17.2 General planar mechanics of a rigid-object

We now apply the kinematics ideas of the last section to the general mechanics principles in Table I in the inside cover. The goal is to understand the relation between forces and motion for a planar object in general 2-D motion. The simple measures① of motion will be the displacement of one reference point $0'$ on the object and the rotation of the object. We also need the first and second time derivatives of the displacement and rotation, altogether we use $\vec{r}_{0'}, \vec{v}_{0'}, \vec{a}_{0'}, \theta, \vec{\omega}$ and $\vec{\alpha}$.

We will treat all bodies as if they are squished into the plane (planar) and moving in the plane (planar motion). But the analysis is sensible for an object that is symmetric with respect to the plane containing the velocities (see Box 17.1 on page 7).

①**Advanced aside:** What we call "simple measures" are examples of "generalized coordinates" in more advanced books. The idea sounds intimidating, but is simply this: If something can only move in a few ways, you should only keep track of the motion with that many variables. The kinematics of a rigid object (Sect. 17.1) allow us to "evaluate" the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change in terms of these "simple measures". By "evaluate" we mean express the motion quantities in terms of these measures. The alternative is as sums over Avogadro's number of particles (There are on the order of 10^{23} atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we would still be stuck with integrals over complicated regions if we did not describe the motion with as few variables as possible. In the case of 2-D rigid object motion, the position of a reference point (x and y) with the rotation θ is called a set of *minimal coordinates*. These, and their time derivatives are the minimal information needed to describe all important mechanics motion quantities.

The balance laws for a rigid object

As always, once you have defined the system and the forces acting on it by drawing a free-body diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\text{Linear momentum balance:} \qquad \sum \vec{F}_i = \dot{\vec{L}} \qquad \text{and}$$
$$\text{Angular momentum balance:} \qquad \sum \vec{M}_{i/O} = \dot{\vec{H}}_{/O}.$$

The same point 0, any point, is used on both sides of the angular momentum balance equation. We also have power balance which, for a system with no internal energy or dissipation, is

$$\text{Power balance:} \qquad P = \dot{E}_{\text{K}}.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one object or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power P only depends on the position of the forces and the motions of the material points at those positions. But the kinetic energy E_{K} and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the positions and accelerations are known (chapter 13);
- Systems where all points have the same acceleration. That is, the system moves like a rigid object that does not rotate (chapter 14); and

- Systems where all points move in circles about the same fixed axis in 2D (chapter 15).

Now we go on to consider the general 2-D motions of a planar rigid object. It's now a little harder to evaluate $\vec{L}, \dot{\vec{L}}, \vec{H}_{/O}, \dot{\vec{H}}_{/O}, E_K$ and $\dot{E}_K$. But not much.

Summary of the motion quantities

Table I in the back of the book describes the motion quantities for various special cases, including the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, are the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for any system with any motion, by the motion of the center-of-mass

$$\vec{L} = m_{\text{tot}} \vec{v}_{\text{cm}} \quad \text{and} \quad \dot{\vec{L}} = m_{\text{tot}} \vec{a}_{\text{cm}}, \tag{17.16}$$

and the angular momentum has two contributions, one from the motion of the center-of-mass and one from rotation of the object about the center of mass,

$$\vec{H}_{/O} = \overbrace{\vec{r}_{\text{cm/O}} \times (m_{\text{tot}} \vec{v}_{\text{cm}})}^{\boxed{\text{Angular momentum due to motion of the center-of-mass}}} + \overbrace{I_{zz}^{\text{cm}} \vec{\omega}}^{\boxed{\text{Angular momentum due to motion relative to the center-of-mass}}} \tag{17.17}$$

$$\text{and} \quad \dot{\vec{H}}_{/O} = \vec{r}_{\text{cm/O}} \times (m_{\text{tot}} \vec{a}_{\text{cm}}) + I_{zz}^{\text{cm}} \dot{\vec{\omega}}. \tag{17.18}$$

These simplifications, for 2D objects moving in the plane, are discussed in box 17.2 on page 8. An important special case for the angular momentum evaluation is when the reference point is coincident with the center-of-mass. Then the angular momentum and its rate of change simplify to

$$\vec{H}_{\text{cm}} = I_{zz}^{\text{cm}} \vec{\omega} \qquad \text{and} \qquad \dot{\vec{H}}_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\vec{\omega}}. \tag{17.19}$$

