17.1 Description of motion: planar rigid-object kinematics

As a rigid object moves, how do the points on it move? There are two reasons to ask this question. First, velocities and accelerations of mass points are needed to apply the momentum-balance equations. Second, formulas for positions, velocities and accelerations of points are useful to understand *mechanisms*, machines where various parts (each one usually idealized as a rigid object) are connected to each other with hinges and bearings of one type or another.

The central observation in all rigid-object kinematics is the definition of a rigid object:

> All pairs of points on a single rigid object keep constant distance from each other as the object moves.

In this section you will learn how to use rigidity to calculate positions, velocities and accelerations of all points (millions and billions of them) on a rigid object given only a few numbers (9 of them). This goal is achieved by putting together the ideas from Chapter 11 (arbitrary motion of one particle), Chapter 14 (straight-line motion), and Chapter chapter 15 (circular motion of a rigid object in a plane).

Displacement and rotation

When a planar object (say a machine part) $\mathcal{B}$ moves from one configuration in the plane to another it has a displacement and a rotation. It starts in some reference configuration *. We mark a reference point on the body that, in the reference configuration, coincides with a fixed reference point, say 0①. We also mark a (directed) line on the body that, in the reference configuration, coincides with a fixed reference line, say the positive x axis.

①The body never has to pass through this reference position, however. For example, the position of an airplane flying from New York to Mumbai is measured relative to a point in the Gulf of Guinea 1000 miles west of Gabon (the location of O° longitude and O° latitude) even though the airplane never goes there (nor does anyone want it to).

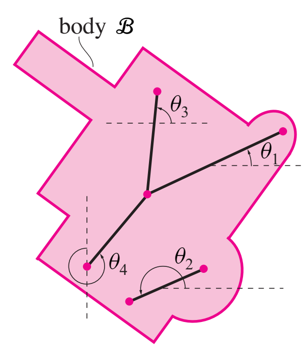

Figure 17.4: Rotation of object $\mathcal{B}$ is measured by the rotation of real or imagined lines marked on the object. The lines make different angles: $\theta_1 \neq \theta_2$, $\theta_2 \neq \theta_3$ etc, but $\dot\theta_1 = \dot\theta_2 = \dot\theta_3 = \dots$. Angular velocity is defined as $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ with $\omega \equiv \dot\theta_1 = \dot\theta_2 = \dot\theta_3 = \dots$.

We could measure rotation by measuring the rotation of any line segment connecting any pair of points fixed to the object. For each line we keep track of the angle that line makes with a line fixed in space, say the positive x or y axis. It's simplest to stick to the convention that counterclockwise rotations are positive (fig. 17.4). The angles $\theta_1, \theta_2 \dots$, all change with time and are all different from each other. But all the angles change the same amount, just like in section 16.1. We can pick any one line we like for definiteness and measure the object rotation by the rotation of that line. So

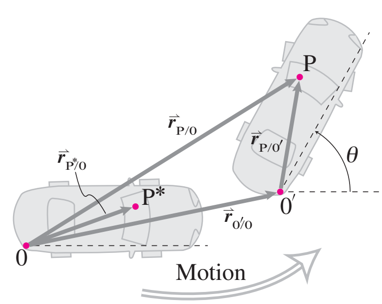

Figure 17.5: The displacement and rotation of a planar object relative to a reference configuration.

> The net motion of a rigid planar object is described by *translation*, the vector displacement of a reference point from a reference posi-

tion $\vec{\boldsymbol{r}}_{o'/o} = \vec{\boldsymbol{r}}_{oo'}$, and a scalar *rotation* θ of the reference line from its reference orientation.

The position of a point on a moving rigid object.

Given that P on the object is at $\vec{\boldsymbol{r}}_{P/0}{}^*$ in the reference (*) configuration, where is it (What is $\vec{\boldsymbol{r}}_{P/0}$?) after the object has been displaced by $\vec{\boldsymbol{r}}_{0'/0}$ and rotated an angle θ?

The base-independent or direct vector notation. An easy way to treat this is to write the new position of P as (see fig. 17.5), in the *base-independent* or *direct vector* representation of the position of P,

$$\vec{\boldsymbol{r}}_{P/0} = \vec{\boldsymbol{r}}_{0'/0} + \vec{\boldsymbol{r}}_{P/0'}. \tag{17.1}$$

The vector $\vec{\boldsymbol{r}}_{0'/0}$ describes *translation*, that's half the story. The other term $\vec{\boldsymbol{r}}_{P/0'}$ we find by rotating $\vec{\boldsymbol{r}}_{P/0}{}^*$ as we did in Section 16.2. This 'base-independent' formula is correct no matter what base vectors are used to represent the vectors in the formula (*e.g.,* $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ or $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$).

