16.4 Dynamics of a rigid object in planar circular motion

We now want to find what we can about the forces on and motions of a single planar object that is hinged at one point. The most famous examples are pendula and gears. But many things move like pendula, like people or towers falling over or boats rocking in the water. And many machine parts besides gears rotate about a bearing including propellers, lawn mower blades, and printing-press rollers. None of these things are strictly planar objects, but, for many analyses, the planar approximation is more or less accurate ①.

①**Accuracy of planar analysis.** The planar analysis might be accurate for one of two different reasons: 1) the objects are nearly planar, or 2) the projection of the 3D equations to 2D gives the planar equations.

The basic method, as always, is to use a free-body diagram and kinematics results to evaluate terms in the linear momentum, angular momentum and energy balance equations. Then these equations are used to calculate unknown forces or to find differential equations of motion. The terms involving force and moment are found from a free-body diagram *exactly* as in statics. The terms involving motion, namely momentum, angular momentum and energy, are evaluated using the various tools developed earlier in this chapter.

Mechanics and the motion quantities

We can evaluate all the momentum and energy terms in the equations of motion (inside cover), namely: $\vec{\boldsymbol{L}}, \dot{\vec{\boldsymbol{L}}}, \vec{\boldsymbol{H}}_{/\mathrm{C}}, \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}, E_{\mathrm{K}}$ and $\dot{E}_{\mathrm{K}}$ (for any reference point C of our choosing) if we can calculate the velocity and acceleration of every point in the system. For circular motion of a rigid object about point C, we know from sec. 16.2 that the velocities and accelerations of a point at $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{/\mathrm{C}}$ are

$$
\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}, \\
\vec{\boldsymbol{a}} &= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}), \\
&= \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}} - \omega^2 \vec{\boldsymbol{r}}
\end{aligned}
$$

where $\vec{\boldsymbol{\omega}}$ is the angular velocity of the object relative to a fixed frame. This situation is a little more complex than the special case of straight-line motion in chapter 14, where all points in a system had the same acceleration as each other, but is still quite manageable. The most general case of 2D rigid-object circular motion is an arbitrarily shaped 2D rigid object with arbitrary ω and $\dot{\omega}$. The situation shown in fig. 16.47 shows the general case (but in that example exactly two forces are applied, as opposed to an arbitrary number).

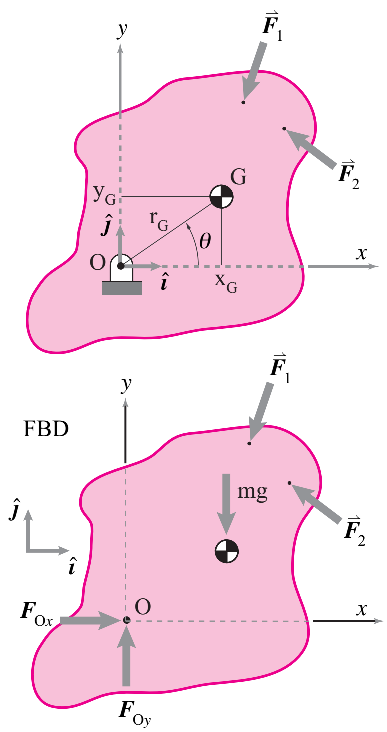

Figure 16.47: **a)** A rigid object rotates about 0. It has center of mass at G. Forces $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ are applied. **b)** The free-body diagram shows all the forces acting on the object, including the reaction force ($\vec{\boldsymbol{F}}_0 = F_{0x}\hat{\imath} + F_{0y}\hat{\jmath}$) at 0 and the gravity force mg at G.