The kinetic energy and its rate of change are given by

$$E_{\mathrm{K}} = \overbrace{\frac{1}{2} m_{\mathrm{tot}} \underbrace{v_{cm}^2}_{\vec{v}_{\mathrm{cm}} \cdot \vec{v}_{\mathrm{cm}}}}^{\boxed{\text{kinetic energy from center-of-mass motion}}} + \overbrace{\frac{1}{2} I_{zz}^{\mathrm{cm}} \omega^2}^{\boxed{\text{kinetic energy relative to the center-of-mass}}} \tag{17.20}$$

$$\text{and} \quad \dot{E}_{\mathrm{K}} = m_{\mathrm{tot}} \underbrace{v_{cm} \dot{v}_{cm}}_{\vec{v}_{\mathrm{cm}} \cdot \vec{a}_{\mathrm{cm}}} + I_{zz}^{\mathrm{cm}} \omega \dot{\omega} \tag{17.21}$$

The relations above are easily derived from the general center of mass theorems (see box 17.2 on page 8 for some of these derivations).

Equations of motion

Putting together the general balance equations and the expressions for the motion quantities we can now write linear momentum balance, angular momentum balance and power balance as:

$$\begin{aligned}
\text{LMB}: \quad & \sum \vec{F}_i = m_{\mathrm{tot}} \vec{a}_{\mathrm{cm}}, && \text{(a)} \\
\text{AMB}: \quad & \sum \vec{M}_{\mathrm{O}} = \vec{r}_{\mathrm{cm/O}} \times (m_{\mathrm{tot}} \vec{a}_{\mathrm{cm}}) + I_{zz}^{\mathrm{cm}} \dot{\vec{\omega}} && \text{(b)} \\
\text{or} \quad & \sum \vec{M}_{\mathrm{cm}} = I_{zz}^{\mathrm{cm}} \dot{\vec{\omega}}, \quad \text{and} && \\
\text{Power}: \quad & \vec{F}_{\mathrm{tot}} \cdot \vec{v}_{\mathrm{cm}} + \vec{\omega} \cdot \vec{M}_{\mathrm{cm}} = m_{\mathrm{tot}} v \dot{v} + I_{zz}^{\mathrm{cm}} \omega \dot{\omega}. && \text{(c)}
\end{aligned} \tag{17.22}$$

Independent equations?

Equations are only independent if no one of them can be derived from the others. When counting equations and unknowns one needs to make sure one is writing independent equations. How many independent equations are in the set eqns. (17.22)abc applied to one free object diagram? The short answer is 3.

The linear momentum balance equation 17.22a yields two independent equations by dotting with any two non-parallel vectors (say, $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$). Dotting with a third vector yields a dependent equation.

For any one reference point the angular momentum equation 17.22b yields one scalar equation. It is a vector equation but always has zero components in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions. But angular momentum equation can yield up to three independent equations by being applied to any set of three non-collinear points.

The power balance equation is one scalar equation.

In total, however, the full set of equations above only makes up a set of three independent equations.

To avoid thinking about what is or is not an independent set of equations some people prefer to stick with one of the canonical sets of independent equations:

- The coordinate based set ("old standard")
  - {LMB}$\cdot\hat{\boldsymbol{\imath}}$ or, equivalently, $\sum F_x = m_{\text{tot}} a_{\text{cm}_x}$,
  - {LMB}$\cdot\hat{\boldsymbol{\jmath}}$ or, equivalently, $\sum F_y = m_{\text{tot}} a_{\text{cm}_y}$, and
  - {AMB}$\cdot\hat{\boldsymbol{k}}$ or, equivalently, $\sum M_{\text{cm}} = I_{zz}^{\text{cm}}\dot{\omega}$.
- Moment only (good for eliminating unknown reaction forces)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {AMB about pt C}$\cdot\hat{\boldsymbol{k}}$ (C is a third point not on the line AB)
- Two moments and a force component
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}$ (where $\hat{\boldsymbol{\lambda}}$ is not perpendicular to the line AB)
- Two force components and a moment (also good for eliminating unknown forces)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_1$ (where $\hat{\boldsymbol{\lambda}}_1$ is any unit vector)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_2$ (where $\hat{\boldsymbol{\lambda}}_2$ is any other unit vector)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)

Any of these will always do the job. The power balance equation is often used as a consistency check rather than an independent equation.

