# 17.2 General planar mechanics of a rigid-object

We now apply the kinematics ideas of the last section to the general mechanics principles in Table I in the inside cover. The goal is to understand the relation between forces and motion for a planar object in general 2-D motion. The simple measures①of motion will be the displacement of one reference point $0'$ on the object and the rotation of the object. We also need the first and second time derivatives of the displacement and rotation, altogether we use $\vec{\boldsymbol{r}}_{0'}, \vec{\boldsymbol{v}}_{0'}, \vec{\boldsymbol{a}}_{0'}, \theta, \vec{\boldsymbol{\omega}}$ and $\vec{\boldsymbol{\alpha}}$.

We will treat all bodies as if they are squished into the plane (planar) and moving in the plane (planar motion). But the analysis is sensible for an object that is symmetric with respect to the plane containing the velocities (see Box 17.1 on page 7).

①**Advanced aside:** What we call "simple measures" are examples of "generalized coordinates" in more advanced books. The idea sounds intimidating, but is simply this: If something can only move in a few ways, you should only keep track of the motion with that many variables. The kinematics of a rigid object (Sect. 17.1) allow us to "evaluate" the motion quantities, namely linear momentum, angular momentum, kinetic energy, and their rates of change in terms of these "simple measures". By "evaluate" we mean express the motion quantities in terms of these measures. The alternative is as sums over Avogadro's number of particles (There are on the order of $10^{23}$ atoms in a typical engineering part.). Even neglecting atoms and viewing matter as continuous we would still be stuck with integrals over complicated regions if we did not describe the motion with as few variables as possible. In the case of 2-D rigid object motion, the position of a reference point ($x$ and $y$) with the rotation $\theta$ is called a set of *minimal coordinates.* These, and their time derivatives are the minimal information needed to describe all important mechanics motion quantities.

## The balance laws for a rigid object

As always, once you have defined the system and the forces acting on it by drawing a free-body diagram, the basic momentum balance equations are applicable (and exact for engineering purposes). Namely,

$$\text{Linear momentum balance:} \qquad \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \qquad \text{and}$$

$$\text{Angular momentum balance:} \qquad \sum \vec{\boldsymbol{M}}_{i/O} = \dot{\vec{\boldsymbol{H}}}_{/O}.$$

The same point 0, any point, is used on both sides of the angular momentum balance equation. We also have power balance which, for a system with no internal energy or dissipation, is

$$\text{Power balance:} \qquad P = \dot{E}_\text{K}.$$

The left hand sides of the momentum balance equations are evaluated the same way, whether the system is composed of one object or many, whether the bodies are deformable or not, and whether the points move in straight lines, circles, hither and thither, or not at all. It is the right hand sides of the momentum equations that involve the motion. Similarly, in the energy balance equations the applied power $P$ only depends on the position of the forces and the motions of the material points at those positions. But the kinetic energy $E_\text{K}$ and its rate of change depend on the motion of the whole system. You already know how to evaluate the momenta and energy, and their rates of change, for a variety of special cases, namely

- Systems composed of particles where all the positions and accelerations are known (chapter 13);
- Systems where all points have the same acceleration. That is, the system moves like a rigid object that does not rotate (chapter 14); and

- Systems where all points move in circles about the same fixed axis in 2D (chapter 15).

Now we go on to consider the general 2-D motions of a planar rigid object. It's now a little harder to evaluate $\vec{\boldsymbol{L}}, \dot{\vec{\boldsymbol{L}}}, \vec{\boldsymbol{H}}_{/\mathrm{O}}, \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}, E_\mathrm{K}$ and $\dot{E}_\mathrm{K}$. But not much.

## Summary of the motion quantities

Table I in the back of the book describes the motion quantities for various special cases, including the planar motions we consider in this chapter. Most relevant is row (7).

The basic idea is that the momenta for general motion, which involves translation and rotation, are the sum of the momenta (both linear and angular, and their rates of change too) from those two effects. Namely, the linear momentum is described, as for any system with any motion, by the motion of the center-of-mass

$$\vec{\boldsymbol{L}} = m_\mathrm{tot}\vec{\boldsymbol{v}}_\mathrm{cm} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_\mathrm{tot}\vec{\boldsymbol{a}}_\mathrm{cm}, \tag{17.16}$$

and the angular momentum has two contributions, one from the motion of the center-of-mass and one from rotation of the object about the center of mass,

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = \overbrace{\vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{O}} \times (m_\mathrm{tot}\vec{\boldsymbol{v}}_\mathrm{cm})}^{\text{Angular momentum due to motion of the center-of-mass}} + \overbrace{I_{zz}^\mathrm{cm}\vec{\boldsymbol{\omega}}}^{\text{Angular momentum due to motion relative to the center-of-mass}} \tag{17.17}$$

$$\text{and} \quad \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{r}}_{\mathrm{cm}/\mathrm{O}} \times (m_\mathrm{tot}\vec{\boldsymbol{a}}_\mathrm{cm}) + I_{zz}^\mathrm{cm}\dot{\vec{\boldsymbol{\omega}}}. \tag{17.18}$$

These simplifications, for 2D objects moving in the plane, are discussed in box 17.2 on page 8. An important special case for the angular momentum evaluation is when the reference point is coincident with the center-of-mass. Then the angular momentum and its rate of change simplify to

$$\vec{\boldsymbol{H}}_\mathrm{cm} = I_{zz}^\mathrm{cm}\vec{\boldsymbol{\omega}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_\mathrm{cm} = I_{zz}^\mathrm{cm}\dot{\vec{\boldsymbol{\omega}}}. \tag{17.19}$$

The kinetic energy and its rate of change are given by

$$E_{\mathrm{K}} = \overbrace{\frac{1}{2} m_{\mathrm{tot}} \underbrace{v_{cm}^2}_{\vec{v}_{\mathrm{cm}} \cdot \vec{v}_{\mathrm{cm}}}}^{\boxed{\text{kinetic energy from center-of-mass motion}}} + \overbrace{\frac{1}{2} I_{zz}^{\mathrm{cm}} \omega^2}^{\boxed{\text{kinetic energy relative to the center-of-mass}}} \tag{17.20}$$

$$\text{and} \quad \dot{E}_{\mathrm{K}} = m_{\mathrm{tot}} \underbrace{v_{cm} \dot{v}_{cm}}_{\vec{v}_{\mathrm{cm}} \cdot \vec{a}_{\mathrm{cm}}} + I_{zz}^{\mathrm{cm}} \omega \dot{\omega} \tag{17.21}$$

The relations above are easily derived from the general center of mass theorems (see box 17.2 on page 8 for some of these derivations).

## Equations of motion

Putting together the general balance equations and the expressions for the motion quantities we can now write linear momentum balance, angular momentum balance and power balance as:

$$\begin{aligned}
\text{LMB}: \quad & \sum \vec{F}_i = m_{\mathrm{tot}} \vec{a}_{\mathrm{cm}}, & \text{(a)} \\
\text{AMB}: \quad & \sum \vec{M}_{\mathrm{O}} = \vec{r}_{\mathrm{cm/O}} \times (m_{\mathrm{tot}} \vec{a}_{\mathrm{cm}}) + I_{zz}^{\mathrm{cm}} \dot{\vec{\omega}} & \text{(b)} \\
\text{or} \quad & \sum \vec{M}_{\mathrm{cm}} = I_{zz}^{\mathrm{cm}} \dot{\vec{\omega}}, \quad \text{and} & \\
\text{Power}: \quad & \vec{F}_{\mathrm{tot}} \cdot \vec{v}_{\mathrm{cm}} + \vec{\omega} \cdot \vec{M}_{\mathrm{cm}} = m_{\mathrm{tot}} v \dot{v} + I_{zz}^{\mathrm{cm}} \omega \dot{\omega}. & \text{(c)}
\end{aligned} \tag{17.22}$$

## Independent equations?

Equations are only independent if no one of them can be derived from the others. When counting equations and unknowns one needs to make sure one is writing independent equations. How many independent equations are in the set eqns. (17.22)abc applied to one free object diagram? The short answer is 3.

The linear momentum balance equation 17.22a yields two independent equations by dotting with any two non-parallel vectors (say, $\hat{\imath}$ and $\hat{\jmath}$). Dotting with a third vector yields a dependent equation.

For any one reference point the angular momentum equation 17.22b yields one scalar equation. It is a vector equation but always has zero components in the $\hat{\imath}$ and $\hat{\jmath}$ directions. But angular momentum equation can yield up to three independent equations by being applied to any set of three non-collinear points.

The power balance equation is one scalar equation.

In total, however, the full set of equations above only makes up a set of three independent equations.