Fixed basis or component representation. For a given basis, say $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ associated with x and y, we can write the coordinates of a point as,

$$\left[\vec{\boldsymbol{r}}_{P/0}\right]_{xy} = \underbrace{\left[\vec{\boldsymbol{r}}_{0'/0}\right]_{xy}}_{\text{displacement}} + \underbrace{\left[R(\theta)\right]\left[\vec{\boldsymbol{r}}_{P/0'}\right]_{x'y'}}_{\text{rotation}} \tag{17.2}$$

or, writing out all the components of the vectors and matrices, We get the *fixed basis* or *component* representation of the motion:

$$\begin{bmatrix} x_P \\ y_P \end{bmatrix} = \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x^*_{P/0} \\ y^*_{P/0} \end{bmatrix}. \tag{17.3}$$

As the motion progresses the displacement $\begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix}$ changes with time as does the rotation angle θ. In this *fixed basis* or *component* representation of the motion of eqn. (17.3) we get the components of the position in terms of base vectors that are fixed in space.

Example:
If in the reference position a particle on a rigid object is at $\vec{\boldsymbol{r}}_{P/0} = (1\hat{\boldsymbol{\imath}} + 2\hat{\boldsymbol{\jmath}})\,\text{m}$ and the object displaces by $\vec{\boldsymbol{r}}_{0'/0} = (3\hat{\boldsymbol{\imath}} + 4\hat{\boldsymbol{\jmath}})\,\text{m}$ and rotates by $\theta = \pi/3\,\text{rad} = 60°$

relative to that configuration, then its new position is:

$$
\begin{aligned}
\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0}\right]_{xy} &= \left[\vec{\boldsymbol{r}}_{0'/0}\right]_{xy} + [R(\theta)]\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0'}\right]_{x'y'} \\
&= \begin{bmatrix} x_{0'/0} \\ y_{0'/0} \end{bmatrix} + \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{P}/0} \\ y^*_{\mathrm{P}/0} \end{bmatrix} \\
&= \left\{\begin{bmatrix} 3 \\ 4 \end{bmatrix} + \begin{bmatrix} \cos\pi/3 & -\sin\pi/3 \\ \sin\pi/3 & \cos\pi/3 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}\right\} \mathrm{m} \\
&= \begin{bmatrix} 3.5-\sqrt{3} \\ 5+\sqrt{3}/2 \end{bmatrix} \mathrm{m} \\
\Rightarrow \vec{\boldsymbol{r}}_{\mathrm{P}/0} &= \left((3.5-\sqrt{3})\hat{\boldsymbol{\imath}} + (5+\sqrt{3}/2)\hat{\boldsymbol{\jmath}}\right) \mathrm{m}
\end{aligned}
$$

Changing-base representation. Finally, the *changing base* representation uses base vectors $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$ that are aligned with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ in the reference configuration but which are glued to the rotating object. If we define x' and y' as the x and y components of P in the reference (*) configuration we have that

$$
\begin{aligned}
\left[\vec{\boldsymbol{r}}_{\mathrm{P}/0}{}^*\right]_{xy} &= \left[\vec{\boldsymbol{r}}_{\mathrm{P}/0'}{}^*\right]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix} \\
\text{so} \quad \vec{\boldsymbol{r}}_{\mathrm{P}/0} &= \left(x_{0'/0}\hat{\boldsymbol{\imath}} + y_{0'/0}\hat{\boldsymbol{\jmath}}\right) + \left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right). \qquad (17.4)
\end{aligned}
$$

The second equation above is the *changing base* representation.

Although all three notations are in some sense equivalent, they have their best uses depending on context. The base-independent or direct-vector notation eqn. (17.1) is the most direct, least ornate and shortest to write. The component or fixed base representation eqn. (17.3) is the best for computer calculations. And the changing-base notation eqn. (17.4) is the most explicit for pencil and paper work.