From a theoretical point of view one might ask the related question of which of the equations of motion can be derived from the others. There are many answers. Here are some of them:

- Power balance follows from LMB and AMB,
- AMB about three non-collinear points implies LMB, and
- LMB and power balance yield AMB

Interestingly, there is no way to derive angular momentum balance from linear momentum balance without the questionable microscopic assumptions discussed in box B.3 on page 1203.

Some simple examples

Here we consider some simple examples of unconstrained motion of a rigid object.

Example: The simplest case: no force and no moment.
If the net force and moment applied to an object are zero we have:

$$\begin{aligned} \text{LMB} \quad &\Rightarrow \vec{\mathbf{0}} &=& \; m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}} \quad \text{and} \\ \text{AMB} \quad &\Rightarrow \vec{\mathbf{0}} &=& \; I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \end{aligned}$$

so $\vec{\boldsymbol{a}}_{\text{cm}} = \vec{\mathbf{0}}$ and $\dot{\omega} = 0$ and the object moves at constant speed in a constant direction with a constant rate of rotation, all determined by the initial conditions. Throw an object in space and its center-of-mass goes in a straight line and it spins at constant rate (subject to the 2-D restrictions of this chapter).

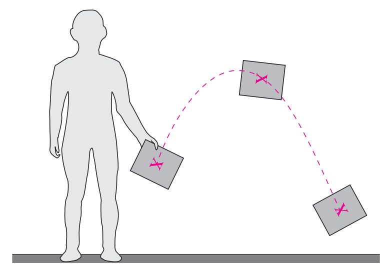


Figure 17.22: The X marked at the center-of-mass of a thrown spinning clipboard follows a parabolic trajectory.

Example: Constant force applied to the center-of-mass.
In this case angular momentum balance about the center-of-mass again yields that the rotation rate is constant. Linear momentum balance is now the same as for a particle at the center-of-mass, *i.e.,* the center-of-mass has a parabolic trajectory.

Near-earth (constant) gravity provides a simple example. An 'X' marked at the center-of-mass of a clipboard tossed across a room follows a parabolic trajectory (see fig. 17.22).

Example: Constant force not at the center-of-mass.
Assume the only force applied to an object is a constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ at A (see fig. 17.23). Then linear momentum balance gives us that

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad F\hat{\boldsymbol{\imath}} = m\vec{\boldsymbol{a}}_{\text{G}} \quad \Rightarrow \quad \vec{\boldsymbol{a}}_{\text{G}} = F/m\hat{\boldsymbol{\imath}} = \text{constant}.$$

So if the object starts at rest, the point G will move in a straight line in the $\hat{\boldsymbol{\imath}}$ direction (The common intuition that point G will be pulled up is incorrect). Angular momentum about the center-of-mass gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\text{cm}\,i} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \quad &\Rightarrow \quad \left\{\vec{\boldsymbol{r}}_{\text{A/G}} \times F\hat{\boldsymbol{\imath}} = I_{zz}^{\text{cm}}\ddot{\theta}\hat{\boldsymbol{k}}\right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \ddot{\theta} + \frac{F\ell}{I_{zz}^{\text{cm}}}\sin\theta = 0, \end{aligned}$$

with $\ell = |\vec{\boldsymbol{r}}_{\text{A/G}}|$, which is the pendulum equation. That is, the object can swing back and forth about $\theta = 0$ just like a pendulum, approximately sinusoidally if the angle θ starts small and with $\dot{\theta}$ initially also small. [One might wonder how to do this experiment. One way would be with a jet on a space craft that keeps re-orienting itself to keep in a constant spatial direction as the object changes orientation. Another would be with a string tied to A and pulled from a great distance.]

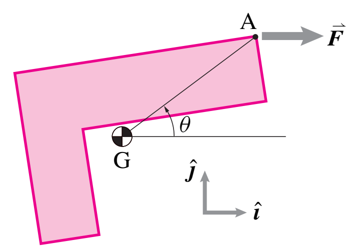


Figure 17.23: The only force applied to the object is the constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ applied at point A. The resulting motion is a constant acceleration of the center-of-mass $\vec{\boldsymbol{a}}_{\text{G}} = (F/m)\hat{\boldsymbol{\imath}}$ and an oscillatory motion of θ identical to that of a pendulum hinged at G. Note that the center of mass goes in a straight line.