To avoid thinking about what is or is not an independent set of equations some people prefer to stick with one of the canonical sets of independent equations:

- The coordinate based set ("old standard")
  - {LMB}$\cdot\hat{\boldsymbol{\imath}}$ or, equivalently, $\sum F_x = m_{\text{tot}} a_{\text{cm}_x}$,
  - {LMB}$\cdot\hat{\boldsymbol{\jmath}}$ or, equivalently, $\sum F_y = m_{\text{tot}} a_{\text{cm}_y}$, and
  - {AMB}$\cdot\hat{\boldsymbol{k}}$ or, equivalently, $\sum M_{\text{cm}} = I_{zz}^{\text{cm}} \dot{\omega}$.
- Moment only (good for eliminating unknown reaction forces)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {AMB about pt C}$\cdot\hat{\boldsymbol{k}}$ (C is a third point not on the line AB)
- Two moments and a force component
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)
  - {AMB about pt B}$\cdot\hat{\boldsymbol{k}}$ (B is any other point)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}$ (where $\hat{\boldsymbol{\lambda}}$ is not perpendicular to the line AB)
- Two force components and a moment (also good for eliminating unknown forces)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_1$ (where $\hat{\boldsymbol{\lambda}}_1$ is any unit vector)
  - {LMB}$\cdot\hat{\boldsymbol{\lambda}}_2$ (where $\hat{\boldsymbol{\lambda}}_2$ is any other unit vector)
  - {AMB about pt A}$\cdot\hat{\boldsymbol{k}}$ (A is any point, on or off the object)

Any of these will always do the job. The power balance equation is often used as a consistency check rather than an independent equation.

From a theoretical point of view one might ask the related question of which of the equations of motion can be derived from the others. There are many answers. Here are some of them:

- Power balance follows from LMB and AMB,
- AMB about three non-collinear points implies LMB, and
- LMB and power balance yield AMB

Interestingly, there is no way to derive angular momentum balance from linear momentum balance without the questionable microscopic assumptions discussed in box B.3 on page 1203.

## Some simple examples

Here we consider some simple examples of unconstrained motion of a rigid object.

**Example: The simplest case: no force and no moment.**
If the net force and moment applied to an object are zero we have:

$$\begin{aligned} \text{LMB} \quad &\Rightarrow \vec{\mathbf{0}} &=& \quad m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}} \quad \text{and} \\ \text{AMB} \quad &\Rightarrow \vec{\mathbf{0}} &=& \quad I_{zz}^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \end{aligned}$$

so $\vec{\boldsymbol{a}}_{\text{cm}} = \vec{\mathbf{0}}$ and $\dot{\omega} = 0$ and the object moves at constant speed in a constant direction with a constant rate of rotation, all determined by the initial conditions. Throw an object in space and its center-of-mass goes in a straight line and it spins at constant rate (subject to the 2-D restrictions of this chapter).

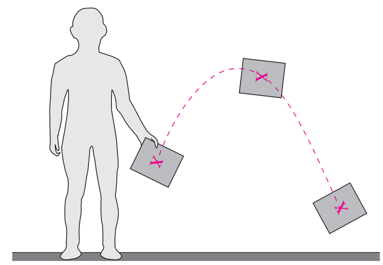



Figure 17.22: The X marked at the center-of-mass of a thrown spinning clipboard follows a parabolic trajectory.

**Example: Constant force applied to the center-of-mass.**
In this case angular momentum balance about the center-of-mass again yields that the rotation rate is constant. Linear momentum balance is now the same as for a particle at the center-of-mass, *i.e.,* the center-of-mass has a parabolic trajectory.

Near-earth (constant) gravity provides a simple example. An 'X' marked at the center-of-mass of a clipboard tossed across a room follows a parabolic trajectory (see fig. 17.22).

**Example: Constant force not at the center-of-mass.**
Assume the only force applied to an object is a constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ at A (see fig. 17.23). Then linear momentum balance gives us that

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad F\hat{\boldsymbol{\imath}} = m\vec{\boldsymbol{a}}_{\text{G}} \quad \Rightarrow \quad \vec{\boldsymbol{a}}_{\text{G}} = F/m\hat{\boldsymbol{\imath}} = \text{constant}.$$

So if the object starts at rest, the point G will move in a straight line in the $\hat{\boldsymbol{\imath}}$ direction (The common intuition that point G will be pulled up is incorrect). Angular momentum about the center-of-mass gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\text{cm i}} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} \quad &\Rightarrow \quad \left\{\vec{\boldsymbol{r}}_{\text{A/G}} \times F\hat{\boldsymbol{\imath}} = I_{zz}^{\text{cm}}\ddot{\theta}\hat{\boldsymbol{k}}\right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad &\Rightarrow \quad \ddot{\theta} + \frac{F\ell}{I_{zz}^{\text{cm}}}\sin\theta = 0, \end{aligned}$$

with $\ell = |\vec{\boldsymbol{r}}_{\text{A/G}}|$, which is the pendulum equation. That is, the object can swing back and forth about $\theta = 0$ just like a pendulum, approximately sinusoidally if the angle $\theta$ starts small and with $\dot{\theta}$ initially also small. [One might wonder how to do this experiment. One way would be with a jet on a space craft that keeps re-orienting itself to keep in a constant spatial direction as the object changes orientation. Another would be with a string tied to A and pulled from a great distance.]

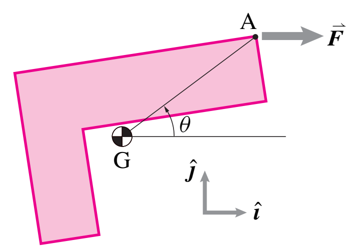



Figure 17.23: The only force applied to the object is the constant force $\vec{\boldsymbol{F}} = F\hat{\boldsymbol{\imath}}$ applied at point A. The resulting motion is a constant acceleration of the center-of-mass $\vec{\boldsymbol{a}}_{\text{G}} = (F/m)\hat{\boldsymbol{\imath}}$ and an oscillatory motion of $\theta$ identical to that of a pendulum hinged at G. Note that the center of mass goes in a straight line.

**Example: The flight of an arrow or rocket.**
As a primitive model of an arrow or rocket assume that the only force is from drag on the fins at C and that this force opposes motion according to

$$\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}_{\text{C}}$$

where c is a drag coefficient (see fig. 17.24). From linear momentum balance we have:

$$\begin{aligned} \sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} \quad \Rightarrow \quad \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -c\vec{\boldsymbol{v}}_{\text{C}} &= m\dot{\vec{\boldsymbol{v}}} \\ m\dot{\vec{\boldsymbol{v}}} &= -c\left(\vec{\boldsymbol{v}} + \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{C/G}}\right) \\ &= -c\left(\vec{\boldsymbol{v}} + \dot{\theta}\hat{\boldsymbol{k}} \times (-\ell\hat{\boldsymbol{\lambda}})\right) \\ (\hat{\boldsymbol{k}} \times \hat{\boldsymbol{\lambda}} = \hat{\boldsymbol{n}}) \quad \Rightarrow \quad \dot{\vec{\boldsymbol{v}}} &= \frac{c}{m}\left(\dot{\theta}\ell\hat{\boldsymbol{n}} - \vec{\boldsymbol{v}}\right). \end{aligned}$$

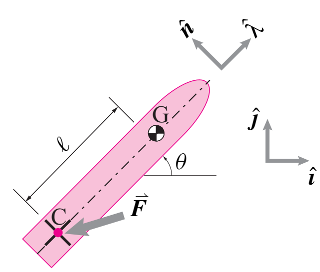



Figure 17.24: A rocket is pointed in the direction $\hat{\boldsymbol{\lambda}}$ which makes an angle $\theta$ with the positive $x$ axis. The position and velocity of the center-of-mass at G are called $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$. The velocity of the tail is $\vec{\boldsymbol{v}}_{\text{C}}$

So if $\vec{\boldsymbol{v}}, \theta$ and $\dot{\theta}$ are known the acceleration $\dot{\vec{\boldsymbol{v}}}$ is calculated by the formula above.

Similarly angular momentum balance about G gives

$$\begin{array}{rcrcl} \sum \vec{\boldsymbol{M}}_{\mathrm{G}} = \dot{\vec{\boldsymbol{H}}}_{\mathrm{G}} & \Rightarrow & \{\vec{\boldsymbol{r}}_{\mathrm{C/G}} \times \vec{\boldsymbol{F}} & = & I_{zz}^{\mathrm{cm}} \dot{\omega} \hat{\boldsymbol{k}}\} \\ \{\} \cdot \hat{\boldsymbol{k}} & \Rightarrow & I_{zz}^{\mathrm{cm}} \dot{\omega} & = & \vec{\boldsymbol{r}}_{\mathrm{C/G}} \times \vec{\boldsymbol{F}} \cdot \hat{\boldsymbol{k}}. \end{array}$$

Then, making the same substitutions as before for $\vec{\boldsymbol{r}}_{\mathrm{C/G}}$ and $\vec{\boldsymbol{F}}$ we get

$$\dot{\omega} = \frac{c\ell}{I_{zz}^{\mathrm{cm}}} \left( \hat{\boldsymbol{\lambda}} \times \vec{\boldsymbol{v}} \cdot \hat{\boldsymbol{k}} - \dot{\theta}\ell \right)$$

which determines the rate of change of $\omega$ if the present values of $\vec{\boldsymbol{v}}, \theta$ and $\dot{\theta}$ are known.

## Setting up differential equations for solution

If one knows the forces and torques on an object in terms of its position, velocity, orientation and angular velocity one then has a 'closed' set of differential equations. That is, one has enough information to define the equations for a mathematician or a computer to solve them.

