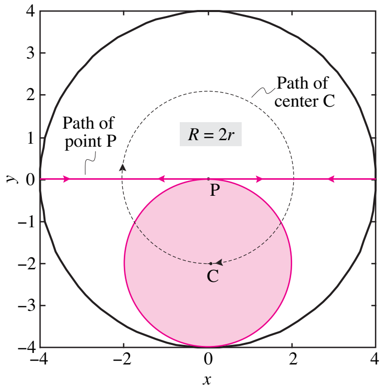


Figure 17.48: Path of point P and the center C of the disk for $R = 2r$.

A curiosity. We just discovered something simple about the path of a point on the edge of a circle rolling in another circle that is twice as big. The edge point moves in a straight line. In contrast one might think about the motion of the center G of a *straight* line segment that *slides* against two *straight* walls as in sample 17.23. A problem couldn't be more different. Naturally the path of point G is a circle (as you can check physically by looking at the middle of a ruler as you hold it sliding against a wall-floor corner).

17.3 Kinematics of rolling and sliding

17.3.1 A stone in a wheel. A round wheel rolls to the right. At time $t = 0$ it picks up a stone from the road. The stone is stuck in the edge of the wheel. You want to know the direction of the rock's motion just before and after it next hits the ground. Here are some candidate answers:

- When the stone approaches the ground its motion is tangent to the ground.
- The stone approaches the ground at angle x (you name it).
- When the stone approaches the ground its motion is perpendicular to the ground.
- The stone approaches the ground at various angles depending on the following conditions(...you list the conditions.)

Although you could address this question analytically, you are to try to get a clear answer by looking at computer generated plots. In particular, you are to plot the pebble's path for a small interval of time near when the stone next touches the ground. You should pick the parameters that make your case for an answer the strongest. You may make more than one plot.

Here are some steps to follow:

a) Assuming the wheel has radius R_w and the pebble is a distance R_p from the center (not necessarily equal to R_w). The pebble is directly below the center of the wheel at time $t = 0$. The wheel spins at constant clockwise rate ω. The x-axis is on the ground and $x(t = 0) = 0$. The wheel rolls without slipping. Using a clear well labeled drawing (use a compass and ruler or a computer drawing program), show that

$$x(t) = \omega t R_w - R_p \sin(\omega t)$$

$$y(t) = R_w - R_p \cos(\omega t)$$

b) Using this relation, write a program to make a plot of the path of the pebble as the wheel makes a little more than one revolution. Also show the outline of the wheel and the pebble itself at some intermediate time of interest. [Use any software and computer that pleases you.]

c) Change whatever you need to change to make a good plot of the pebble's path for a small amount of time as the pebble approaches and leaves the road. Also show the wheel and the pebble at some time in this interval.

d) In this configuration the pebble moves a *very* small distance in a small time so your axes need to be scaled. But make sure your x- and y- axes have the same scale so that the path of the pebble and the outline of the wheel will not be distorted.

e) How does your computer output buttress your claim that the pebble approaches and leaves the ground at the angles you claim?

f) Think of something about the pebble in the wheel that was not explicitly asked in this problem and explain it using the computer, and/or hand calculation and/or a drawing.

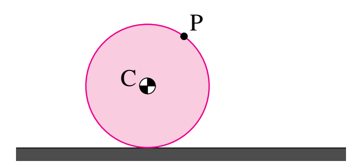
17.3.2 A uniform disk of radius r rolls at a constant rate without slip. A small ball of mass m is attached to the outside edge of the disk. What is the force required to hold the disk in place when the mass is above the center of the disk?

17.3.3 Rolling at constant rate. A round disk rolls on the ground at constant rate. It rolls $1\frac{1}{4}$ revolutions over the time of interest.

a) **Particle paths.** Accurately plot the paths of three points: the center of the disk C, a point on the outer edge that is initially on the ground, and a point that is initially half way between the former two points. [Hint: Write a parametric equation for the position of the points. First find a relation between ω and v_C. Then note that the position of a point is the position of the center plus the position of the point relative to the center.] Draw the paths on the computer, make sure x and y scales are the same.

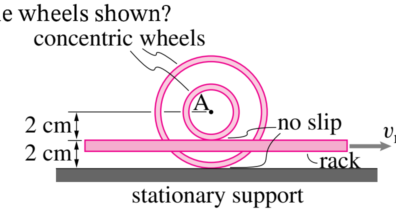
b) **Velocity of points.** Find the velocity of the points at a few instants in the motion: after $\frac{1}{4}$, $\frac{1}{2}$, $\frac{3}{4}$, and 1 revolution. Draw the velocity vector (by hand) on your plot. Draw the direction accurately and draw the lengths of the vectors in proportion to their magnitude. You can find the velocity by differentiating the position vector or by using relative motion formulas appropriately. Draw the disk at its position after one quarter revolution. Note that the velocity of the points is perpendicular to the line connecting the points to the ground contact.

c) **Acceleration of points.** Do the same as above but for acceleration. Note that the acceleration of the points is parallel to the line connecting the points to the center of the disk.



Problem 17.3.3

17.3.4 The concentric wheels are welded to each other and roll without slip on the rack and stationary support. The rack moves to the right at $v_r = 1\,\mathrm{m/s}$. What is the velocity of point A in the middle of the wheels shown?



Problem 17.3.4

17.3.5 Questions (a) - (e) refer to the cylinders in the configuration shown in the figure. Question (f) is closely related. Answer the questions in terms of the given quantities (and any other quantities you define if needed).

a) What is the speed (magnitude of velocity) of point c?

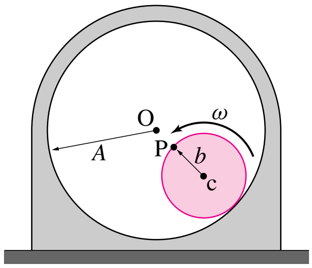
b) What is the speed of point P?

c) What is the magnitude of the acceleration of point c?

d) What is the magnitude of the acceleration of point P?

e) What is the radius of curvature of the path of the particle P at the point of interest?

f) In the special case of $A = 2b$ what is the curve which particle P traces (for all time)? Sketch the path.



Problem 17.3.5: A little cylinder (with outer radius b and center at point c) rolls without slipping inside a bigger fixed cylinder (with inner radius A and center at point O). The *absolute* angular velocity of the little cylinder ω is *constant.* P is attached to the outside edge of the little cylinder. At the instant of interest, P is on the line between O and c.