Example: The flight of an arrow or rocket.
As a primitive model of an arrow or rocket assume that the only force is from drag on the fins at C and that this force opposes motion according to

$$\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}_{\text{C}}$$

where c is a drag coefficient (see fig. 17.24). From linear momentum balance we have:

$$\begin{aligned} \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -c\vec{\boldsymbol{v}}_{\text{C}} &= m\dot{\vec{\boldsymbol{v}}} \\ m\dot{\vec{\boldsymbol{v}}} &= -c\left(\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{C/G}}\right) \\ &= -c\left(\vec{\boldsymbol{v}} + \dot{\theta}\hat{\boldsymbol{k}} \times (-\ell\hat{\boldsymbol{\lambda}})\right) \\ (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{n}}) \quad \Rightarrow \quad \dot{\vec{\boldsymbol{v}}} &= \frac{c}{m}\left(\dot{\theta}\ell\hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}\right). \end{aligned}$$

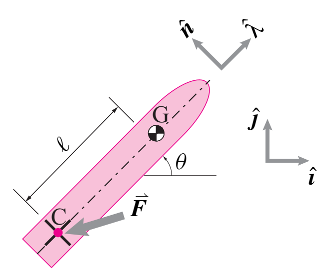


Figure 17.24: A rocket is pointed in the direction $\hat{\boldsymbol{\lambda}}$ which makes an angle θ with the positive x axis. The position and velocity of the center-of-mass at G are called $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$. The velocity of the tail is $\vec{\boldsymbol{v}}_{\text{C}}$

So if $\vec{\boldsymbol{v}}, \theta$ and $\dot{\theta}$ are known the acceleration $\dot{\vec{\boldsymbol{v}}}$ is calculated by the formula above.

Similarly angular momentum balance about G gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\text{G}} = \dot{\vec{\boldsymbol{H}}}_{\text{G}} \quad &\Rightarrow \quad \{\vec{\boldsymbol{r}}_{\text{C/G}} \times \vec{\boldsymbol{F}} &= I_{zz}^{\text{cm}} \dot{\omega} \hat{\boldsymbol{k}}\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad I_{zz}^{\text{cm}} \dot{\omega} &= \vec{\boldsymbol{r}}_{\text{C/G}} \times \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{k}}. \end{aligned}$$

Then, making the same substitutions as before for $\vec{\boldsymbol{r}}_{\text{C/G}}$ and $\vec{\boldsymbol{F}}$ we get

$$\dot{\omega} = \frac{c\ell}{I_{zz}^{\text{cm}}} \left(\hat{\boldsymbol{\lambda}} \times \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}} - \dot{\theta} \ell \right)$$

which determines the rate of change of ω if the present values of $\vec{\boldsymbol{v}}, \theta$ and $\dot{\theta}$ are known.

Setting up differential equations for solution

If one knows the forces and torques on an object in terms of its position, velocity, orientation and angular velocity one then has a 'closed' set of differential equations. That is, one has enough information to define the equations for a mathematician or a computer to solve them.

The full set of differential equations is gathered from linear and angular momentum balance and also from simple kinematics. Namely, one has the following set of 6 first order differential equations:

$$\begin{aligned} \dot{x} &= v_x, \\ \dot{v}_x &= F_x/m, \\ \dot{y} &= v_y, \\ \dot{v}_y &= F_y/m, \\ \dot{\theta} &= \omega, \text{ and} \\ \dot{\omega} &= M_{\text{cm}}/I_{zz}^{\text{cm}}, \end{aligned}$$

where the positions and velocities are the positions and velocities of the center-of-mass. The expressions for F_x, F_y, and M_{cm} may well be complicated, as in the rocket example above.

17.1 2-D mechanics makes sense in a 3-D world

The math for two-dimensional mechanics analysis is simpler than the math for three-dimensional analysis. And thus easier to learn first. But we do actually live in a three-dimensional world. You might wonder at the utility of learning something that is not right. There are three answers.

1. Two-dimensional analysis can give partial information about the three-dimensional system that is exactly the same as the three-dimensional analysis would give by *projection*, no matter what the motion;
2. if the motion is planar the 2-D kinematics can be used; and
3. if the object is planar or symmetric about the motion plane, and any constraints that hold the object are also symmetric about the motion plane, the 2-D analysis is not only correct, but complete.