The full set of differential equations is gathered from linear and angular momentum balance and also from simple kinematics. Namely, one has the following set of 6 first order differential equations:

$$\begin{array}{rcl} \dot{x} & = & v_x, \\ \dot{v}_x & = & F_x/m, \\ \dot{y} & = & v_y, \\ \dot{v}_y & = & F_y/m, \\ \dot{\theta} & = & \omega, \text{ and} \\ \dot{\omega} & = & M_{\mathrm{cm}}/I_{zz}^{\mathrm{cm}}, \end{array}$$

where the positions and velocities are the positions and velocities of the center-of-mass. The expressions for $F_x, F_y$, and $M_{\mathrm{cm}}$ may well be complicated, as in the rocket example above.

# 17.1 2-D mechanics makes sense in a 3-D world

The math for two-dimensional mechanics analysis is simpler than the math for three-dimensional analysis. And thus easier to learn first. But we do actually live in a three-dimensional world. You might wonder at the utility of learning something that is not right. There are three answers.

1. Two-dimensional analysis can give partial information about the three-dimensional system that is exactly the same as the three-dimensional analysis would give by *projection*, no matter what the motion;
2. if the motion is planar the 2-D kinematics can be used; and
3. if the object is planar or symmetric about the motion plane, and any constraints that hold the object are also symmetric about the motion plane, the 2-D analysis is not only correct, but complete.

Of course no machine is exactly planar or exactly symmetric, but if the approximation seems reasonable most engineers will accept a small loss in accuracy for great gain in simplicity.

## a) Projection

First let's relax our assumption of 2-D motion. Consider arbitrary 3-D motions of an arbitrarily complex system. If we take the dot product of the linear momentum equations with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ and the angular momentum balance equation with $\hat{\boldsymbol{k}}$ we get

$$\begin{aligned}
&\{\textstyle\sum \vec{\boldsymbol{F}}_i = \sum m_i \vec{\boldsymbol{a}}_i\} \cdot \hat{\boldsymbol{\imath}} \;\Rightarrow \textstyle\sum F_{i_x} = \sum m_i a_{i_x}, && \text{(a)}\\
&\{\textstyle\sum \vec{\boldsymbol{F}}_i = \sum m_i \vec{\boldsymbol{a}}_i\} \cdot \hat{\boldsymbol{\jmath}} \Rightarrow \textstyle\sum F_{i_y} = \sum m_i a_{i_y}, \quad \text{and} && \text{(b)}\\
&\{\textstyle\sum \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i = \sum \vec{\boldsymbol{r}}_i \times m_i \vec{\boldsymbol{a}}_i\} \cdot \hat{\boldsymbol{k}} &&\\
&\qquad \Rightarrow \textstyle\sum r_{i_x} F_{i_y} - r_{i_y} F_{i_x} = \sum m_i (r_{i_x} a_{i_y} - r_{i_y} a_{i_x}). && \text{(c)}
\end{aligned} \tag{17.23}$$

These are exactly the equations of 2-D mechanics. That is, if we only consider the planar components of the forces, the planar components of the positions, and the planar components of the motions, we get a correct but partial set of the 3-D equations. In this sense 2-D analysis is correct but incomplete.

## b) Planar motion

If all the velocities of the parts of a 3-D system have no $z$ component the motion is planar (in the $xy$ plane). Thus the right-hand sides of eqns. (17.23) are not just projections, but the whole story. Further, in the case of rigid-object motion, the 2-D kinematics equation

$$\vec{\boldsymbol{v}}_{\mathrm{P}} = \vec{\boldsymbol{v}}_{\mathrm{G}} + \omega \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{P/G}} = \vec{\boldsymbol{v}}_{\mathrm{G}} + \omega \hat{\boldsymbol{k}} \times \left( r_{\mathrm{P/G}_x} \hat{\boldsymbol{\imath}} + r_{\mathrm{P/G}_y} \hat{\boldsymbol{\jmath}} \right) \tag{17.24}$$

also applies (the $z$ component of the position drops out of the cross product) and the expression for, say, the $z$ component of the angular momentum of an object about its center-of-mass is

$$H_{\mathrm{cm}_z} = I_{zz}^{\mathrm{cm}} \omega.$$

Differentiating, or adding up the $m_i \vec{\boldsymbol{a}}_i$ terms we get,

$$\dot{H}_{\mathrm{cm}_z} = I_{zz}^{\mathrm{cm}} \dot{\omega}.$$

Similarly, the $z$ component of the full angular momentum balance equation for a 3-D rigid object in planar motion is the same as the $z$ component of eqn. (17.22)b.

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} \cdot \hat{\boldsymbol{k}} = \left( \vec{\boldsymbol{r}}_{\mathrm{cm/O}} \times (m_{\mathrm{tot}} \vec{\boldsymbol{a}}_{\mathrm{cm}}) \right) \cdot \hat{\boldsymbol{k}} + I_{zz}^{\mathrm{cm}} \dot{\omega}$$

So for planar motion of 3-D rigid bodies one can do a correct 2-D analysis with the full ease of analyzing a planar object.

But this result is deceptively simple. The free object diagram in 3-D most likely shows forces in the $z$ direction, pairs of forces in the $x$ or $y$ directions that are applied at points with the same $x$ and $y$ coordinates but different $z$ values, or moments with components in the $x$ or $y$ directions. Full information about these force and moment components can't be found from 2-D analysis. That is,

> the nature of the forces that it takes to *keep* a system in planar motion can't be found from a planar analysis.

For example, a system rotating about the $z$ axis which is statically balanced but is dynamically imbalanced has no *net* $x$ or $y$ reaction force, as a planar analysis would reveal, yet the bearing reaction forces are not zero.

Another example would be a plan view of a car in a turn (assuming a stiff suspension). A 2-D analysis could be accurate, but would not be complete enough to describe the tire reaction forces needed to keep the car flat.

## c) Symmetric bodies and planar bodies

If the rigid object has all its mass in the $xy$ plane, or its mass is symmetrically distributed about the $xy$ plane, and it is in planar motion in the $xy$ plane then

$$\begin{gathered}\textstyle\sum m_i a_{i_z} = 0 \quad \text{and}\\
\{\textstyle\sum \vec{\boldsymbol{r}}_i \times m_i \vec{\boldsymbol{a}}_i\} \cdot \hat{\boldsymbol{\imath}} = 0 \quad \text{and} \quad \{\textstyle\sum \vec{\boldsymbol{r}}_i \times m_i \vec{\boldsymbol{a}}_i\} \cdot \hat{\boldsymbol{\jmath}} = 0\end{gathered}$$

where $\vec{\boldsymbol{r}}$ is measured relative to any point in the plane. Thus, by linear and angular momentum balance,

$$\begin{gathered}\textstyle\sum F_z = 0 \quad \text{and}\\
\{\textstyle\sum \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i\} \cdot \hat{\boldsymbol{\imath}} = 0 \quad \text{and} \quad \{\textstyle\sum \vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i\} \cdot \hat{\boldsymbol{\jmath}} = 0\end{gathered}$$

so

> A planar object or a symmetric object in planar motion needs no force in the $z$ direction and no moment in the $x$ or $y$ direction to keep it in the plane.

Systems that are symmetric or flat and moving in an approximately planar manner, are thus both accurately and completely modeled with a 2-D analysis. A slight generalization of the result is to any object or collection of objects whose centers of mass are on the plane and each of which is dynamically balanced for rotation about a $\hat{\boldsymbol{k}}$ axis through its center-of-mass.

# 17.2 The center-of-mass theorems for 2-D rigid bodies

That all the particles in a system are part of one planar object in planar motion (in that plane) allows highly useful simplification of the expressions for the motion quantities, namely Eqns. 17.16 to 17.20. We can derive these expressions from the center-of-mass theorems from chapter B. For completeness, we repeat some of those derivations here. To save space, we only use the integral ($\int$) forms for the general expressions; the derivations with sums ($\sum$) are similar. In all cases position, velocity, and acceleration are relative to a fixed point in space (that is $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$, and $\vec{\boldsymbol{a}}$ mean $\vec{\boldsymbol{r}}_{/0}$, $\vec{\boldsymbol{v}}_{/0}$, and $\vec{\boldsymbol{a}}_{/0}$ respectively).

## Linear momentum.

Here we show that to evaluate linear momentum and its rate of change you only need to know the motion of the center of mass.

$$\begin{aligned}\vec{\boldsymbol{L}} &\equiv \int \vec{\boldsymbol{v}}\,dm = \int \frac{d}{dt}\vec{\boldsymbol{r}}\,dm = \frac{d}{dt}\int \vec{\boldsymbol{r}}\,dm = \frac{d}{dt}(m_{\text{tot}}\vec{\boldsymbol{r}}_{\text{cm}})\\ &= m_{\text{tot}}\frac{d}{dt}\vec{\boldsymbol{r}}_{\text{cm}} = m_{\text{tot}}\vec{\boldsymbol{v}}_{\text{cm}}\end{aligned}$$

By identical reasoning, or by differentiating the expression above with respect to time,

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{\text{cm}}$$

Thus for linear momentum balance one need not pay heed to rotation, only the center-of-mass motion matters.

