SAMPLE 17.6 Free planar motion. A rigid rod of length $\ell = 1\,\mathrm{m}$ and mass $m_\mathrm{r} = 1\,\mathrm{kg}$, and a rigid square plate of side 1 m and mass $m_\mathrm{p} = 10\,\mathrm{kg}$ are launched in motion on a frictionless plane (*e.g.,* an ice hockey rink) with exactly the same initial velocities $\vec{\boldsymbol{v}}_\mathrm{cm}(0) = 10\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + 1\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{\omega}}(0) = 1\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. Both the rod and the plate have their center-of-mass at the baseline at $t = 0$.

1. Which of the two is farther from the base line in 3 seconds and which one has undergone a greater number of revolutions?
2. Find and draw the position of the bar at $t = 1$ sec and at $t = 3$ sec.

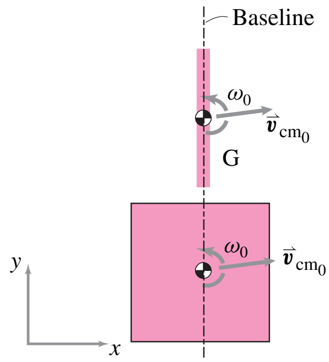

Figure 17.25

Solution

1. The free-body diagram of the rod is shown in fig. 17.26. There are no forces acting on the rod in the xy-plane. Although there is force of gravity and the normal reaction of the surface acting on the rod, these forces are inconsequential since they act normal to the xy-plane. Therefore, we do not include these forces in our free-body diagram . The linear momentum balance for the rod gives

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m_\mathrm{r}\vec{\boldsymbol{a}}_\mathrm{cm} \\ \vec{\boldsymbol{0}} &= m_\mathrm{r}\dot{\vec{\boldsymbol{v}}}_\mathrm{cm} \\ \Rightarrow \quad \vec{\boldsymbol{v}}_\mathrm{cm} &= \int \vec{\boldsymbol{0}}\,dt = \text{constant} = \vec{\boldsymbol{v}}_{\mathrm{cm}\,0} \\ \Rightarrow \quad \vec{\boldsymbol{r}}_\mathrm{cm} &= \int \vec{\boldsymbol{v}}_{\mathrm{cm}\,0}\,dt = \vec{\boldsymbol{r}}_{\mathrm{cm}\,0} + \vec{\boldsymbol{v}}_{\mathrm{cm}\,0}t \end{aligned} \tag{17.30}$$

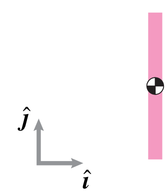

Figure 17.26

It is clear from the analysis above that in the absence of any applied forces, the position of the body depends only on the initial position and the initial velocity. Since both the plate and the rod start from the same base line with the same initial velocity, they travel the same distance from the base line during any given time period; mass or its geometric distribution plays no role in the motion. Thus the center-of-mass of the rod and the plate will be exactly the same distance ($|\vec{\boldsymbol{r}}_\mathrm{cm}(t) - \vec{\boldsymbol{r}}_{\mathrm{cm}\,0}| = |\vec{\boldsymbol{v}}_{\mathrm{cm}\,0}t|$) at time t. Similarly, the angular momentum balance about the center-of-mass of the rod gives

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_\mathrm{cm} &= \dot{\vec{\boldsymbol{H}}}_\mathrm{cm} \\ \vec{\boldsymbol{0}} &= I_{zz}^\mathrm{cm}\dot{\vec{\boldsymbol{\omega}}} \\ \Rightarrow \quad \vec{\boldsymbol{\omega}} &= \int \vec{\boldsymbol{0}}\,dt = \text{constant} = \vec{\boldsymbol{\omega}}_0 = \dot{\theta}_0\hat{\boldsymbol{k}} \\ \Rightarrow \quad \theta &= \int \dot{\theta}_0\,dt = \theta_0 + \dot{\theta}_0 t \end{aligned} \tag{17.31}$$

Thus the angular position of the body is also, as expected, independent of the mass and mass distribution of the body, and depends entirely on the initial position and the initial angular velocity. Therefore, both the rod and the plate undergo exactly the same amount of rotation $\left(\theta(t) - \theta_0 = \dot{\theta}_0 t\right)$ during any given time.