Of course no machine is exactly planar or exactly symmetric, but if the approximation seems reasonable most engineers will accept a small loss in accuracy for great gain in simplicity.

a) Projection

First let's relax our assumption of 2-D motion. Consider arbitrary 3-D motions of an arbitrarily complex system. If we take the dot product of the linear momentum equations with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ and the angular momentum balance equation with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned}
&\left\{\sum \vec{\boldsymbol{F}}_i = \sum m_i \vec{\boldsymbol{a}}_i\right\}\cdot\hat{\boldsymbol{\imath}} \ \Rightarrow \sum F_{i_x} = \sum m_i a_{i_x}, && \text{(a)}\\
&\left\{\sum \vec{\boldsymbol{F}}_i = \sum m_i \vec{\boldsymbol{a}}_i\right\}\cdot\hat{\boldsymbol{\jmath}} \Rightarrow \sum F_{i_y} = \sum m_i a_{i_y}, \quad \text{and} && \text{(b)}\\
&\left\{\sum \vec{\boldsymbol{r}}_i\times\vec{\boldsymbol{F}}_i = \sum \vec{\boldsymbol{r}}_i\times m_i\vec{\boldsymbol{a}}_i\right\}\cdot\hat{\boldsymbol{k}} &&\\
&\qquad\Rightarrow \sum r_{i_x}F_{i_y} - r_{i_y}F_{i_x} = \sum m_i(r_{i_x}a_{i_y} - r_{i_y}a_{i_x}). && \text{(c)}
\end{aligned} \tag{17.23}$$

These are exactly the equations of 2-D mechanics. That is, if we only consider the planar components of the forces, the planar components of the positions, and the planar components of the motions, we get a correct but partial set of the 3-D equations. In this sense 2-D analysis is correct but incomplete.

b) Planar motion

If all the velocities of the parts of a 3-D system have no z component the motion is planar (in the xy plane). Thus the right-hand sides of eqns. (17.23) are not just projections, but the whole story. Further, in the case of rigid-object motion, the 2-D kinematics equation

$$\vec{\boldsymbol{v}}_{\mathrm{P}} = \vec{\boldsymbol{v}}_{\mathrm{G}} + \omega\hat{\boldsymbol{k}}\times\vec{\boldsymbol{r}}_{\mathrm{P/G}} = \vec{\boldsymbol{v}}_{\mathrm{G}} + \omega\hat{\boldsymbol{k}}\times\left(r_{\mathrm{P/G}_x}\hat{\boldsymbol{\imath}} + r_{\mathrm{P/G}_y}\hat{\boldsymbol{\jmath}}\right) \tag{17.24}$$

also applies (the z component of the position drops out of the cross product) and the expression for, say, the z component of the angular momentum of an object about its center-of-mass is

$$H_{\mathrm{cm}_z} = I_{zz}^{\mathrm{cm}}\omega.$$

Differentiating, or adding up the $m_i\vec{\boldsymbol{a}}_i$ terms we get,

$$\dot{H}_{\mathrm{cm}_z} = I_{zz}^{\mathrm{cm}}\dot{\omega}.$$

Similarly, the z component of the full angular momentum balance equation for a 3-D rigid object in planar motion is the same as the z component of eqn. (17.22)b.

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}}\cdot\hat{\boldsymbol{k}} = \left(\vec{\boldsymbol{r}}_{\mathrm{cm/O}}\times(m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}})\right)\cdot\hat{\boldsymbol{k}} + I_{zz}^{\mathrm{cm}}\dot{\omega}$$

So for planar motion of 3-D rigid bodies one can do a correct 2-D analysis with the full ease of analyzing a planar object.

But this result is deceptively simple. The free object diagram in 3-D most likely shows forces in the z direction, pairs of forces in the x or y directions that are applied at points with the same x and y coordinates but different z values, or moments with components in the x or y directions. Full information about these force and moment components can't be found from 2-D analysis. That is,

> the nature of the forces that it takes to *keep* a system in planar motion can't be found from a planar analysis.

For example, a system rotating about the z axis which is statically balanced but is dynamically imbalanced has no *net* x or y reaction force, as a planar analysis would reveal, yet the bearing reaction forces are not zero.