## Angular momentum.

Here we attempt a derivation like the one above but get slightly more complicated results. For simplicity we evaluate angular momentum and its rate of change relative to the origin, but a very similar derivation would hold relative to any fixed point C.

$$\begin{aligned}\vec{\boldsymbol{H}}_{/\text{O}} &\equiv \int \vec{\boldsymbol{r}}\times\vec{\boldsymbol{v}}\,dm\\
&= \int (\vec{\boldsymbol{r}}-\vec{\boldsymbol{r}}_{\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}})\times(\vec{\boldsymbol{v}}-\vec{\boldsymbol{v}}_{\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}})\,dm\\
&= \int \left(\vec{\boldsymbol{r}}_{/\text{cm}}+\vec{\boldsymbol{r}}_{\text{cm}}\right)\times\left(\vec{\boldsymbol{v}}_{/\text{cm}}+\vec{\boldsymbol{v}}_{\text{cm}}\right)dm\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm\\
&\quad + \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\,dm + \int \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}\int dm\\
&\quad + \underbrace{\left(\int \vec{\boldsymbol{r}}_{/\text{cm}}\,dm\right)}_{\vec{\boldsymbol{0}}}\times\vec{\boldsymbol{v}}_{\text{cm}} + \vec{\boldsymbol{r}}_{\text{cm}}\times\underbrace{\left(\int \vec{\boldsymbol{v}}_{/\text{cm}}\,dm\right)}_{\vec{\boldsymbol{0}}}\\
&= \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\vec{\boldsymbol{v}}_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.\end{aligned}$$

This much is true for any system in any motion. For a rigid object we know about the motions of the parts. Using the center-of-mass as a reference point we know that for all points on the object $\vec{\boldsymbol{v}}_{/\text{cm}} = \vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}$. Thus we can continue the derivation above, following the same reasoning as was used in chapter 7 for circular motion of rigid bodies:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \int \vec{\boldsymbol{r}}_{/\text{cm}}\times\left(\vec{\boldsymbol{\omega}}\times\vec{\boldsymbol{r}}_{/\text{cm}}\right)dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Using the identity for the triple cross product (see box 17.4 on page 10) or using the geometry of the cross product directly with $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ as in chapters 7 and 8 we get

$$\vec{\boldsymbol{H}}_{/\text{O}} = \omega\hat{\boldsymbol{k}}\int r^2_{/\text{cm}}\,dm + \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}.$$

Then defining $I^{\text{cm}}_{zz} \equiv \int r^2_{/\text{cm}}\,dm$ we get the desired result:

$$\vec{\boldsymbol{H}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

A similar derivation, or differentiation of the result above (and using that $(\frac{d}{dt}\vec{\boldsymbol{r}})\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{v}}\times\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$) gives

$$\dot{\vec{\boldsymbol{H}}}_{/\text{O}} = \vec{\boldsymbol{r}}_{\text{cm}}\times\vec{\boldsymbol{a}}_{\text{cm}}m_{\text{tot}} + I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

The results above hold for any reference point, not just the origin of the fixed coordinate system. Thus, relative to a point instantaneously coinciding with the center-of-mass

$$\vec{\boldsymbol{H}}_{\text{cm}} = \underbrace{\vec{\boldsymbol{r}}_{\text{cm/cm}}}_{\vec{\boldsymbol{0}}\times\vec{\boldsymbol{v}}_{\text{cm}}m_{\text{tot}}+I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}} = I^{\text{cm}}_{zz}\omega\hat{\boldsymbol{k}}.$$

and similarly

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}_{zz}\dot{\omega}\hat{\boldsymbol{k}}.$$

## Kinetic energy.

Unsurprisingly the expression for kinetic energy and its rate of change are also simplified by derivations very similar to those above. Skipping the details (or leaving them as an exercise for the peppy reader):

$$\begin{aligned}E_{\text{K}} &\equiv \int \frac{1}{2}\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{v}}\,dm\\ &= \frac{1}{2}m_{\text{tot}}v^2_{cm} + \frac{1}{2}I^{\text{cm}}_{zz}\omega^2\end{aligned}$$

and

$$\begin{aligned}\dot{E}_{\text{K}} &\equiv \frac{d}{dt}E_{\text{K}}\\ &= m_{\text{tot}}v\dot{v} + I^{\text{cm}}_{zz}\omega\dot{\omega}.\end{aligned}$$

# 17.3 The work of a moving force and of a couple

The work of a force acting on an object from state one to state two is

$$W_{12} = \int_{t_1}^{t_2} P dt.$$

But sometimes we like to think not of the time integral of the power, but of the path integral of the moving force. So we rearrange this integral as follows.

$$\begin{aligned} W_{12} &= \int_{t_1}^{t_2} P dt \\ &= \int_{t_1}^{t_2} \vec{F} \cdot \underbrace{\vec{v} dt}_{d\vec{r}} \\ &= \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} \qquad (17.25) \end{aligned}$$

The validity of equation 17.25 depends on the force acting on the same material point of the moving object as it moves from position 1 to position 2; i.e., the force moves with the object. If the material point of force application changes with time, eqn. (17.25) is senseless and should be replaced with the following more generally applicable equation:

$$W_{12} \equiv \int_{t_1}^{t_2} P\, dt = \int_{t_1}^{t_2} \vec{F} \cdot \vec{v} dt \qquad (17.26)$$

where $\vec{v}$ is the velocity of the material point at the instantaneous location of the applied force.

## Hand drags on a passing train: a subtlety

There is a subtle distinction between eqn. (17.25) and eqn. (17.26). As an example think of standing still and dragging your hand on a passing train. Your hand slows down the train with the force

$$\vec{F}_{\text{hand on train}}.$$

It might seem that the work of the hand on the train is zero because your hand doesn't move; work is force times distance and the distance is zero and eqn. (17.25) superficially evaluates to

$$\int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = 0.$$

But we have violated the condition for the validity of eqn. (17.25): the force be applied to a fixed material point as time progresses. Whereas on the train your hand smears a whole line of material points.

Clearly your hand does slow the train, so it must do (negative) work on the train, as eqn. (17.26) correctly shows because

$$P_{\text{force on train}} = \vec{F}_{\text{hand on train}} \cdot \vec{v}_{\text{train}} \neq 0.$$

The power of the hand force on the train is the force on the train dotted with the velocity of the train (*not* with the velocity of your hand). Thus, your hand does negative work on the train. eqn. (17.26) applies to the train and eqn. (17.25) does not.

On the other hand (so to speak) if one looks at the power of the force on the hand we have:

$$\vec{F}_{\text{train on hand}} = -\vec{F}_{\text{hand on train}}$$

while the velocity of the hand is zero so

$$P_{\text{force on hand}} = \vec{F}_{\text{train on hand}} \cdot \vec{v}_{\text{hand}} = 0.$$

So the train does *no* work on your hand while your hand does (negative) work on the train. The difference, of course, is mechanical energy lost to heat.

## Work of an applied torque

By thinking of an applied torque as really a distribution of forces, the work of an applied torque is the sum of the contributions of the applied forces. If a collection of forces equivalent to a torque is applied to one rigid object the power of these forces turns out to be $\vec{M} \cdot \vec{\omega}$. At a given angular velocity a bigger torque applies more power. And a given torque applies more power to a faster spinning object.

Here's a quick derivation for a collection of forces $\vec{F}_i$ that add to zero acting at points with positions $\vec{r}_i$ relative to a reference point on the object o′.

$$\begin{aligned} P &= \sum \vec{F}_i \cdot \vec{v}_i \\ &= \sum \vec{F}_i \cdot \left(\vec{v}_{o'} + \vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \vec{v}_{o'} \cdot \underbrace{\sum \vec{F}_i}_{\vec{0}} + \sum \vec{F}_i \cdot \left(\vec{\omega} \times \vec{r}_{i/o'}\right) \\ &= \sum \vec{\omega} \cdot \left(\vec{r}_{i/o'} \times \vec{F}_i\right) \\ &= \vec{\omega} \cdot \sum \vec{r}_{i/o'} \times \vec{F}_i \\ &= \vec{\omega} \cdot \vec{M}_{o'} \qquad (17.27) \end{aligned}$$

## Work of a general force distribution

A general force distribution has, by reasoning close to that above, a power of:

$$P = \vec{F}_{\text{tot}} \cdot \vec{v}_{o'} + \vec{\omega} \cdot \vec{M}_{o'}. \qquad (17.28)$$

For a given force system applied to a given object in a given motion any point o′ can be used. The terms in the formula above will depend on o′, but the sum does not. Besides the center-of-mass, another convenient locations for o′ is a fixed hinge, in which case the applied force has no power.

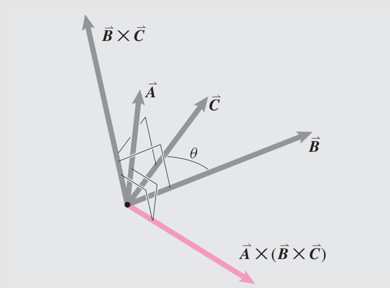
# 17.4 The vector triple product $\vec{A} \times (\vec{B} \times \vec{C})$

The formula

$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}. \qquad (17.29)$$

can be verified by writing each of the vectors in terms of its orthogonal components (*e.g.*, $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$) and checking equality of the 27 terms on the two sides of the equations (only 12 are non-zero). If this 20 minute proof seems tedious it can be replaced by a more abstract geometric argument partly presented below that surely takes more than 20 minutes to grasp.