2. We can find the position of the rod at $t = 1\,\mathrm{s}$ and $t = 3\,\mathrm{s}$ by substituting these values of t in eqns. (17.30) and 17.31. For convenience, let us assume that $\vec{\boldsymbol{r}}_{\mathrm{cm}\,0} = \vec{\boldsymbol{0}}$. From the initial configuration of the rod, we also know that $\theta_0 = 0$.

$$\begin{aligned} \vec{\boldsymbol{r}}_\mathrm{cm}(t = 1\,\mathrm{s}) &= \vec{\boldsymbol{v}}_{\mathrm{cm}\,0}\cdot(1\,\mathrm{s}) = (10\,\mathrm{m/s}\hat{\boldsymbol{\imath}} + 1\,\mathrm{m/s}\hat{\boldsymbol{\jmath}})\cdot(1\,\mathrm{s})10\,\mathrm{m}\hat{\boldsymbol{\imath}} + 1\,\mathrm{m}\hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{r}}_\mathrm{cm}(t = 3\,\mathrm{s}) &= \vec{\boldsymbol{v}}_{\mathrm{cm}\,0}\cdot(3\,\mathrm{s}) = 30\,\mathrm{m}\hat{\boldsymbol{\imath}} + 3\,\mathrm{m}\hat{\boldsymbol{\jmath}}. \\ \theta(t = 1\,\mathrm{s}) &= \dot{\theta}_0\cdot(1\,\mathrm{s}) = (1\,\mathrm{rad/s})\cdot(1\,\mathrm{s}) = 1\,\mathrm{rad} \\ \theta(t = 3\,\mathrm{s}) &= \dot{\theta}_0\cdot(3\,\mathrm{s}) = 3\,\mathrm{rad}. \end{aligned}$$

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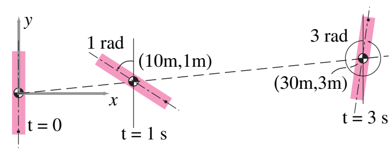

Figure 17.27

Accordingly, we show the position of the rod in fig. 17.27.

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SAMPLE 17.7 A passive rigid diver. An experimental model of a rigid diver is to be launched from a diving board that is 3 m above the water level. Say that the initial velocity of the center-of-mass and the initial angular velocity of the diver can be controlled at launch. The diver is launched into the dive in almost vertical position, and it is required to be as vertical as possible at the very end of the dive (which is marked by the position of the diver's center-of-mass at 1 m above the water level). If the initial vertical velocity of the diver's center-of-mass is 3 m/s, find the required initial angular velocity for the vertical entry of the diver into the water.

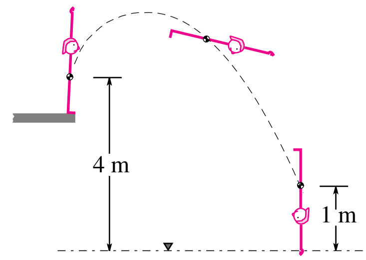

Figure 17.28

Solution Once the diver leaves the diving board, it is in free flight under gravity, *i.e.*, the only force acting on it is the force due to gravity. The free-body diagram of the diver is shown in fig. 17.29. The linear momentum balance for the diver gives

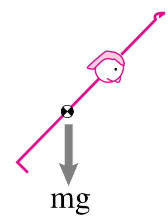

Figure 17.29

$$\begin{aligned}
\sum \vec{F} &= m\vec{a}_{\text{cm}} \\
-mg\hat{\jmath} &= m\ddot{y}\hat{\jmath} \\
\Rightarrow \quad \ddot{y} &= -g \\
\sum \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\
\vec{0} &= I_{zz}^{\text{cm}}\ddot{\theta}\hat{k} \\
\Rightarrow \quad \ddot{\theta} &= 0.
\end{aligned}$$