Angular velocity

Because all lines on object $\mathcal{B}$ rotate at the same rate (at a given instant) $\mathcal{B}$'s rotation rate is the single number we call $\omega_{\mathcal{B}}$ ('omega b'). As for the case of pure rotation, we define a vector angular velocity with magnitude $\omega_{\mathcal{B}}$ which is perpendicular to the xy plane:

$$\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \underbrace{\omega_{\mathcal{B}}}_{\dot{\theta}}\,\hat{\boldsymbol{k}}$$

where $\dot{\theta}$ is the rate of change of the angle of any line marked on object $\mathcal{B}$.

So long as you are careful to define angular velocity by the rotation of line segments and not by the motion of individual particles, the concept of angular velocity in general motion is defined exactly as for an object rotating about a fixed axis. The key kinematic fact is, in words:

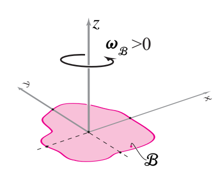

Figure 17.6: It is generally best to take positive ω to be counterclockwise when viewed from the positive z axis.

On a rigid object in 2D any given point is moving in circles about any other given point (relative to that point).

So you can think of the general planar motion of a rigid object as the general motion of a point plus uniform circular motion about that point. When a rigid object moves it always has an angular velocity (possibly zero). If we call the object $\mathcal{B}$ (script B for body), we then call the object's angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$. Again, it is generally best to use the CCW sign convention that when $\omega_{\mathcal{B}} > 0$ the object is rotating counterclockwise when viewed looking in from a point outside the object on the positive z axis (that is, 'looking down', see fig. 17.6).

The angular velocity vector $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$ of an object $\mathcal{B}$ describes its rate and direction of rotation. For planar motions $\vec{\boldsymbol{\omega}}_{\mathcal{B}} = \omega_{\mathcal{B}}\hat{\boldsymbol{k}}$.

Relative velocity of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$ the relative velocity $\vec{\boldsymbol{v}}_{\mathrm{B/A}}$ of the points ('the velocity of B relative to A') is given by the cross product of the angular velocity of the object with the relative position of the two points (fig. 17.7):

$$\vec{\boldsymbol{v}}_{B/A} \equiv \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}. \tag{17.5}$$

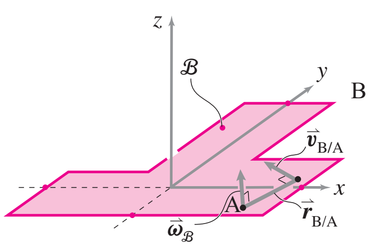

Figure 17.7: The relative velocity of points A and B is in the xy plane and perpendicular to the line segment AB.

That is, the relative velocity of two points on a rigid object is the same as would be predicted for one of the points if the other were stationary. The derivation of this formula is the same as for planar circular motion.

Note that we use three-dimensional cross products even though we are doing planar 2D kinematics. Generally the plane of motion is the xy plane and $\vec{\boldsymbol{\omega}}$ will be in the z direction. Because $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ must be perpendicular to $\vec{\boldsymbol{\omega}}$ it is perpendicular to the z axis. So the three-dimensional cross product $\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ gives a vector that is perpendicular to $\vec{\boldsymbol{r}}$ and is in the xy plane .

We can also represent the relative velocity in the changing base notation as

$$\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} &= \frac{d}{dt}\left(x'_{\mathrm{B/A}}\hat{\boldsymbol{\imath}}' + y'_{\mathrm{B/A}}\hat{\boldsymbol{\jmath}}'\right) \\
&= x'_{\mathrm{B/A}}\frac{d}{dt}\hat{\boldsymbol{\imath}}' + y'_{\mathrm{B/A}}\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'_{\mathrm{B/A}}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\imath}}'\right) + y'_{\mathrm{B/A}}\left(\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \hat{\boldsymbol{\jmath}}'\right).
\end{aligned}$$

That is, the relative velocity is the same as the relative position, but with the base vectors substituted with their rates of change.

Finally, we can use the fixed-base or component notation for position and differentiate the terms (see box 16.5 on page 773):

$$
\begin{aligned}
\left[\vec{\boldsymbol{v}}_{\mathrm{B/A}}\right]_{xy} &= \frac{d}{dt}\begin{bmatrix} x_{B/A} \\ y_{B/A} \end{bmatrix} \\
&= \frac{d}{dt}\left\{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix}\right\} \\
&= \begin{bmatrix} -\dot\theta\sin\theta & -\dot\theta\cos\theta \\ \dot\theta\cos\theta & -\dot\theta\sin\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix} \\
&= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} x^*_{\mathrm{B/A}} \\ y^*_{\mathrm{B/A}} \end{bmatrix}
\end{aligned}
$$

where $x^*_{B/A}$ and $y^*_{B/A}$ are the components of the position of B with respect to A in the reference configuration and hence do not change with time.