## Geometry of the vector triple product




Because $\vec{B} \times \vec{C}$ is perpendicular to both $\vec{B}$ and $\vec{C}$ it is perpendicular to the plane of $\vec{B}$ and $\vec{C}$, that is, it is 'normal' to the plane $BC$. $\vec{A} \times (\vec{B} \times \vec{C})$ is perpendicular to both $\vec{A}$ and $\vec{B} \times \vec{C}$, so it is perpendicular to the normal to the plane of $BC$. That is, it must be in the plane of $\vec{B}$ and $\vec{C}$. But any vector in the plane of $\vec{B}$ and $\vec{C}$ must be a combination of $\vec{B}$ and $\vec{C}$. Also, the vector triple product must be proportional in magnitude to each of $\vec{A}$, $\vec{B}$ and $\vec{C}$. Finally, the triple cross product of $\vec{A}\times(\vec{B}\times\vec{C})$ must be the negative of $\vec{A}\times(\vec{C}\times\vec{B})$ because $\vec{B} \times \vec{C} = -\vec{C} \times \vec{B}$. So the identity

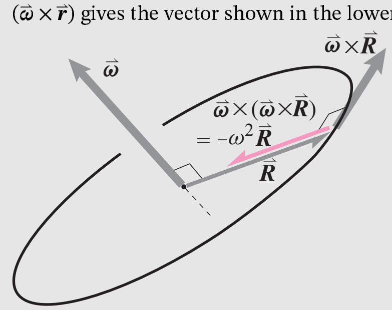
$$\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{A} \cdot \vec{C})\vec{B} - (\vec{A} \cdot \vec{B})\vec{C}$$

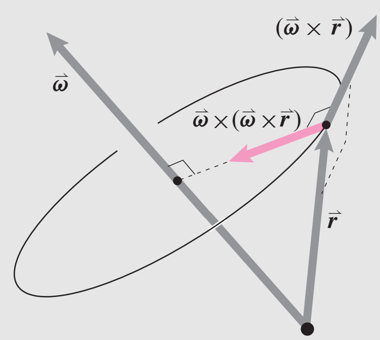
is almost natural: The expression above is almost the only expression that is a linear combination of $\vec{A}$ and $\vec{B}$ that is linear in both, also linear in $\vec{C}$ and switches sign if $\vec{B}$ and $\vec{C}$ are interchanged. These properties would be true if the whole expression were multiplied by any constant scalar. But a test of the equation with three unit vectors shows that such a multiplicative constant must be one. This reasoning constitutes an informal derivation of the identity 17.29.

## Using the triple cross product in dynamics equations

We will use identity 17.29 for two purposes in the development of dynamics equations:

1. In the 2D expression for acceleration, the centripetal acceleration is given by $\vec{\omega}\times(\vec{\omega}\times\vec{R})$ simplifies to $-\omega^2\vec{R}$ if $\vec{\omega} \perp \vec{R}$. This equation follows by setting $\vec{A} = \vec{\omega}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{R}$ in equation 17.29 and using $\vec{R}\cdot\vec{\omega} = 0$ if $\vec{\omega} \perp \vec{R}$. In 3D $\vec{\omega}\times(\vec{\omega} \times \vec{r})$ gives the vector shown in the lower figure below.




2. The term $\vec{r} \times (\vec{\omega} \times \vec{r})$ will appear in the calculation of the angular momentum of a rigid body. By setting $\vec{A} = \vec{r}$, $\vec{B} = \vec{\omega}$ and $\vec{C} = \vec{r}$, in equation 17.29 and use $\vec{r} \cdot \vec{r} = r^2$ because $\vec{r} \parallel \vec{r}$ we get the useful result that $\vec{r}\times(\vec{\omega}\times\vec{r}) = r^2\vec{\omega} - (\vec{r}\cdot\vec{\omega})\vec{r}$.

**SAMPLE 17.6 Free planar motion.** A rigid rod of length $\ell = 1\,\text{m}$ and mass $m_\text{r} = 1\,\text{kg}$, and a rigid square plate of side 1 m and mass $m_\text{p} = 10\,\text{kg}$ are launched in motion on a frictionless plane (*e.g.,* an ice hockey rink) with exactly the same initial velocities $\vec{\boldsymbol{v}}_\text{cm}(0) = 10\,\text{m/s}\hat{\boldsymbol{\imath}} + 1\,\text{m/s}\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{\omega}}(0) = 1\,\text{rad/s}\hat{\boldsymbol{k}}$. Both the rod and the plate have their center-of-mass at the baseline at $t = 0$.

1. Which of the two is farther from the base line in 3 seconds and which one has undergone a greater number of revolutions?
2. Find and draw the position of the bar at $t = 1$ sec and at $t = 3$ sec.

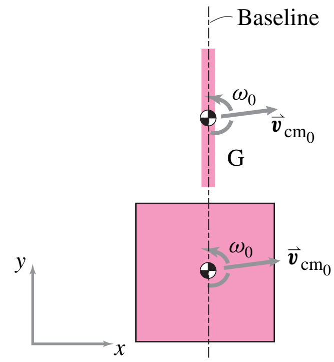



Figure 17.25

### Solution

1. The free-body diagram of the rod is shown in fig. 17.26. There are no forces acting on the rod in the $xy$-plane. Although there is force of gravity and the normal reaction of the surface acting on the rod, these forces are inconsequential since they act normal to the $xy$-plane. Therefore, we do not include these forces in our free-body diagram . The linear momentum balance for the rod gives

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m_\text{r}\vec{\boldsymbol{a}}_\text{cm} \\ \vec{\boldsymbol{0}} &= m_\text{r}\dot{\vec{\boldsymbol{v}}}_\text{cm} \\ \Rightarrow \quad \vec{\boldsymbol{v}}_\text{cm} &= \int \vec{\boldsymbol{0}}\,dt = \text{constant} = \vec{\boldsymbol{v}}_{\text{cm}\,0} \\ \Rightarrow \quad \vec{\boldsymbol{r}}_\text{cm} &= \int \vec{\boldsymbol{v}}_{\text{cm}\,0}\,dt = \vec{\boldsymbol{r}}_{\text{cm}\,0} + \vec{\boldsymbol{v}}_{\text{cm}\,0}t \end{aligned} \tag{17.30}$$

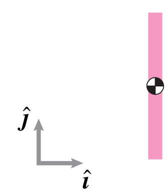



Figure 17.26

It is clear from the analysis above that in the absence of any applied forces, the position of the body depends only on the initial position and the initial velocity. Since both the plate and the rod start from the same base line with the same initial velocity, they travel the same distance from the base line during any given time period; mass or its geometric distribution plays no role in the motion. Thus the center-of-mass of the rod and the plate will be exactly the same distance ($|\vec{\boldsymbol{r}}_\text{cm}(t) - \vec{\boldsymbol{r}}_{\text{cm}\,0}| = |\vec{\boldsymbol{v}}_{\text{cm}\,0}t|$) at time $t$. Similarly, the angular momentum balance about the center-of-mass of the rod gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_\text{cm} &= \dot{\vec{\boldsymbol{H}}}_\text{cm} \\ \vec{\boldsymbol{0}} &= I_{zz}^\text{cm}\dot{\vec{\boldsymbol{\omega}}} \\ \Rightarrow \quad \vec{\boldsymbol{\omega}} &= \int \vec{\boldsymbol{0}}\,dt = \text{constant} = \vec{\boldsymbol{\omega}}_0 = \dot{\theta}_0\hat{\boldsymbol{k}} \\ \Rightarrow \quad \theta &= \int \dot{\theta}_0\,dt = \theta_0 + \dot{\theta}_0 t \end{aligned} \tag{17.31}$$

Thus the angular position of the body is also, as expected, independent of the mass and mass distribution of the body, and depends entirely on the initial position and the initial angular velocity. Therefore, both the rod and the plate undergo exactly the same amount of rotation $\left(\theta(t) - \theta_0 = \dot{\theta}_0 t\right)$ during any given time.

2. We can find the position of the rod at $t = 1$ s and $t = 3$ s by substituting these values of $t$ in eqns. (17.30) and 17.31. For convenience, let us assume that $\vec{\boldsymbol{r}}_{\text{cm}\,0} = \vec{\boldsymbol{0}}$. From the initial configuration of the rod, we also know that $\theta_0 = 0$.

$$\begin{aligned} \vec{\boldsymbol{r}}_\text{cm}(t = 1\,\text{s}) &= \vec{\boldsymbol{v}}_{\text{cm}\,0} \cdot (1\,\text{s}) = (10\,\text{m/s}\hat{\boldsymbol{\imath}} + 1\,\text{m/s}\hat{\boldsymbol{\jmath}}) \cdot (1\,\text{s})10\,\text{m}\hat{\boldsymbol{\imath}} + 1\,\text{m}\hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{r}}_\text{cm}(t = 3\,\text{s}) &= \vec{\boldsymbol{v}}_{\text{cm}\,0} \cdot (3\,\text{s}) = 30\,\text{m}\hat{\boldsymbol{\imath}} + 3\,\text{m}\hat{\boldsymbol{\jmath}}. \\ \theta(t = 1\,\text{s}) &= \dot{\theta}_0 \cdot (1\,\text{s}) = (1\,\text{rad/s}) \cdot (1\,\text{s}) = 1\,\text{rad} \\ \theta(t = 3\,\text{s}) &= \dot{\theta}_0 \cdot (3\,\text{s}) = 3\,\text{rad}. \end{aligned}$$

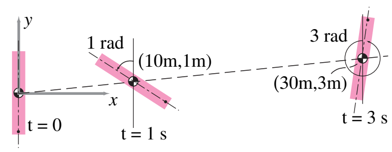



Figure 17.27

Accordingly, we show the position of the rod in fig. 17.27.