From these equations of motion, it is clear that the linear and the angular motions of the diver are uncoupled. We can easily solve the equations of motion to get

$$\begin{aligned}
y(t) &= y_0 + \dot{y}_0 t - \frac{1}{2}gt^2 \\
\theta(t) &= \theta_0 + \dot{\theta}_0 t.
\end{aligned}$$

We need to find the initial angular speed $\dot{\theta}_0$ such that $\theta = \pi$ when $y = 1\,\text{m}$ (the center-of-mass position at the water entry). From the expression for $\theta(t)$, we get, $\dot{\theta}_0 = \pi/t$. Thus we need to find the value of t at the instant of water entry. We can find t from the expression for $y(t)$ since we know that $y = 1\,\text{m}$ at that instant, and that $y_0 = 3\,\text{m}$ and $\dot{y}_0 = 3\,\text{m/s}$. We have,

$$\begin{aligned}
y &= y_0 + \dot{y}_0 - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \frac{\dot{y}_0 \pm \sqrt{\dot{y}_0^2 + 2g(y_0 - y)}}{g} \\
&= \frac{3\,\text{m/s} \pm \sqrt{(3\,\text{m/s})^2 + 2 \cdot 9.8\,\text{m/s}^2 \cdot (3\,\text{m} - 1\,\text{m})}}{9.8\,\text{m/s}^2} \\
&= 1.15 \text{ or } -0.53\,\text{s}.
\end{aligned}$$

We reject the negative value of time as meaningless in this context. Thus it takes the diver 1.15 s to complete the dive. Since, the diver must rotate by π during this time, we have

$$\dot{\theta}_0 = \pi/t = \pi/(1.15\,\text{s}) = 2.73\,\text{rad/s}.$$

$$\boxed{\dot{\theta}_0 = 2.73\,\text{rad/s}}$$

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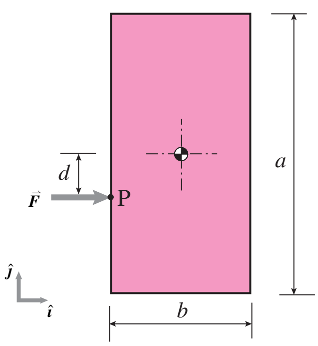

Figure 17.30

SAMPLE 17.8 A plate tumbling in space. A rectangular plate of mass $m = 0.5\,\text{kg}, I_{zz}^{\text{cm}} = 2.08\times 10^{-3}\,\text{kg}\cdot\text{m}^2$, and dimensions $a = 200\,\text{mm}$ and $b = 100\,\text{mm}$ is pushed by a force $\vec{F} = 0.5\,\text{N}\hat{\imath}$, acting at $d = 30\,\text{mm}$ away from the mass-center, as shown in the figure. Assume that the force remains constant in magnitude and direction but remains attached to the material point P of the plate. There is no gravity.

1. Find the initial acceleration of the mass-center.
2. Find the initial angular acceleration of the plate.
3. Write the equations of motion of the plate (for both linear and angular motion).

Solution The only force acting on the plate is the applied force $\vec{F}$. Thus, fig. 17.30 is also the free-body diagram of the plate at the start of motion.

1. From the linear momentum balance we get,

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_{\text{cm}} \\ \Rightarrow \quad \vec{a}_{\text{cm}} &= \frac{\sum \vec{F}}{m} = \frac{0.5\,\text{N}\hat{\imath}}{0.5\,\text{kg}} = 1\,\text{m/s}^2\hat{\imath}. \end{aligned}$$

which is the initial acceleration of the mass-center.