Another example would be a plan view of a car in a turn (assuming a stiff suspension). A 2-D analysis could be accurate, but would not be complete enough to describe the tire reaction forces needed to keep the car flat.

c) Symmetric bodies and planar bodies

If the rigid object has all its mass in the xy plane, or its mass is symmetrically distributed about the xy plane, and it is in planar motion in the xy plane then

$$\sum m_i a_{i_z} = 0 \quad \text{and}$$
$$\left\{\sum \vec{\boldsymbol{r}}_i\times m_i\vec{\boldsymbol{a}}_i\right\}\cdot\hat{\boldsymbol{\imath}} = 0 \quad \text{and} \quad \left\{\sum \vec{\boldsymbol{r}}_i\times m_i\vec{\boldsymbol{a}}_i\right\}\cdot\hat{\boldsymbol{\jmath}} = 0$$

where $\vec{\boldsymbol{r}}$ is measured relative to any point in the plane. Thus, by linear and angular momentum balance,

$$\sum F_z = 0 \quad \text{and}$$
$$\left\{\sum \vec{\boldsymbol{r}}_i\times\vec{\boldsymbol{F}}_i\right\}\cdot\hat{\boldsymbol{\imath}} = 0 \quad \text{and} \quad \left\{\sum \vec{\boldsymbol{r}}_i\times\vec{\boldsymbol{F}}_i\right\}\cdot\hat{\boldsymbol{\jmath}} = 0$$

so

> A planar object or a symmetric object in planar motion needs no force in the z direction and no moment in the x or y direction to keep it in the plane.

Systems that are symmetric or flat and moving in an approximately planar manner, are thus both accurately and completely modeled with a 2-D analysis. A slight generalization of the result is to any object or collection of objects whose centers of mass are on the plane and each of which is dynamically balanced for rotation about a $\hat{\boldsymbol{k}}$ axis through its center-of-mass.

17.2 The center-of-mass theorems for 2-D rigid bodies

That all the particles in a system are part of one planar object in planar motion (in that plane) allows highly useful simplification of the expressions for the motion quantities, namely Eqns. 17.16 to 17.20. We can derive these expressions from the center-of-mass theorems from chapter B. For completeness, we repeat some of those derivations here. To save space, we only use the integral ($\int$) forms for the general expressions; the derivations with sums ($\sum$) are similar. In all cases position, velocity, and acceleration are relative to a fixed point in space (that is $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}$ mean $\vec{\boldsymbol{r}}_{/0}$, $\vec{\boldsymbol{v}}_{/0}$, and $\vec{\boldsymbol{a}}_{/0}$ respectively).

Linear momentum.

Here we show that to evaluate linear momentum and its rate of change you only need to know the motion of the center of mass.

$$\begin{aligned}\vec{\boldsymbol{L}} &\equiv \int \vec{\boldsymbol{v}}\,dm = \int \frac{d}{dt}\vec{\boldsymbol{r}}\,dm = \frac{d}{dt}\int \vec{\boldsymbol{r}}\,dm = \frac{d}{dt}(m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}})\\ &= m_{\text{tot}}\frac{d}{dt}\vec{\boldsymbol{r}}_{\text{cm}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}}\end{aligned}$$

By identical reasoning, or by differentiating the expression above with respect to time,

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}$$

Thus for linear momentum balance one need not pay heed to rotation, only the center-of-mass motion matters.

Angular momentum.

Here we attempt a derivation like the one above but get slightly more complicated results. For simplicity we evaluate angular momentum and its rate of change relative to the origin, but a very similar derivation would hold relative to any fixed point C.

$$\begin{aligned}\vec{\boldsymbol{H}}_{/\text{O}} &\equiv \int \vec{\boldsymbol{r}}\times\vec{\boldsymbol{v}}\,dm\\
&= \int (\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_{\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}})\times(\vec{\boldsymbol{v}}-\vec{\boldsymbol{v}}_{\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}})\,dm\\
&= \int \left(\vec{\boldsymbol{r}}_{/\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}}\right)\times\left(\vec{\boldsymbol{v}}_{/\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}}\right)dm\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm\\
&\quad + \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\int dm\\
&\quad + \underbrace{\left(\int \vec{\boldsymbol{r}}_{/\text{cm}}\,dm\right)}_{\vec{0}}\times\vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}}\times\underbrace{\left(\int \vec{\boldsymbol{v}}_{/\text{cm}}\,dm\right)}_{\vec{0}}\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.\end{aligned}$$

This much is true for any system in any motion. For a rigid object we know about the motions of the parts. Using the center-of-mass as a reference point we know that for all points on the object $\vec{\boldsymbol{v}}_{/\text{cm}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}$. Thus we can continue the derivation above, following the same reasoning as was used in chapter 7 for circular motion of rigid bodies:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\left(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}\right)dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Using the identity for the triple cross product (see box 17.4 on page 10) or using the geometry of the cross product directly with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ as in chapters 7 and 8 we get

$$\vec{\boldsymbol{H}}_{/\text{O}} = \omega\hat{\boldsymbol{k}}\int r^2_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Then defining $I^{\text{cm}}_{zz} \equiv \int r^2_{/\text{cm}}\,dm$ we get the desired result:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