**SAMPLE 17.7** **A passive rigid diver.** An experimental model of a rigid diver is to be launched from a diving board that is 3 m above the water level. Say that the initial velocity of the center-of-mass and the initial angular velocity of the diver can be controlled at launch. The diver is launched into the dive in almost vertical position, and it is required to be as vertical as possible at the very end of the dive (which is marked by the position of the diver's center-of-mass at 1 m above the water level). If the initial vertical velocity of the diver's center-of-mass is 3 m/s, find the required initial angular velocity for the vertical entry of the diver into the water.

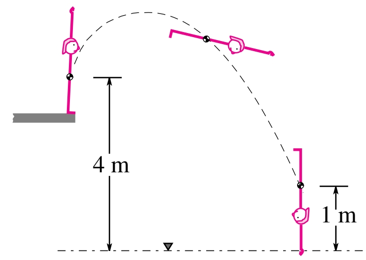


Figure 17.28

**Solution** Once the diver leaves the diving board, it is in free flight under gravity, *i.e.,* the only force acting on it is the force due to gravity. The free-body diagram of the diver is shown in fig. 17.29. The linear momentum balance for the diver gives

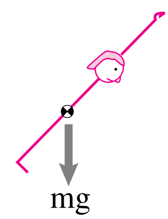


Figure 17.29

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a}_{\mathrm{cm}} \\
-mg\hat{\jmath} &= m\ddot{y}\hat{\jmath} \\
\Rightarrow \quad \ddot{y} &= -g \\
\sum \vec{M}_{\mathrm{cm}} &= \dot{\vec{H}}_{\mathrm{cm}} \\
\vec{0} &= I_{zz}^{\mathrm{cm}}\ddot{\theta}\hat{k} \\
\Rightarrow \quad \ddot{\theta} &= 0.
\end{aligned}
$$

From these equations of motion, it is clear that the linear and the angular motions of the diver are uncoupled. We can easily solve the equations of motion to get

$$
\begin{aligned}
y(t) &= y_0 + \dot{y}_0 t - \frac{1}{2} g t^2 \\
\theta(t) &= \theta_0 + \dot{\theta}_0 t.
\end{aligned}
$$

We need to find the initial angular speed $\dot{\theta}_0$ such that $\theta = \pi$ when $y = 1\,\mathrm{m}$ (the center-of-mass position at the water entry). From the expression for $\theta(t)$, we get, $\dot{\theta}_0 = \pi/t$. Thus we need to find the value of $t$ at the instant of water entry. We can find $t$ from the expression for $y(t)$ since we know that $y = 1\,\mathrm{m}$ at that instant, and that $y_0 = 3\,\mathrm{m}$ and $\dot{y}_0 = 3\,\mathrm{m/s}$. We have,

$$
\begin{aligned}
y &= y_0 + \dot{y}_0 - \frac{1}{2} g t^2 \\
\Rightarrow \quad t &= \frac{\dot{y}_0 \pm \sqrt{\dot{y}_0^2 + 2g(y_0 - y)}}{g} \\
&= \frac{3\,\mathrm{m/s} \pm \sqrt{(3\,\mathrm{m/s})^2 + 2 \cdot 9.8\,\mathrm{m/s^2} \cdot (3\,\mathrm{m} - 1\,\mathrm{m})}}{9.8\,\mathrm{m/s^2}} \\
&= 1.15 \text{ or } -0.53\,\mathrm{s}.
\end{aligned}
$$

We reject the negative value of time as meaningless in this context. Thus it takes the diver 1.15 s to complete the dive. Since, the diver must rotate by $\pi$ during this time, we have

$$\dot{\theta}_0 = \pi/t = \pi/(1.15\,\mathrm{s}) = 2.73\,\mathrm{rad/s}.$$

$$\boxed{\dot{\theta}_0 = 2.73\,\mathrm{rad/s}}$$

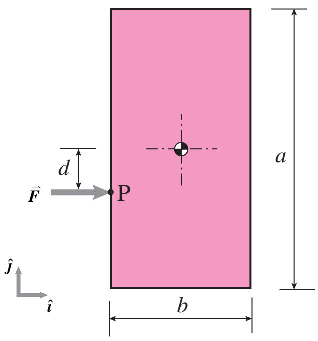



Figure 17.30

**SAMPLE 17.8** **A plate tumbling in space.** A rectangular plate of mass $m = 0.5\,\text{kg}, I_{zz}^{\text{cm}} = 2.08 \times 10^{-3}\,\text{kg}\cdot\text{m}^2$, and dimensions $a = 200\,\text{mm}$ and $b = 100\,\text{mm}$ is pushed by a force $\vec{F} = 0.5\,\text{N}\hat{\imath}$, acting at $d = 30\,\text{mm}$ away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

1. Find the initial acceleration of the mass-center.
2. Find the initial angular acceleration of the plate.
3. Write the equations of motion of the plate (for both linear and angular motion).

**Solution** The only force acting on the plate is the applied force $\vec{F}$. Thus, fig. 17.30 is also the free-body diagram of the plate at the start of motion.

1. From the linear momentum balance we get,

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \quad \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} = \frac{0.5\,\text{N}\hat{\imath}}{0.5\,\text{kg}} = 1\,\text{m/s}^2\hat{\imath}. \end{aligned}$$

which is the initial acceleration of the mass-center.

$$\vec{a}_{\text{cm}} = 1\,\text{m/s}^2\hat{\imath}$$

2. From the angular momentum balance about the mass-center, we get

$$\begin{aligned} \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\ Fd\hat{k} &= I_{zz}^{\text{cm}}\dot{\vec{\omega}} \\ \Rightarrow \quad \dot{\vec{\omega}} &= \frac{Fd}{I_{zz}^{\text{cm}}}\hat{k} = \frac{0.5\,\text{N}\cdot 0.03\,\text{m}}{2.08\,\text{kg}\cdot\text{m}^2} = 7.2\,\text{rad/s}^2\hat{k} \end{aligned}$$

which is the initial angular acceleration of the plate.

$$\dot{\vec{\omega}} = 7.2\,\text{rad/s}^2\hat{k}$$

3. To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration, $\vec{a}_{\text{cm}} = F/m\hat{\imath}$, and the angular acceleration, $\dot{\vec{\omega}} = Fd/I_{zz}^{\text{cm}}\hat{k}$, qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

   Let us first draw a free-body diagram of the plate in a general configuration during its motion (see fig. 17.31). Assume the center-of-mass to be displaced by $x\hat{\imath}$ and $y\hat{\jmath}$, and the longitudinal axis of the plate to be rotated by $\theta\hat{k}$ with respect to the vertical (inertial $y$-axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

$$\begin{aligned} &\sum \vec{F} = m\vec{a}_{\text{cm}} &&\Rightarrow &&\vec{a}_{\text{cm}} = \frac{\sum \vec{F}}{m} \\ \text{or} \quad &\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \frac{F}{m}\hat{\imath} &&\Rightarrow &&\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0. \end{aligned}$$

   Since $F/m$ is constant, the equations of motion of the center-of-mass indicate that the acceleration is constant and that the mass-center moves in the $x$-direction.

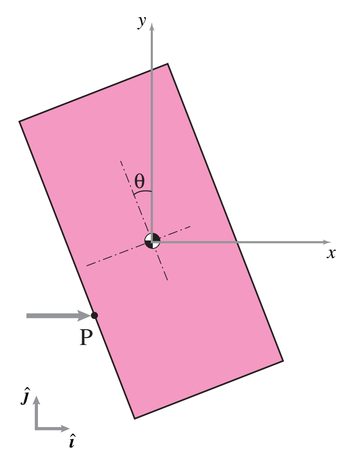



Figure 17.31: Free-body diagram of the plate at some instant $t$ when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed vertical axis.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned}\vec{\boldsymbol{M}}_{\mathrm{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\mathrm{cm}}\\ \vec{\boldsymbol{r}}_{\mathrm{P/cm}} \times \vec{\boldsymbol{F}} &= I_{zz}^{\mathrm{cm}}\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Now,

$$\begin{aligned}\vec{\boldsymbol{r}}_{\mathrm{P/cm}} &= -r[\cos(\theta+\alpha)\hat{\boldsymbol{\imath}} + \sin(\theta+\alpha)\hat{\boldsymbol{\jmath}}]\\ \vec{\boldsymbol{F}} &= F\hat{\boldsymbol{\imath}}\\ \Rightarrow \quad \vec{\boldsymbol{r}}_{\mathrm{P/cm}} \times \vec{\boldsymbol{F}} &= Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fr}{I_{zz}^{\mathrm{cm}}}\sin(\theta+\alpha)$$

where $r = \sqrt{d^2 + (b/2)^2}$ and $\alpha = \tan^{-1}(2d/b)$.