Linear momentum: $\vec{L}$ and $\dot{\vec{L}}$

For any system in any motion we have the general result that

$$\vec{\boldsymbol{L}} = m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{\mathrm{cm}} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}} \vec{\boldsymbol{a}}_{\mathrm{cm}}.$$

For a rigid object, the center-of-mass is a particular point G that is fixed relative to the object. So the velocity and acceleration of that point can be expressed the same way as for any other point. So, for an object in planar rotational motion about 0

$$\vec{\boldsymbol{L}} = m_{\text{tot}}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{G}/0}$$

and

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}} \underbrace{\left(\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\text{G}/0} - \omega^2 \vec{\boldsymbol{r}}_{\text{G}/0}\right)}_{\vec{\boldsymbol{a}}_{\text{G}}}.$$

If the center-of-mass is at 0 the momentum and its rate of change are both zero. But if the center-of-mass is off the axis of rotation, there must be a net force on the object with a component parallel to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\omega \neq 0$) and also a component orthogonal to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\dot{\omega} \neq 0$). This net force might be applied at 0 or G or any other place(s) on the object.

Angular momentum: $\vec{\boldsymbol{H}}_{/\text{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$

The angular momentum itself is easy enough to calculate, using the short hand notation that $\vec{\boldsymbol{r}}$ is the position vector $\vec{\boldsymbol{r}}_{/0}$ of a point relative to hinge point O.

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\text{O}} &= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm & \text{(a)} \\
&= \int \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\, dm & \text{(b)} \\
&= \omega \hat{\boldsymbol{k}} \int r^2\, dm & \text{(c)} \\
&= \underbrace{I_{zz}^{O}}_{I_{zz}^{O} \text{ is the `polar' moment of inertia.}} \omega \hat{\boldsymbol{k}} \\
\Rightarrow \quad H_0 &= \omega I_{zz}^{O} & \text{(d)}
\end{aligned} \tag{16.35}$$

Here eqn. (16.35)c is a vector equation. But since both sides are in the $\hat{\boldsymbol{k}}$ direction we can dot both sides with $\hat{\boldsymbol{k}}$ to get the scalar moment equation eqn. (16.35)d, taking both M_{net} and ω as positive when counterclockwise.

The moment of inertia shortcut. Note that an integral over all the mass of $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$ has been reduced to a simple multiplication by using that

$$\int r^2\, dm = I_{zz}^{0}.$$

This shortcut is especially useful because $\int r^2\,dm$ does not change as the object rotates about 0.

However, the moment of inertia is never needed for problem solving. One can just evaluate, for example, $\vec{\boldsymbol{H}}_{/\mathrm{O}}$ by evaluating the integral $\int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\,dm$ for the specific problem at hand.

To get the all important term for angular momentum balance, namely $\dot{\vec{\boldsymbol{H}}}_{/0}$, for this system we can differentiate eqn. (16.35). We can also use the general expression for $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ to write the angular momentum balance equation as follows.

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \text{rate of change of angular momentum}_{/0} && \text{(a)}\\
&= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\,dm && \text{(b)}\\
&= \int \vec{\boldsymbol{r}} \times \left(\underbrace{-\omega^2\vec{\boldsymbol{r}} + \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}}_{\vec{\boldsymbol{a}}}\right) dm && \text{(c)}\\
&= \int \vec{\boldsymbol{r}} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}})\,dm \quad (\text{because } \vec{\boldsymbol{r}} \times \vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}) && \text{(d)}\\
&= \int \vec{\boldsymbol{r}} \times (\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}})\,dm && \text{(e)}\\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\,dm \;=\; \dot{\omega} I_{zz}^0 \hat{\boldsymbol{k}} && \text{(f)}\\
\Rightarrow\quad \dot{H}_{/0} &= \dot{\omega} \int r^2\,dm \;=\; \dot{\omega} I_{zz}^0 && \text{(g)}
\end{aligned}
\tag{16.36}
$$

We get from eqn. (16.36)e to eqn. (16.36)f by noting that $\vec{\boldsymbol{r}}$ is perpendicular to $\hat{\boldsymbol{k}}$. Thus, using the right hand rule twice we get $\vec{\boldsymbol{r}} \times (\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}) = r^2\hat{\boldsymbol{k}}$.