A similar derivation, or differentiation of the result above (and using that $(\frac{d}{dt}\vec{\boldsymbol{r}})\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$) gives

$$\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{a}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

The results above hold for any reference point, not just the origin of the fixed coordinate system. Thus, relative to a point instantaneously coinciding with the center-of-mass

$$\vec{\boldsymbol{H}}_{\text{cm}} = \underbrace{\vec{\boldsymbol{r}}_{\text{cm/cm}}}_{\vec{\boldsymbol{0}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}+I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}} = I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

and similarly

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

Kinetic energy.

Unsurprisingly the expression for kinetic energy and its rate of change are also simplified by derivations very similar to those above. Skipping the details (or leaving them as an exercise for the peppy reader):

$$\begin{aligned}E_{\text{K}} &\equiv \int \frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\,dm\\ &= \frac{1}{2}m_{\text{tot}}v^2_{cm} + \frac{1}{2}I^{\text{cm}}_{zz}\omega^2\end{aligned}$$

and

$$\begin{aligned}\dot{E}_{\text{K}} &\equiv \frac{d}{dt}E_{\text{K}}\\ &= m_{\text{tot}}v\dot{v} + I^{\text{cm}}_{zz}\omega\dot{\omega}.\end{aligned}$$

17.3 The work of a moving force and of a couple

The work of a force acting on an object from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \underbrace{\vec{v} dt}_{d\vec{r}} \\ &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \end{aligned} \tag{17.25}$$

The validity of equation 17.25 depends on the force acting on the same material point of the moving object as it moves from position 1 to position 2; i.e., the force moves with the object. If the material point of force application changes with time, eqn. (17.25) is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P\, dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt \tag{17.26}$$

where $\vec{v}$ is the velocity of the material point at the instantaneous location of the applied force.

Hand drags on a passing train: a subtlety

There is a subtle distinction between eqn. (17.25) and eqn. (17.26). As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train with the force

$$\vec{F}_{\text{hand on train}}.$$

It might seem that the work of the hand on the train is zero because your hand doesn't move; work is force times distance and the distance is zero and eqn. (17.25) superficially evaluates to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0.$$

But we have violated the condition for the validity of eqn. (17.25): the force be applied to a fixed material point as time progresses. Whereas on the train your hand smears a whole line of material points.

Clearly your hand does slow the train, so it must do (negative) work on the train, as eqn. (17.26) correctly shows because

$$P_{\text{force on train}} = \vec{F}_{\text{hand on train}} \cdot \vec{v}_{\text{train}} \neq 0.$$

The power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand). Thus, your hand does negative work on the train. eqn. (17.26) applies to the train and eqn. (17.25) does not.

On the other hand (so to speak) if one looks at the power of the force on the hand we have:

$$\vec{F}_{\text{train on hand}} = -\vec{F}_{\text{hand on train}}$$

while the velocity of the hand is zero so

$$P_{\text{force on hand}} = \vec{F}_{\text{train on hand}} \cdot \vec{v}_{\text{hand}} = 0.$$

So the train does *no* work on your hand while your hand does (negative) work on the train. The difference, of course, is mechanical energy lost to heat.

Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid object the power of these forces turns out to be $\vec{M} \cdot \vec{\omega}$. At a given angular velocity a bigger torque applies more power. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces $\vec{F}_i$ that add to zero acting at points with positions $\vec{r}_i$ relative to a reference point on the object o′.

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= \sum \vec{F}_i \cdot \left(\vec{v}_{o'} + \vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \vec{v}_{o'} \cdot \underbrace{\sum \vec{F}_i}_{\vec{0}} + \sum \vec{F}_i \cdot \left(\vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \sum \vec{\omega} \cdot \left(\vec{r}_{i/o'} \times \vec{F}_i\right) \\ &= \vec{\omega} \cdot \sum \vec{r}_{i/o'} \times \vec{F}_i \\ &= \vec{\omega} \cdot \vec{M}_{o'} \end{aligned} \tag{17.27}$$

Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{F}_{\text{tot}} \cdot \vec{v}_{o'} + \vec{\omega} \cdot \vec{M}_{o'}. \tag{17.28}$$

For a given force system applied to a given object in a given motion any point o′ can be used. The terms in the formula above will depend on o′, but the sum does not. Besides the center-of-mass, another convenient locations for o′ is a fixed hinge, in which case the applied force has no power.