$$\boxed{\vec{a}_{\text{cm}} = 1\,\text{m/s}^2\hat{\imath}}$$

2. From the angular momentum balance about the mass-center, we get

$$\begin{aligned} \vec{M}_{\text{cm}} &= \dot{\vec{H}}_{\text{cm}} \\ Fd\hat{k} &= I_{zz}^{\text{cm}}\dot{\vec{\omega}} \\ \Rightarrow \quad \dot{\vec{\omega}} &= \frac{Fd}{I_{zz}^{\text{cm}}}\hat{k} = \frac{0.5\,\text{N}\cdot 0.03\,\text{m}}{2.08\,\text{kg}\cdot\,\text{m}^2} = 7.2\,\text{rad/s}^2\hat{k} \end{aligned}$$

which is the initial angular acceleration of the plate.

$$\boxed{\dot{\vec{\omega}} = 7.2\,\text{rad/s}^2\hat{k}}$$

3. To find the equations of motion, we can use the linear momentum balance and the angular momentum balance as we have done above. So, why aren't the equations obtained above for the linear acceleration, $\vec{a}_{\text{cm}} = F/m\hat{\imath}$, and the angular acceleration, $\dot{\vec{\omega}} = Fd/I_{zz}^{\text{cm}}\hat{k}$, qualified to be called equations of motion? Because, they are not valid for a general configuration of the plate during its motion. The expressions for the accelerations are valid only in the initial configuration (and hence those are initial accelerations).

   Let us first draw a free-body diagram of the plate in a general configuration during its motion (see fig. 17.31). Assume the center-of-mass to be displaced by $x\hat{\imath}$ and $y\hat{\jmath}$, and the longitudinal axis of the plate to be rotated by $\theta\hat{k}$ with respect to the vertical (inertial y-axis). The applied force remains horizontal and attached to the material point P, as stated in the problem. The linear momentum balance gives

$$\begin{aligned} &\sum \vec{F} = m\vec{a}_{\text{cm}} && \Rightarrow && \vec{a}_{\text{cm}} = \frac{\sum \vec{F}}{m} \\ \text{or} \quad &\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = \frac{F}{m}\hat{\imath} && \Rightarrow && \ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0. \end{aligned}$$

   Since F/m is constant, the equations of motion of the center-of-mass indicate that the acceleration is constant and that the mass-center moves in the x-direction.

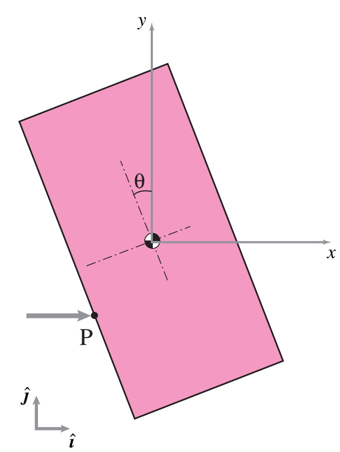

Figure 17.31: Free-body diagram of the plate at some instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed vertical axis.

Similarly, we now use angular momentum balance to determine the rotation (angular motion) of the plate. The angular momentum balance about the mass-center give

$$\begin{aligned}\vec{\boldsymbol{M}}_{\text{cm}} &= \dot{\vec{\boldsymbol{H}}}_{\text{cm}}\\ \vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}} &= I_{zz}^{\text{cm}}\ddot{\theta}\hat{\boldsymbol{k}}.\end{aligned}$$

Now,

$$\begin{aligned}\vec{\boldsymbol{r}}_{\text{P/cm}} &= -r[\cos(\theta+\alpha)\hat{\boldsymbol{\imath}} + \sin(\theta+\alpha)\hat{\boldsymbol{\jmath}}]\\ \vec{\boldsymbol{F}} &= F\hat{\boldsymbol{\imath}}\\ \Rightarrow \quad \vec{\boldsymbol{r}}_{\text{P/cm}} \times \vec{\boldsymbol{F}} &= Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}}.\end{aligned}$$

Thus,

$$\ddot{\theta} = \frac{Fr}{I_{zz}^{\text{cm}}}\sin(\theta+\alpha)$$

where $r = \sqrt{d^2 + (b/2)^2}$ and $\alpha = \tan^{-1}(2d/b)$.