Absolute velocity of a point on a rigid object

If one knows the velocity of one point on a rigid object and one also knows the angular velocity of the object, then one can find the velocity of any other point. How? By addition. Say we know the velocity of point A, the angular velocity of the object, and the relative position of A and B, then

$$
\begin{aligned}
\vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + (\vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A) \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\
&= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \underbrace{\vec{\boldsymbol{r}}_{B/A}}_{\vec{\boldsymbol{r}}_B - \vec{\boldsymbol{r}}_A}.
\end{aligned}
\qquad (17.6)
$$

That is, the absolute velocity of the point B is the absolute velocity of the point A plus the velocity of the point B relative to the point A (fig. 17.8).

Equation (17.6) is a 2D vector equation, and thus equivalent to 2 scalar equations, in 4 vectors which have 7 independent components ($\vec{\boldsymbol{\omega}}$ only has one independent component). Generally one could use eqn. (17.6) to solve for any 2 components in terms of the other 5.

Instantaneous Center of Rotation. Interestingly, for any planar object with non-zero ω there is some point C 'on' the object that has exactly zero velocity (fig. 17.9). By 'on' the object we mean a point that moves with the object in a rigid way. Imagine a giant sheet of clear plexiglass glued to the object and extending way beyond the object. By 'on' the object we

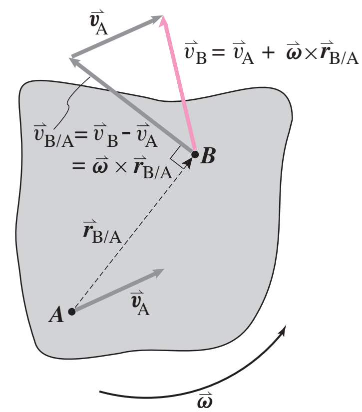

Figure 17.8: Absolute velocity of a point on a rigid object: $\vec{\boldsymbol{v}}_{\mathrm{B}} = \vec{\boldsymbol{v}}_{\mathrm{A}} + \omega \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}$

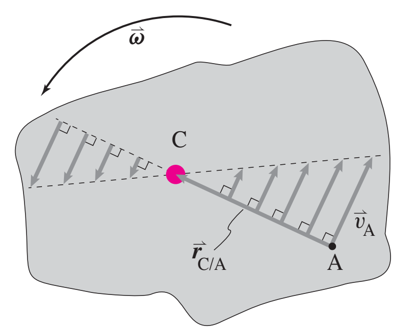

Figure 17.9: The center of rotation C is that point 'on' the object that has zero velocity. So long as $\omega \neq 0$ there always is such a point.

mean a point on the plexiglass. Why must there be such a point? Consider a point A on the object with some velocity. Relative to A the other points are going in circles. Each radial line extending from A has a relative velocity orthogonal to that line. Thus, considering all radial lines, every velocity direction is represented. And the relative velocities have magnitudes from 0 to infinity as points on a radial line are scanned from A outwards. Thus each possible relative velocity is had by some point 'on' the body. And one of those exactly cancels the velocity of A. Thus some point C has no velocity and all other points on the body rotate around it. This point is called the *center of rotation*. For other than purely circular motion, as an object moves, its center of rotation changes with time. See Sample 17.2 on page 850 for more details.

When trying to understand a mechanism it is often useful to think about the locus of locations of the center of rotation of each of the parts.

Angular acceleration

The angular acceleration $\vec{\boldsymbol{\alpha}}$ ('alpha') of a rigid object is the rate of change of angular velocity, $\vec{\boldsymbol{\alpha}} = \dot{\vec{\boldsymbol{\omega}}}$. The angular acceleration of a object $\mathcal{B}$ is $\vec{\boldsymbol{\alpha}}_{\mathcal{B}}$. As for angular velocity, in 2D angular acceleration is perpendicular to the plane of motion:

$$\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{\alpha}} = \alpha\hat{\boldsymbol{k}} = \dot{\omega}\hat{\boldsymbol{k}}.$$

In 2D the angular acceleration is only due to the speeding up or slowing down of the rotation rate; i.e., $\alpha = \dot{\omega} = \ddot{\theta}$.