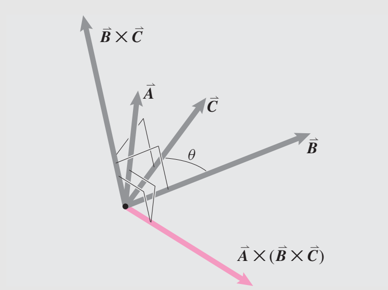
17.4 The vector triple product $\vec{A}\times(\vec{B}\times\vec{C})$

The formula

$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}. \tag{17.29}$$

can be verified by writing each of the vectors in terms of its orthogonal components (*e.g.*, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$) and checking equality of the 27 terms on the two sides of the equations (only 12 are non-zero). If this 20 minute proof seems tedious it can be replaced by a more abstract geometric argument partly presented below that surely takes more than 20 minutes to grasp.

Geometry of the vector triple product



Because $\vec{B}\times\vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ it is perpendicular to the plane of $\vec{B}$ and $\vec{C}$, that is, it is 'normal' to the plane BC. $\vec{A}\times(\vec{B}\times\vec{C})$ is perpendicular to both $\vec{A}$ and $\vec{B}\times\vec{C}$, so it is perpendicular to the normal to the plane of BC. That is, it must be in the plane of $\vec{B}$ and $\vec{C}$. But any vector in the plane of $\vec{B}$ and $\vec{C}$ must be a combination of $\vec{B}$ and $\vec{C}$. Also, the vector triple product must be proportional in magnitude to each of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Finally, the triple cross product of $\vec{A}\times(\vec{B}\times\vec{C})$ must be the negative of $\vec{A}\times(\vec{C}\times\vec{B})$ because $\vec{B}\times\vec{C} = -\vec{C}\times\vec{B}$. So the identity

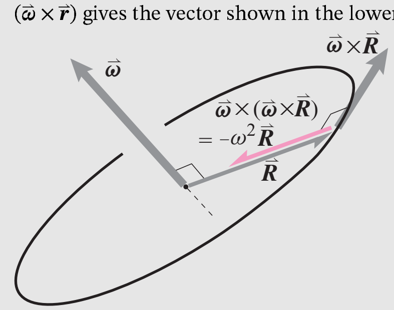
$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\vec{B} - (\vec{A}\cdot\vec{B})\vec{C}$$

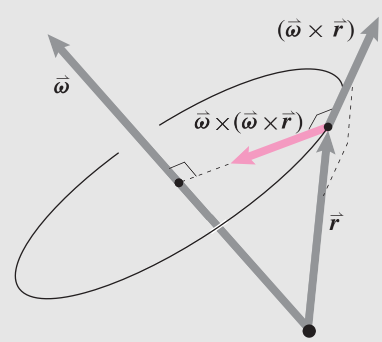
is almost natural: The expression above is almost the only expression that is a linear combination of $\vec{A}$ and $\vec{B}$ that is linear in both, also linear in $\vec{C}$ and switches sign if $\vec{B}$ and $\vec{C}$ are interchanged. These properties would be true if the whole expression were multiplied by any constant scalar. But a test of the equation with three unit vectors shows that such a multiplicative constant must be one. This reasoning constitutes an informal derivation of the identity 17.29.

Using the triple cross product in dynamics equations

We will use identity 17.29 for two purposes in the development of dynamics equations:

1. In the 2D expression for acceleration, the centripetal acceleration is given by $\vec{\omega}\times(\vec{\omega}\times\vec{R})$ simplifies to $-\omega^2\vec{R}$ if $\vec{\omega}\perp\vec{R}$. This equation follows by setting $\vec{A} = \vec{\omega}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{R}$ in equation 17.29 and using $\vec{R}\cdot\vec{\omega} = 0$ if $\vec{\omega}\perp\vec{R}$. In 3D $\vec{\omega}\times(\vec{\omega}\times\vec{r})$ gives the vector shown in the lower figure below.





2. The term $\vec{r}\times(\vec{\omega}\times\vec{r})$ will appear in the calculation of the angular momentum of a rigid body. By setting $\vec{A} = \vec{r}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{r}$, in equation 17.29 and use $\vec{r}\cdot\vec{r} = r^2$ because $\vec{r}\parallel\vec{r}$ we get the useful result that $\vec{r}\times(\vec{\omega}\times\vec{r}) = r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}$.