Eqn. 16.36f and eqn. (16.36)g are the vector and scalar versions for the rate of change of angular momentum with respect to point 0 for rotation of a planar object about 0. Repeating,

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\,dm \;=\; \dot{\omega} I_{zz}^0 \hat{\boldsymbol{k}}\\
\text{and}\qquad \dot{H}_{/0} &= \dot{\omega} \int r^2\,dm \;=\; \dot{\omega} I_{zz}^0.
\end{aligned}
\tag{16.37}
$$

Again, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ can always be evaluated either of two ways: 1) as a sum or integral of $\vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}$ over all the mass or 2) using the shortcut with I_{zz}^0.

Power and Energy

Let's assume that there are a set of point forces applied to an object②.

②Forces distributed on a curve, surface or throughout a volume could be treated similarly with sums replaced by line, surface, or volume integrals.

And, to be contrary, let's assume the mass is continuously distributed (the derivation for rigidly connected point masses would be similar). The power balance equation for one rotating rigid object is (discussed below):

$$
\begin{aligned}
\text{Net power in} &= \text{rate of change of kinetic energy} && \text{(a)}\\
P &= \dot{E}_\mathrm{K} && \text{(b)}\\
\sum_{\text{all applied forces}} \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i &= \frac{d}{dt}\int_{\text{all mass}} \frac{1}{2} v^2\, dm && \text{(c)}\\
\sum \vec{\boldsymbol{F}}_i \cdot (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_i) &= \frac{d}{dt}\int \frac{1}{2}(\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) \cdot (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\, dm && \text{(d)}\\
\sum \vec{\boldsymbol{\omega}} \cdot (\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i) &= \frac{d}{dt}\int \frac{1}{2}\omega^2 r^2\, dm && \text{(e)}\\
\vec{\boldsymbol{\omega}} \cdot \sum (\vec{\boldsymbol{r}}_i \times \vec{\boldsymbol{F}}_i) &= \frac{d}{dt}\left(\frac{1}{2}\omega^2\right)\int r^2\, dm && \text{(f)}\\
\vec{\boldsymbol{\omega}} \cdot \sum \vec{\boldsymbol{M}}_i &= \dot{\omega}\omega \int r^2\, dm && \text{(g)}\\
\vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_\text{tot} &= \dot{\vec{\boldsymbol{\omega}}} \cdot \underbrace{\left(\vec{\boldsymbol{\omega}} \int r^2\, dm\right)}_{\vec{\boldsymbol{H}}_{/0}} && \text{(h)}
\end{aligned}
\tag{16.38}
$$

When not directly labeled, positions and moments are assumed to be relative to the hinge at 0. Derivation 16.38 is two derivations in one. The left side about power and the right side about kinetic energy. Let's discuss one at a time.

Power. On the left side of eqn. (16.38) we note in (c) that the power of each force is the dot product of the force with the velocity of the point it touches. In (d) we use what we know about the velocities of points on rotating rigid bodies. In (e) we use the vector identity $\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{B}} \times \vec{\boldsymbol{C}} = \vec{\boldsymbol{B}} \cdot \vec{\boldsymbol{C}} \times \vec{\boldsymbol{A}}$ (see sec. 1.3, 82). In (f) we note that $\vec{\boldsymbol{\omega}}$ is common to all points so factors out of the sum. In (g) we note that $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{F}}_i$ is the moment of the force about pt O. And in (g) we sum the moments of the forces. So the power of a set of forces acting on a rigid object is the product of their net moment (about 0) and the object angular velocity,

$$P = \vec{\boldsymbol{\omega}} \cdot \vec{\boldsymbol{M}}_\text{tot}. \tag{16.39}$$

Kinetic energy. On the right side of eqn. (16.38) we note in (c) that the kinetic energy is the sum of the kinetic energy of the mass increments. In (d) we use what we know about the velocities of these bits of mass, given that they are on a common rotating object. In (e) we use that the magnitude of the cross product of orthogonal vectors is the product of the magnitudes ($|\vec{\boldsymbol{A}} \times \vec{\boldsymbol{B}}| = AB$) and that the dot product of a vector with itself is its magnitude squared ($\vec{\boldsymbol{A}} \cdot \vec{\boldsymbol{A}} = A^2$). In (f) we factor out ω^2 because it is common to all the mass increments and note that the remaining integral is constant in time for a rigid object. In (g) we carry out the derivative. In (h) we de-simplify the result from (g) in order to show a more general form that we will find later in 3D mechanics. Eqn. (h) follows from (g) because $\vec{\omega}$ is parallel to $\dot{\vec{\omega}}$ for 2D rotations.