Relative acceleration of two points on a rigid object

For any two points A and B glued to a rigid object $\mathcal{B}$, the acceleration of B relative to A is

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \frac{d}{dt}\vec{\boldsymbol{v}}_{\mathrm{B/A}} \\
&= \frac{d}{dt}\left\{\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}\right\} \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{v}}_{\mathrm{B/A}}), \\
&= \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}), \\
&= \alpha_{\mathcal{B}}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A} + (-\omega_{\mathcal{B}}^2\vec{\boldsymbol{r}}_{\mathrm{B/A}}),
\end{aligned} \tag{17.7}$$

This is the base-independent or direct-vector expression for relative acceleration. If point A has no acceleration, this formula for relative acceleration reduces to the total acceleration of a point undergoing circular motion from Chapter 15.

Equation (17.7) could also be derived, with some algebra, by taking two time derivatives of the relative position coordinate expression

$$\left[\vec{\boldsymbol{r}}_{\mathrm{B/A}}\right]_{xy} = \left[R(\theta)\right]\left[\vec{\boldsymbol{r}}_{\mathrm{B/A}}{}^{*}\right]_{x'y'}$$

or by taking two time derivatives of the changing base vector expression

$$\vec{\boldsymbol{r}}_{\mathrm{B/A}} = x'_{B/A}\hat{\boldsymbol{\imath}}' + y'_{B/A}\hat{\boldsymbol{\jmath}}'.$$

Absolute acceleration of a point on a rigid object

If one knows the acceleration of one point on a rigid body *and* the angular velocity and acceleration of the body, then one can find the acceleration of any other point. How? Add the absolute acceleration of one point to the acceleration of the second relative to the first.

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A + (\vec{\boldsymbol{a}}_B - \vec{\boldsymbol{a}}_A) = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} \\
&= \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}) + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A} \\
&= \vec{\boldsymbol{a}}_A - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{B/A} + \alpha_{\mathcal{B}} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A} \qquad (17.8)
\end{aligned}$$

This is the base-independent or direct-vector expression for acceleration. The fixed-base (component) and changing-base notations are somewhat more complex.

Equation 17.8 is often called ‘the three term acceleration formula’ because the acceleration of B is the sum of three terms. First is $\vec{\boldsymbol{a}}_A$, the acceleration of some point A on the object. Second is $\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times (\vec{\boldsymbol{\omega}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A})$, the centripetal acceleration of B going in circles relative to A. It is directed from B towards A. Third is $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}} \times \vec{\boldsymbol{r}}_{B/A}$, due to the change in the magnitude of the angular velocity. The third term is in the direction normal to the line from A to B (tangent to the circle of B going around A).

Example: Robot arm
Given the configuration shown in fig. 17.10 the acceleration of point B can be found by thinking of link AB as the object $\mathcal{B}$ in eqn. (17.8) and using what you know about circular motion to find the acceleration of A:

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \vec{\boldsymbol{a}}_A - \omega_{\mathcal{B}}^2 \vec{\boldsymbol{r}}_{B/A} + \alpha_{\mathcal{B}} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{B/A} \\
&= (-\omega_{\mathrm{OA}}^2 \ell \hat{\boldsymbol{\jmath}} - \dot{\omega}_{\mathrm{OA}} \ell \hat{\boldsymbol{\imath}}) - (\omega_{\mathrm{AB}}^2 \ell \hat{\boldsymbol{\imath}}) + (\dot{\omega}_{\mathrm{AB}} \hat{\boldsymbol{k}} \times (\ell \hat{\boldsymbol{\imath}})) \\
&= -(\dot{\omega}_{\mathrm{OA}} \ell + \omega_{\mathrm{AB}}^2 \ell)\,\hat{\boldsymbol{\imath}} + (-\omega_{\mathrm{OA}}^2 \ell + \dot{\omega}_{\mathrm{AB}} \ell)\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

[Note that $\omega_{\mathrm{AB}} \neq \dot{\theta}$ where θ is the angle between the links. Rather $\omega_{\mathrm{AB}} = \omega_{0\mathrm{A}} + \dot{\theta}$.]