Thus, we have got the equations of motion for both the linear and the angular motion.

$$\ddot{x} = \frac{F}{m}, \quad \ddot{y} = 0, \quad \ddot{\theta} = \frac{Fr}{I_{zz}^{\text{cm}}}\sin(\theta+\alpha)$$

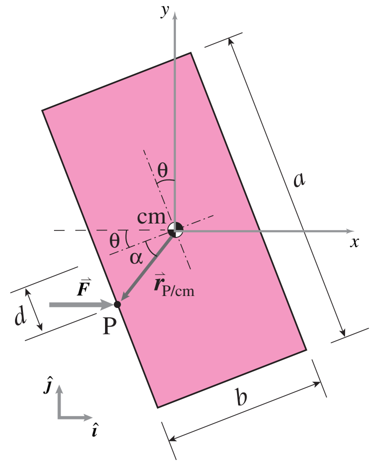

Figure 17.32: Geometry of the plate at the instant t when the longitudinal axis of the plate makes an angle $\theta(t)$ with the fixed y-axis. The position of point P is $\vec{\boldsymbol{r}}_{P/cm}$ which makes a fixed angle $\alpha(= \tan^{-1}\frac{d}{b/2})$ with the transverse axis of the plate. This angle is shown here merely for ease of calculation.

4. The equations of linear motion of the plate are very simple and we can solve them at once to get

$$\begin{aligned}x(t) &= x_0 + \dot{x}_0 t + \frac{1}{2}\frac{F}{m}t^2\\ y(t) &= y_0 + \dot{y}_0 t.\end{aligned}$$

If the plate starts from rest ($\dot{x}_0 = 0$, $\dot{y}_0 = 0$) with the center-of-mass at the origin ($x_0 = 0$, $y_0 = 0$), then we have

$$x(t) = \frac{F}{2m}t^2, \quad \text{and} \quad y(t) = 0.$$

Thus the center-of-mass moves along the x-axis with acceleration F/m.

The equation of angular motion of the plate is not so simple. In fact, it is a nonlinear ODE. It is difficult to get an analytical solution of this equation. However, we can solve it numerically using, say, a Runge-Kutta ODE solver:

```
ODEs = {thetadot = w, wdot = (F*r/Icm)*sin(theta+a)}
IC   = {theta(0) = 0, w(0) = 0}
Set  F=.5, d=0.03; b=0.1; Icm=2.08e-03
compute r = sqrt(d^2+.25*b^2),  a = atan(2*d/b)
Solve ODEs with IC for t=0 to t=10
Plot theta(t)
```

The plot obtained from this calculation is shown in fig. 17.33 which shows that the plate oscillates in its plane as the center of mass races along the x-axis.

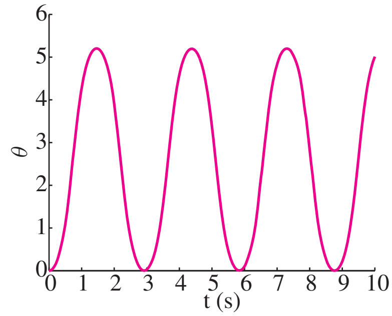

Figure 17.33: Variation of plate orientation θ with time.

The trajectory of point P can now be obtained from $\vec{\boldsymbol{r}}_{\text{P}} = \vec{\boldsymbol{r}}_{\text{cm}} - r\cos(\theta+\alpha)\hat{\boldsymbol{\imath}} - r\sin(\theta+\alpha)\hat{\boldsymbol{\jmath}}$. Using this relationship, the trajectory of point P, along with the trajectory of the center of mass, is shown in fig. 17.34 for the first 3 seconds.