Note that we started here with the basic power balance equation from the front inside cover. Instead, we could have derived power balance from our angular momentum balance expression (see box 16.8 on 5).

Example: Spinning disk

The round flat uniform disk in fig. 16.48 is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{cm} = \vec{\boldsymbol{0}},$$

Which we could also have calculated by evaluating the integral $\dot{\vec{\boldsymbol{L}}} \equiv \int \vec{\boldsymbol{a}}\, dm$ instead of using the general result that $\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{cm}$. From angular momentum balance we have:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{i/O} &= \dot{\vec{\boldsymbol{H}}}_{/O} \\
\Rightarrow \quad \vec{\boldsymbol{M}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\
&= \int_0^{R_0}\int_0^{2\pi} (R\hat{\boldsymbol{e}}_R) \times (-R\omega^2\hat{\boldsymbol{e}}_R)\underbrace{\overbrace{\left(\frac{m_{\text{tot}}}{\pi R_O^2}\right)}^{\rho}\overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int\int \vec{\boldsymbol{0}}\, d\theta\, dR \\
&= \vec{\boldsymbol{0}}.
\end{aligned}$$

So the net force and moment needed are $\vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{M}} = \vec{\boldsymbol{0}}$. Like a particle that moves at constant velocity with no force, a uniform disk rotates at constant rate with no torque (at least in 2D).

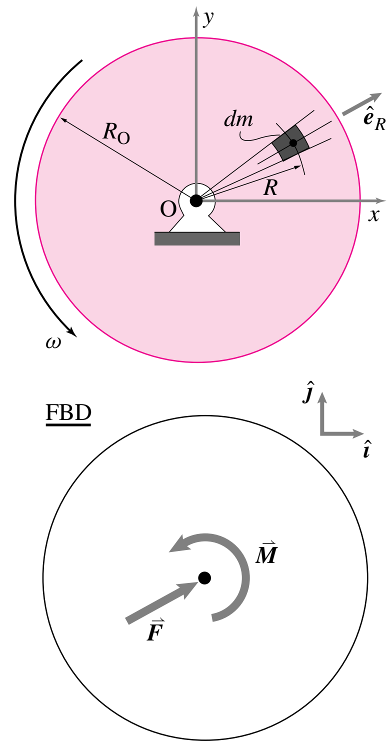

Figure 16.48: A uniform disk turned by a motor at a constant rate. The free-body diagram shows a force and moment equivalent to the force system that acts including gravity, bearing forces, etc. A bit of mass dm occupies the space between the radial lines at θ and $\theta + d\theta$ and between the circles at radii R and $R + dR$.

16.8 The relation between angular momentum balance and power balance

For this system, angular momentum balance can be derived from power balance and vice versa. Thus neither is essentially more fundamental than the other and both are reliable. First we can derive power balance from angular momentum balance as follows:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{\text{net}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \\
\vec{\omega} \cdot \vec{\boldsymbol{M}}_{\text{net}} &= \vec{\omega} \cdot \left(\dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm\right). \\
P &= \dot{E}_{\text{K}}
\end{aligned} \tag{16.40}$$

That is, when we dot both sides of the angular momentum equation with $\vec{\omega}$ we get on the left side a term which we recognize as the power of the forces and on the right side a term which is the rate of change of kinetic energy.

The opposite derivation starts with the power balance fig. 16.38(g)

$$\begin{aligned}
\vec{\omega} \cdot \sum \vec{\boldsymbol{M}}_i &= \dot{\omega}\omega \int r^2\, dm \quad \text{(g)} \\
\Rightarrow \quad \omega\left(\hat{\boldsymbol{k}} \cdot \sum \vec{\boldsymbol{M}}_i\right) &= \dot{\omega}\omega \int r^2\, dm \\
\Rightarrow \quad \left(\hat{\boldsymbol{k}} \cdot \sum \vec{\boldsymbol{M}}_i\right) &= \dot{\omega} \int r^2\, dm
\end{aligned} \tag{16.41}$$

and, assuming $\omega \neq 0$, divide by ω to get the angular momentum equation for planar rotational motion.