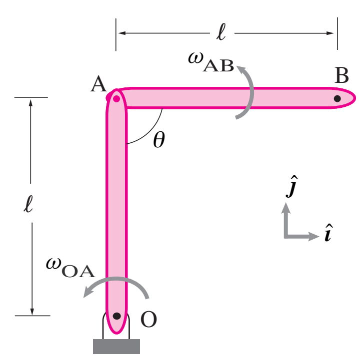

Figure 17.10: A two link robot arm.

Computer graphics

Given one point given by the xy pair $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ we can find out what happens to it by rotation $[R]$ as

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R] \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

For example the point $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$ gets changed by a 45 deg rotation to

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$
$$\approx \begin{bmatrix} .7 & -.7 \\ .7 & .7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}.$$

A translation is just a vector addition. For example the point $\begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix}$ gets translated a distance 2 in the y direction by the addition of $\begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ like this

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\text{translated}} = \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} = \begin{bmatrix} -1.4 \\ 1.4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1.4 \\ 3.4 \end{bmatrix}.$$

Putting these together the point $\begin{bmatrix} x_0 \\ y_0 \end{bmatrix}$ gets rotated and translated by first multiplying by the rotation matrix and then adding the translation:

$$\begin{bmatrix} x \\ y \end{bmatrix} = [R]\begin{bmatrix} x_0 \\ y_0 \end{bmatrix} + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & -.7 \\ .7 & .7 \end{bmatrix}\begin{bmatrix} 0 \\ 2 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix} \approx \begin{bmatrix} -1.4 \\ 3.4 \end{bmatrix}.$$

A collection of points all rotated the same amount and then all translated the same amount keep their relative distances.

A picture is a set of points on a plane. If all the points are rotated and translated the same amount the picture is rotated and translated. Thus a picture of a rigid object described by points is rigidly rotated and translated. On a computer line drawings are often represented as a connect-the-dots picture. The picture is represented by the x and y coordinates of the reference dots at the corners. These can be stored in an array with the first row being the x coordinates and the second row the y coordinates as explained on page 762. Each column of this matrix represents one point of the connect-the-dots picture. Thus a primitive picture of a house at the origin is given by the array

$$[P_0] \equiv [xy \text{ points originally}] = \begin{bmatrix} 0 & 2 & 2 & 1 & 0 & 0 \\ 0 & 0 & 2 & 3 & 2 & 1 \end{bmatrix}$$

with the lower left corner of the house at the origin.

To rotate this picture we rotate each of the columns of the matrix $P_0]$. But this is exactly what is accomplished by the matrix multiplication $[R][P_0]$. To translate the points you add the translation vector to each of the columns of the resulting matrix. Thus the whole picture rotated by 45° and translated up by 1 is given by

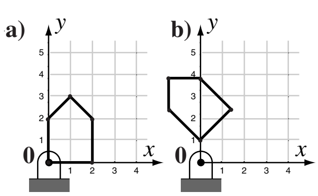

Figure 17.11: (a) A house drawn as 6 dots connected by line segments. The first and last point are the same. (b) The same house but rigidly rotated and translated.

$$[P_{\text{new}}] = [R][P_0] + \begin{bmatrix} x_t \\ y_t \end{bmatrix} \approx \begin{bmatrix} .7 & -.7 \\ .7 & .7 \end{bmatrix} [P_0] + \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

which gives a new array of points that, when connected give the picture shown. We have allowed the informal notation of adding a column matrix to a rectangular matrix, by which we mean adding to each column of the rectangular matrix.

To animate the motion of a house flying in, say, the Wizard of Oz, you would first define the house as the set of points $[P_0]$. Then define, maybe by means of numerical solution of differential equations, a set of rotations and translations. Then for each rotation and translation the picture of the house should be drawn, one after the other. The sequence of such pictures is an animation of a flying and spinning house.

Summary of the kinematics of one rigid object in general 2D motion

You can use the position of one reference point and the rotation of the object as simple kinematic measures of the entire motion of the object. If you know the position, velocity, and acceleration of one point on a rigid object (represented by 6 scalars, say) , and you know the angular rate and angular acceleration (3 scalars) then you can find the position, velocity and acceleration of any point on the object (also given its initial position). In 2D, just 9 numbers tell you the position, velocity, and acceleration of any of the billions of points whose initial positions you know[2].

[2] In 1D it takes just 3 numbers and in 3D just 18. The unusual pattern (3,9,18) comes from rotation being characterized by 0, 1, and 3 numbers in 1, 2, and 3 dimensions, respectively.