Thus, we have got the equations of motion for both the linear and the angular motion.

$$\boxed{\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0, \quad \ddot{\theta} = \frac{Fr}{I_{zz}^{\mathrm{cm}}}\sin(\theta+\alpha)}$$

4. The equations of linear motion of the plate are very simple and we can solve them at once to get

$$\begin{aligned}x(t) &= x_0 + \dot{x}_0 t + \frac{1}{2}\frac{F}{m}t^2\\ y(t) &= y_0 + \dot{y}_0 t.\end{aligned}$$

If the plate starts from rest ($\dot{x}_0 = 0$, $\dot{y}_0 = 0$) with the center-of-mass at the origin ($x_0 = 0$, $y_0 = 0$), then we have

$$x(t) = \frac{F}{2m}t^2, \quad \text{and} \quad y(t) = 0.$$

Thus the center-of-mass moves along the $x$-axis with acceleration $F/m$.

The equation of angular motion of the plate is not so simple. In fact, it is a nonlinear ODE. It is difficult to get an analytical solution of this equation. However, we can solve it numerically using, say, a Runge-Kutta ODE solver:

```
ODEs = {thetadot = w, wdot = (F*r/Icm)*sin(theta+a)}
IC   = {theta(0) = 0, w(0) = 0}
Set  F=.5, d=0.03; b=0.1; Icm=2.08e-03
compute r = sqrt(d^2+.25*b^2),  a = atan(2*d/b)
Solve ODEs with IC for t=0 to t=10
Plot theta(t)
```

The plot obtained from this calculation is shown in fig. 17.33 which shows that the plate oscillates in its plane as the center of mass races along the $x$-axis.

The trajectory of point P can now be obtained from $\vec{\boldsymbol{r}}_{\mathrm{P}} = \vec{\boldsymbol{r}}_{\mathrm{cm}} - r\cos(\theta+\alpha)\hat{\boldsymbol{\imath}} - r\sin(\theta+\alpha)\hat{\boldsymbol{\jmath}}$. Using this relationship, the trajectory of point P, along with the trajectory of the center of mass, is shown in fig. 17.34 for the first 3 seconds.

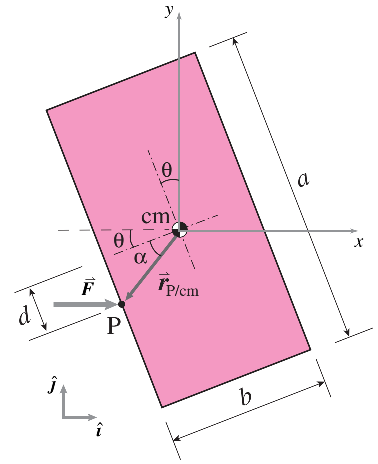



Figure 17.32: Geometry of the plate at the instant $t$ when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed $y$-axis. The position of point P is $\vec{\boldsymbol{r}}_{P/cm}$ which makes a fixed angle $\alpha(=\tan^{-1}\frac{d}{b/2})$ with the transverse axis of the plate. This angle is shown here merely for ease of calculation.

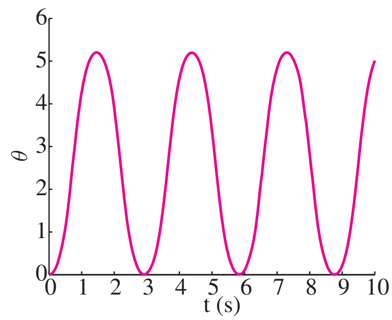



Figure 17.33: Variation of plate orientation $\theta$ with time.

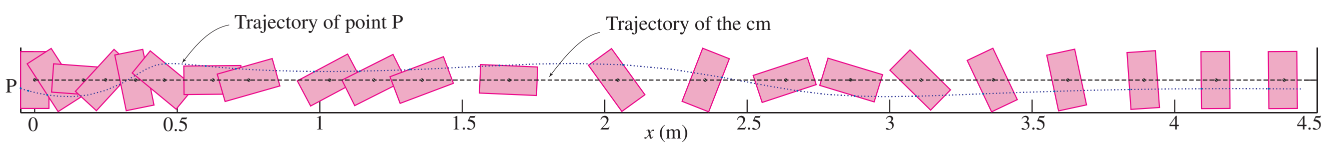



Figure 17.34: Motion of the plate during the first 3 seconds: Position of the plate at various time instants along with the trajectories of the center of mass and point P.

**SAMPLE 17.9 Impulse-momentum.** Consider the plate problem of Sample 17.8 (page 4) again. Assume that the plate is at rest at $t = 0$ in the vertical upright position and that the force acts on the plate for 2 seconds.

1. Find the velocity of the center-of-mass of the plate at the end of 2 seconds.
2. Can you also find the angular velocity of the plate at the end of 2 seconds?

### Solution

1. Since we are interested in finding the velocity at a particular instant $t$, given the velocity at another instant $t = 0$, we can use the impulse-momentum equations to find the desired velocity.

$$\begin{aligned}
\vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt \\
m\vec{\boldsymbol{v}}_{\mathrm{cm}}(t) - m\vec{\boldsymbol{v}}_{\mathrm{cm}}(0) &= \int_0^t \vec{\boldsymbol{F}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\mathrm{cm}}(t) &= \vec{\boldsymbol{v}}_{\mathrm{cm}}(0) + \frac{1}{m}\int_0^2 \vec{\boldsymbol{F}}\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{0.5\,\mathrm{kg}}\int_0^2 (0.5\,\mathrm{N}\hat{\boldsymbol{\imath}})\, dt \\
&= 2\,\mathrm{m/s}\hat{\boldsymbol{\imath}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{\mathrm{cm}}(2\,\mathrm{s}) = 2\,\mathrm{m/s}\hat{\boldsymbol{\imath}}}$$

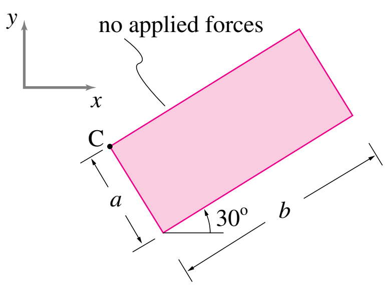
2. Now, let us try to find the angular velocity the same way, using angular impulse-momentum relation. We have,

$$\begin{aligned}
(\vec{\boldsymbol{H}}_{\mathrm{cm}})_2 - (\vec{\boldsymbol{H}}_{\mathrm{cm}})_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}(t) - I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}(0) &= \int_0^t \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{\omega}}(t) &= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
&= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t (\vec{\boldsymbol{r}}_{\mathrm{P/cm}} \times \vec{\boldsymbol{F}})\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t (Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}})\, dt \\
&= \frac{Fr}{I_{zz}^{\mathrm{cm}}}\left(\int_0^t \sin(\theta+\alpha)\, dt\right)\hat{\boldsymbol{k}}.
\end{aligned}$$

Now, we are in trouble; how do we evaluate the integral? In the integrand, we have $\theta$ which is an implicit function of $t$. Unless we know how $\theta$ depends on $t$ we cannot evaluate the integral. To find $\theta(t)$ we have to solve the equation of angular motion we derived in the previous sample. However, we were not able to solve for $\theta(t)$ analytically; we had to resort to a numerical solution. Thus, it is not possible to evaluate the integral above and, therefore, we cannot find the angular velocity of the plate at the end of 2 seconds using impulse-momentum equations. We could, however, find the desired velocity easily from the numerical solution.

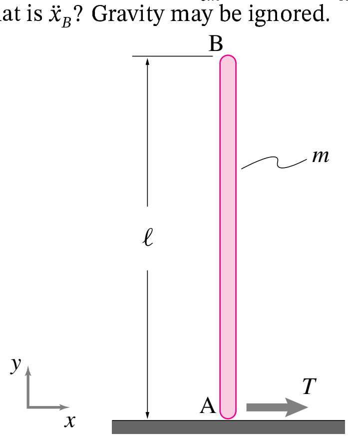
## 17.2 Dynamics of a rigid object

**17.2.1** The uniform rectangle of width $a = 1\,\text{m}$, length $b = 2\,\text{m}$, and mass $m = 1\,\text{kg}$ in the figure is sliding on the $xy$-plane with no friction. At the moment in question, point C is at $x_C = 3\,\text{m}$ and $y_C = 2\,\text{m}$. The linear momentum is $\vec{\boldsymbol{L}} = 4\hat{\imath} + 3\hat{\jmath}\ (\text{kg}{\cdot}\text{m/s})$ and the angular momentum about the center of mass is $\vec{\boldsymbol{H}}_{cm} = 5\hat{\boldsymbol{k}}(\text{kg}{\cdot}\text{m/s}^2)$. Find the acceleration of any point on the body that you choose. (Mark it.) [Hint: You have been given some redundant information.]