Using moment-of-inertia in 2-D circular motion dynamics

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from earlier chapters. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. As you recall from earlier chapters, the linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center-of-mass of the system of interest. One wishes for something so simple for the calculation of angular momentum.

It turns out that we are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification. For general motion of non-rigid bodies there is no simplification to be had. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to evaluate $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ but once this calculation is done, one need not work with the integrals while worrying about the dynamics. At this point we will assume that you are comfortable calculating and looking-up moments of inertia. We proceed to use it for the purposes of studying mechanics. For constant rate rotation, we can calculate the velocity and acceleration of various points on a rigid body using $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$. So we can calculate the various motion quantities of interest: linear momentum $\vec{\boldsymbol{L}}$, rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$, angular momentum $\vec{\boldsymbol{H}}$, rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$, and kinetic energy E_{K}.

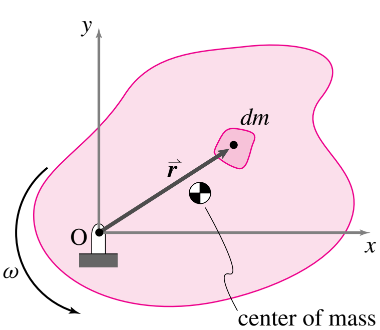

Figure 16.49: A two-dimensional body is rotating around the point O at constant rate ω. A differential bit of mass dm is shown. The center-of-mass is also shown.

Consider a two-dimensional rigid body like that shown in fig. 16.49. Now let us consider the various motion quantities in turn. First the linear momentum $\vec{\boldsymbol{L}}$. The linear momentum of any system in any motion is $\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate ω about point O (using $\vec{\boldsymbol{\omega}} = \omega \hat{\boldsymbol{k}}$):

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o} m_{tot}.$$

Similarly, for any system, we can calculate the rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$ as $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate,

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o}) m_{tot}.$$

That is, the linear momentum is correctly calculated for this special motion, as it is for all motions, by thinking of the body as a point mass at the center-of-mass.

Unlike the calculation of linear momentum, the angular momentum turns out to be something different than would be calculated by using a point mass at the center of mass. You can remember this important fact by looking at the case when the rotation is about the center-of-mass (point

O coincides with the center-of-mass). In this case one can intuitively see that the angular momentum of a rigid body *is not* zero even though the center-of-mass is not moving. Here's the calculation just to be sure:

$$
\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{O}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm && \text{(by definition of } \vec{\boldsymbol{H}}_{/\mathrm{O}}\text{)}\\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O})\, dm && \text{(using } \vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}\text{)}\\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \left[(\omega\hat{\boldsymbol{k}}) \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}})\right] dm && \text{(substituting } \vec{\boldsymbol{r}}_{/O} \text{ and } \vec{\boldsymbol{\omega}}\text{)}\\
&= \{\int (x^2_{/O} + y^2_{/O})\, dm\}\omega\hat{\boldsymbol{k}} && \text{(doing cross products)}\\
&= \{\int r^2_{/O}\, dm\}\omega\hat{\boldsymbol{k}}\\
&= \underbrace{I^O_{zz}}_{\boxed{I^O_{zz} \text{ is the 'polar' moment of inertia.}}}\ \omega\hat{\boldsymbol{k}}
\end{aligned}
$$

We have defined the 'polar' moment of inertia as $I^o_{zz} = \int r^2_{/o}\, dm$. In order to calculate I^o_{zz} for a specific body, assuming uniform mass distribution for example, one must convert the differential quantity of mass dm into a differential of geometric quantities. For a line or curve, $dm = \rho d\ell$; for a plate or surface, $dm = \rho dA$, and for a 3-D region, $dm = \rho dV$. $d\ell$, dA, and dV are differential line, area, and volume elements, respectively. In each case, ρ is the mass density per unit length, per unit area, or per unit volume, respectively. To avoid clutter, we do not define a different symbol for the density in each geometric case. The differential elements must be further defined depending on the coordinate systems chosen for the calculation; *e.g.,* for rectangular coordinates, $dA = dxdy$ or, for polar coordinates, $dA = rdrd\theta$.