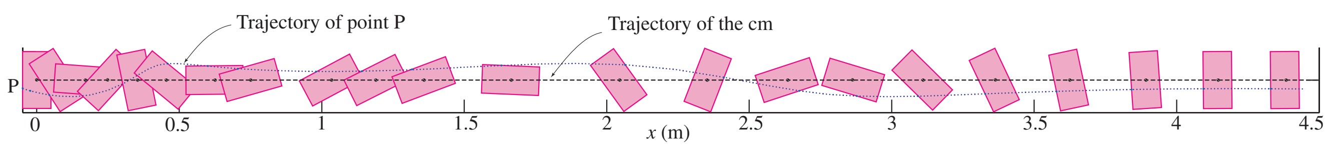

Figure 17.34: Motion of the plate during the first 3 seconds: Position of the plate at various time instants along with the trajectories of the center of mass and point P.

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SAMPLE 17.9 Impulse-momentum. Consider the plate problem of Sample 17.8 (page 4) again. Assume that the plate is at rest at $t = 0$ in the vertical upright position and that the force acts on the plate for 2 seconds.

1. Find the velocity of the center-of-mass of the plate at the end of 2 seconds.
2. Can you also find the angular velocity of the plate at the end of 2 seconds?

Solution

1. Since we are interested in finding the velocity at a particular instant t, given the velocity at another instant $t = 0$, we can use the impulse-momentum equations to find the desired velocity.

$$
\begin{aligned}
\vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{F}}\, dt \\
m\vec{\boldsymbol{v}}_{\mathrm{cm}}(t) - m\vec{\boldsymbol{v}}_{\mathrm{cm}}(0) &= \int_0^t \vec{\boldsymbol{F}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\mathrm{cm}}(t) &= \vec{\boldsymbol{v}}_{\mathrm{cm}}(0) + \frac{1}{m}\int_0^2 \vec{\boldsymbol{F}}\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{0.5\,\mathrm{kg}}\int_0^2 (0.5\,\mathrm{N}\hat{\boldsymbol{\imath}})\, dt \\
&= 2\,\mathrm{m/s}\hat{\boldsymbol{\imath}}.
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{v}}_{\mathrm{cm}}(2\,\mathrm{s}) = 2\,\mathrm{m/s}\hat{\boldsymbol{\imath}}}$$

2. Now, let us try to find the angular velocity the same way, using angular impulse-momentum relation. We have,

$$
\begin{aligned}
(\vec{\boldsymbol{H}}_{\mathrm{cm}})_2 - (\vec{\boldsymbol{H}}_{\mathrm{cm}})_1 &= \int_{t_1}^{t_2} \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}(t) - I_{zz}^{\mathrm{cm}}\vec{\boldsymbol{\omega}}(0) &= \int_0^t \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
\Rightarrow \quad \vec{\boldsymbol{\omega}}(t) &= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t \sum \vec{\boldsymbol{M}}_{\mathrm{cm}}\, dt \\
&= \vec{\boldsymbol{\omega}}(0) + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t (\vec{\boldsymbol{r}}_{\mathrm{P/cm}} \times \vec{\boldsymbol{F}})\, dt \\
&= \vec{\boldsymbol{0}} + \frac{1}{I_{zz}^{\mathrm{cm}}}\int_0^t (Fr\sin(\theta+\alpha)\hat{\boldsymbol{k}})\, dt \\
&= \frac{Fr}{I_{zz}^{\mathrm{cm}}}\left(\int_0^t \sin(\theta+\alpha)\, dt\right)\hat{\boldsymbol{k}}.
\end{aligned}
$$

Now, we are in trouble; how do we evaluate the integral? In the integrand, we have θ which is an implicit function of t. Unless we know how θ depends on t we cannot evaluate the integral. To find $\theta(t)$ we have to solve the equation of angular motion we derived in the previous sample. However, we were not able to solve for $\theta(t)$ analytically; we had to resort to a numerical solution. Thus, it is not possible to evaluate the integral above and, therefore, we cannot find the angular velocity of the plate at the end of 2 seconds using impulse-momentum equations. We could, however, find the desired velocity easily from the numerical solution.