Problem 17.2.1

**17.2.2** The vertical pole AB of mass $m$ and length $\ell$ is initially, *at rest* on a frictionless surface. A tension $T$ is suddenly applied at A. What is $\ddot{x}_{cm}$? What is $\ddot{\theta}_{AB}$? What is $\ddot{x}_B$? Gravity may be ignored.




Problem 17.2.2

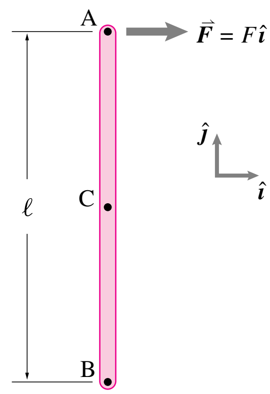
**17.2.3 Force on a stick in space. 2-D . No gravity.** A uniform thin stick with length $\ell$ and mass $m$ is, at the instant of interest, parallel to the $y$ axis and has no velocity and no angular velocity. The force $\vec{\boldsymbol{F}} = F\hat{\imath}$ with $F > 0$ is suddenly applied at point $A$. The questions below concern the instant after the force $\vec{\boldsymbol{F}}$ is applied.

a) What is the acceleration of point $C$, the center of mass? *

b) What is the angular acceleration of the stick? *

c) What is the acceleration of the point $A$? *

d) (relatively harder) What additional force would have to be applied to point $B$ to make point $B$'s acceleration zero? *




Problem 17.2.3

**17.2.4** A uniform thin rod of length $\ell$ and mass $m$ stands vertically, with one end resting on a frictionless surface and the other held by someone's hand. The rod is released from rest, displaced slightly from the vertical. No forces are applied during the release. There is gravity.

a) Find the path of the center of mass.

b) Find the force of the floor on the end of the rod just before the rod is horizontal.

**17.2.5** A uniform disk, with mass center labeled as point G, is sitting motionless on the frictionless $xy$ plane. A massless peg is attached to a point on its perimeter. This disk has radius of 1 m and mass of 10 kg. A constant force of $F = 1000\,\text{N}\boldsymbol{\imath}$ is applied to the peg for .0001 s (one ten-thousandth of a second).

a) What is the velocity of the center of mass of the center of the disk after the force is applied?

b) Assuming that the idealizations named in the problem statement are exact is your answer to (a) exact or approximate?

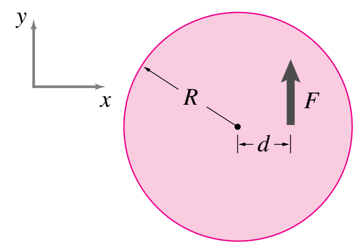
c) What is the angular velocity of the disk after the force is applied?

d) Assuming that the idealizations named in the problem statement are exact is your answer to (c) exact or approximate?

**17.2.6** A uniform thin flat disc is floating in space. It has radius $R$ and mass $m$. A force $F$ is applied to it at a distance $d$ from the center in the $y$ direction. Treat this problem as two-dimensional.

a) What is the acceleration of the center of the disc? *

b) What is the angular acceleration of the disk? *




Problem 17.2.6

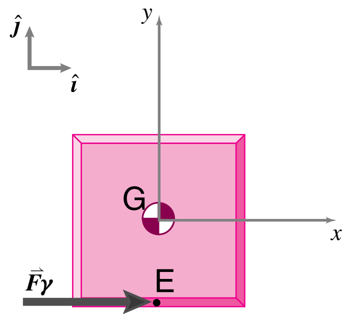
**17.2.7** A uniform 1kg plate that is one meter on a side is initially at rest in the position shown. A constant force $\vec{\boldsymbol{F}} = 1\,\text{N}\hat{\boldsymbol{\imath}}$ is applied at $t = 0$ and maintained henceforth. If you need to calculate any quantity that you don't know, but can't do the calculation to find it, assume that the value is given.

a) Find the position of G as a function of time (the answer should have numbers and units).

b) Find a differential equation, and initial conditions, that when solved would give $\theta$ as a function of time. $\theta$ is the counterclockwise rotation of the plate from the configuration shown.

c) Write computer commands that would generate a drawing of the outline of the plate at $t = 1\,\text{s}$. You can use hand calculations or the computer for as many of the intermediate commands as you like. Hand work and sketches should be provided as needed to justify or explain the computer work.

d) Run your code and show clear output with labeled plots. Mark output by hand to clarify any points.

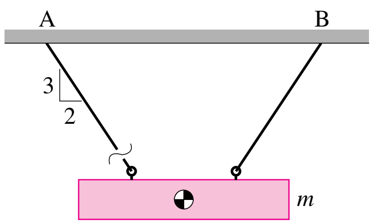



Problem 17.2.7

**17.2.8** A uniform rectangular metal beam of mass $m$ hangs symmetrically by two strings as shown in the figure.

a) Draw a free-body diagram of the beam and evaluate $\sum \vec{F}$.

b) Repeat (a) immediately after the left string is cut.



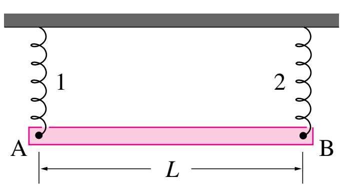

Problem 17.2.8

**17.2.9** A uniform slender bar AB of mass $m$ is suspended from two springs (each of spring constant $K$) as shown. Immediately after spring 2 breaks, determine

a) the angular acceleration of the bar,

b) the acceleration of point $A$, and

c) the acceleration of point $B$.




Problem 17.2.9

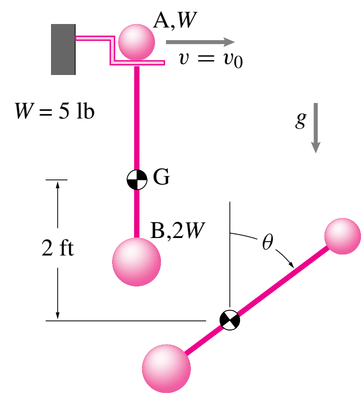
**17.2.10** Two small spheres A and B are connected by a rigid rod of length $\ell$ = 1.0 ft and negligible mass. The assembly is hung from a hook, as shown. Sphere A is struck, suddenly breaking its contact with the hook and giving it a horizontal velocity $v_0$ = 3.0 ft/s which sends the assembly into free fall. Determine the angular momentum of the assembly about its mass center at point G immediately after A is hit. After the center of mass has fallen two feet, determine:

a) the angle $\theta$ through which the rod has rotated,

b) the velocity of sphere A,

c) the total kinetic energy of the assembly of spheres A and B and the rod, and

d) the acceleration of sphere A.




Problem 17.2.10

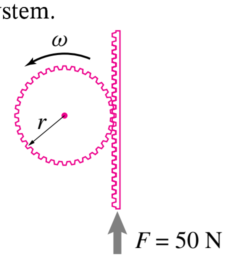
The next several problems concern Work, power and energy

**17.2.11** Verify that the expressions for work done by a force $F$, $W = F\Delta S$, and by a moment $M$, $W = M\Delta\theta$, are dimensionally correct if $\Delta S$ and $\Delta\theta$ are linear and angular displacements respectively.

**17.2.12** A uniform disc of mass $m$ and radius $r$ rotates with angular velocity $\omega\hat{\mathbf{k}}$. Its center of mass translates with velocity $\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}$ in the $xy$-plane. What is the total kinetic energy of the disk?

**17.2.13** Calculate the energy stored in a spring using the expression $E_P = \frac{1}{2}k\delta^2$ if the spring is compressed by 100 mm and the spring constant is 100 N/m.

**17.2.14** In a rack and pinion system, the rack is acted upon by a constant force $F$ = 50 N and has speed $v$ = 2 m/s in the direction of the force. Find the input power to the system.

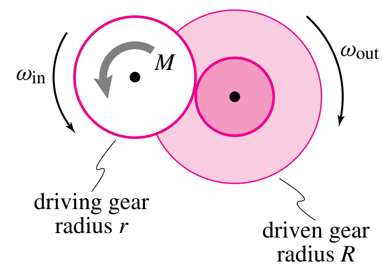



Problem 17.2.14

**17.2.15** The driving gear in a compound gear train rotates at constant speed $\omega_0$. The driving torque is $M_{in}$. If the driven gear rotates at a constant speed $\omega_{out}$, find:

a) the input power to the system, and

b) the output torque of the system assuming there is no power loss in the system; i.e., power in = power out.

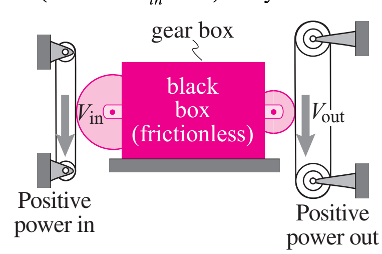



Problem 17.2.15

**17.2.16** An elaborate frictionless gear box has an input and output roller with $V_{in} = const$. Assuming that $V_{out} = 7V_{in}$ and the force between the left belt and roller is $F_{in}$ = 3 lb:

a) What is $F_{out}$ (draw a picture defining the signs of $F_{in}$ and $F_{out}$)?. *

b) Is $F_{out}$ greater or less than the $F_{in}$? (Assume $F_{in} > 0$.) Why? *




Problem 17.2.16