Since $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{\omega}}$ always point in the $\hat{\boldsymbol{k}}$ direction for two dimensional problems people often just think of angular momentum as a scalar and write the equation above simply as '$H = I\omega$,' the form usually seen in elementary physics courses.

The derivation above has a feature that one might not notice at first sight. The quantity called I^{O}_{zz} does not depend on the rotation of the body. That is, the value of the integral does not change with time, so I^{O}_{zz} is a constant. So, perhaps unsurprisingly, a two-dimensional body spinning about the z-axis through O has constant angular momentum about O if it spins at a constant rate. (3)

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \vec{\boldsymbol{0}}.$$

Now, of course we could find this result about constant rate motion of 2-D bodies somewhat more cumbersomely by plugging in the general formula

(3) Note tha
about some
be constant
does not acc

for rate of change of angular momentum as follows:

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/O} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O}))\, dm \\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \left[\omega\hat{\boldsymbol{k}} \times (\omega\hat{\boldsymbol{k}} \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}))\right] dm \\
&= \vec{\boldsymbol{0}}.
\end{aligned}
\tag{16.42}
$$

Using moment of inertia about the center of mass. Often it is easier to think of the motion as composed of two parts, motion of the center of mass and motion relative to the center of mass, as explained in box 16.9 on page 9. Thus we have two terms for angular momentum $\vec{\boldsymbol{H}}_{/O}$ and its rate of change $\dot{\vec{\boldsymbol{H}}}_{/O}$:

$$\vec{\boldsymbol{H}}_{/O} = \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{v}}_{cm} + I_{zz}^{cm}\omega\hat{\boldsymbol{k}} \tag{16.43}$$

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{a}}_{cm} + I_{zz}^{cm}\alpha\hat{\boldsymbol{k}} \tag{16.44}$$

Kinetic energy. Finally, we can calculate the kinetic energy by adding up $\frac{1}{2}m_i v_i^2$ for all the bits of mass on a 2-D body spinning about the z-axis:

$$E_K = \int \frac{1}{2}v^2\, dm = \int \frac{1}{2}(\omega r)^2\, dm = \frac{1}{2}\omega^2 \int r^2\, dm = \frac{1}{2}I_{zz}^{o}\omega^2 \quad . \tag{16.45}$$

The kinetic energy can also be written as a sum of contributions of motion of the center of mass and motion relative to the center of mass,

$$E_K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{zz}^{cm}\omega^2 \quad . \tag{16.46}$$

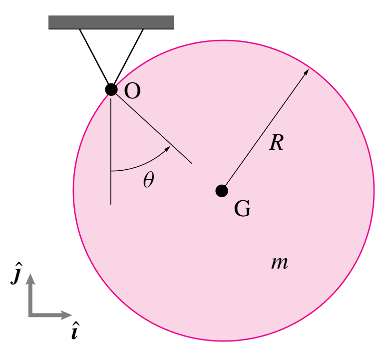

Figure 16.50

Example: Pendulum disk

For the disk shown in fig. 16.50, we can calculate the rate of change of angular momentum about point O as

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/O} &= \vec{\boldsymbol{r}}_{G/O} \times m\vec{\boldsymbol{a}}_{cm} + I_{zz}^{cm}\alpha\hat{\boldsymbol{k}} \\
&= R^2 m\ddot{\theta}\hat{\boldsymbol{k}} + I_{zz}^{cm}\ddot{\theta}\hat{\boldsymbol{k}} \\
&= (I_{zz}^{cm} + R^2 m)\ddot{\theta}\hat{\boldsymbol{k}}.
\end{aligned}
$$

Alternatively, we could calculate directly

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/O} &= I_{zz}^{O}\alpha\hat{\boldsymbol{k}} \\
&= \underbrace{(I_{zz}^{cm} + R^2 m)}_{\text{by the parallel axis theorem}}\ddot{\theta}\hat{\boldsymbol{k}}.
\end{aligned}
$$

Note that we are using the planarity of the objects and of their motion for our calculations④.

④**2D *vs* 3D.** Beware of falling into the common misconception that the formula $M = I\alpha$ applies in three dimensions by just thinking of the scalars as vectors and matrices. In 3D the formula

$$\dot{\vec{\boldsymbol{H}}}_{/O} = [\boldsymbol{I}^O] \cdot \underbrace{\dot{\vec{\boldsymbol{\omega}}}}_{\vec{\alpha}}$$

is only correct when $\vec{\boldsymbol{\omega}}$ is zero or when $\vec{\boldsymbol{\omega}}$ is an eigen vector of $[I_{/O}]$. That is, the vector equation

$$\sum \text{Moments about O} = [\boldsymbol{I}^O] \cdot \vec{\boldsymbol{\alpha}}$$

is generally wrong in 3D.

The equation for linear momentum balance is the same as always, we just need to calculate the acceleration of the center-of-mass of the spinning body.

$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm} = m_{tot}\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}_{cm/O}) + \dot{\vec{\omega}}\times\vec{r}_{cm/O}\right] \tag{16.47}$$

Finally, the kinetic energy for a planar rigid body rotating in the plane is:

$$E_K = \frac{1}{2}\vec{\omega}\cdot([I^{cm}]\cdot\vec{\omega}) + \frac{1}{2}m\,\underbrace{v_{cm}^2}_{\vec{v}_{cm} = \vec{\omega}\times\vec{r}_{cm/O}}.$$

16.9 Simplifying $\vec{H}_{/C}$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_{/C} = \sum\vec{r}_{i/C}\times m_i\vec{v}_i.$$

If we rewrite $\vec{v}_i$ as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned}
\vec{H}_{/C} &= \sum\left(\vec{r}_{cm} + \vec{r}_{i/cm}\right)\times\left[\vec{v}_{cm} + \vec{v}_{i/cm}\right]m_i. \\
&= \sum\vec{r}_{cm}\times\vec{v}_{cm}m_i + \sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}m_i \\
&\quad + \sum\vec{r}_{cm}\times\vec{v}_{i/cm}m_i + \sum\vec{r}_{i/cm}\times\vec{v}_{cm}m_i \\
&= \vec{r}_{cm}\times\vec{v}_{cm}m_{tot} + \sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}\,m_i \\
&\quad + \vec{r}_{cm}\times\underbrace{\left[\sum\vec{v}_{i/cm}\,m_i\right]}_{\vec{0}} + \underbrace{\left[\sum\vec{r}_{i/cm}\,m_i\right]}_{\vec{0}}\times\vec{v}_{cm}
\end{aligned}$$

So,

$$\vec{H}_{/C} = \underbrace{\vec{r}_{cm}\times\vec{v}_{cm}m_{tot}}_{\text{contribution of center of mass motion}} + \underbrace{\sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}\,m_i}_{\text{contribution of motion relative to center-of-mass}}.$$

The reason $\sum\vec{r}_{i/cm}\,m_i = \vec{0}$ is somewhat intuitive. It is what you would calculate if you were looking for the center-of-mass relative to the center of mass. More formally,

$$\begin{aligned}
\sum\vec{r}_{i/cm}\,m_i &= \sum(\vec{r}_i - \vec{r}_{cm})\,m_i \\
&= \underbrace{\sum\vec{r}_i\,m_i}_{m_{tot}\vec{r}_{cm}} - m_{tot}\vec{r}_{cm} \\
&= \vec{0}.
\end{aligned}$$

Similarly, $\sum\vec{v}_{i/cm}\,m_i = \vec{0}$ because it is what you would calculate if you were looking for the velocity of the center-of-mass relative to the center of mass.

The central result of this box is that

angular momentum of any system is that due to motion *of* the center-of-mass *plus* motion *relative to* the center-of-mass.