16.4 Dynamics of a rigid object in planar circular motion

We now want to find what we can about the forces on and motions of a single planar object that is hinged at one point. The most famous examples are pendula and gears. But many things move like pendula, like people or towers falling over or boats rocking in the water. And many machine parts besides gears rotate about a bearing including propellers, lawn mower blades, and printing-press rollers. None of these things are strictly planar objects, but, for many analyses, the planar approximation is more or less accurate ①.

①**Accuracy of planar analysis.** The planar analysis might be accurate for one of two different reasons: 1) the objects are nearly planar, or 2) the projection of the 3D equations to 2D gives the planar equations.

The basic method, as always, is to use a free-body diagram and kinematics results to evaluate terms in the linear momentum, angular momentum and energy balance equations. Then these equations are used to calculate unknown forces or to find differential equations of motion. The terms involving force and moment are found from a free-body diagram *exactly* as in statics. The terms involving motion, namely momentum, angular momentum and energy, are evaluated using the various tools developed earlier in this chapter.

Mechanics and the motion quantities

We can evaluate all the momentum and energy terms in the equations of motion (inside cover), namely: $\vec{\boldsymbol{L}}, \dot{\vec{\boldsymbol{L}}}, \vec{\boldsymbol{H}}_{/C}, \dot{\vec{\boldsymbol{H}}}_{/C}, E_K$ and $\dot{E}_K$ (for any reference point C of our choosing) if we can calculate the velocity and acceleration of every point in the system. For circular motion of a rigid object about point C, we know from sec. 16.2 that the velocities and accelerations of a point at $\vec{\boldsymbol{r}} = \vec{\boldsymbol{r}}_{/C}$ are

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\omega} \times \vec{\boldsymbol{r}}, \\
\vec{\boldsymbol{a}} &= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}), \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} - \omega^2 \vec{\boldsymbol{r}}
\end{aligned}$$

where $\vec{\omega}$ is the angular velocity of the object relative to a fixed frame. This situation is a little more complex than the special case of straight-line motion in chapter 14, where all points in a system had the same acceleration as each other, but is still quite manageable. The most general case of 2D rigid-object circular motion is an arbitrarily shaped 2D rigid object with arbitrary ω and $\dot{\omega}$. The situation shown in fig. 16.47 shows the general case (but in that example exactly two forces are applied, as opposed to an arbitrary number).

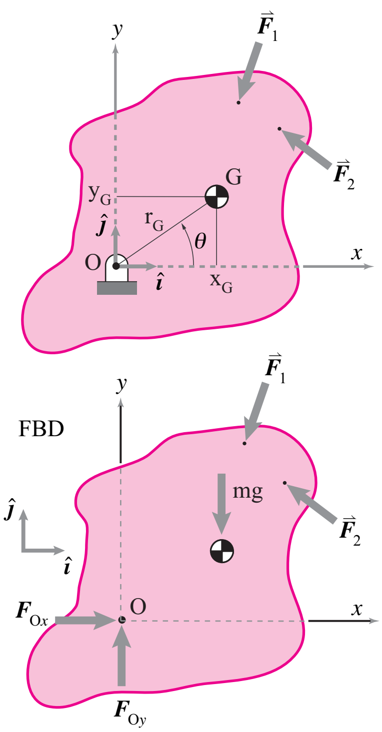

Figure 16.47: **a)** A rigid object rotates about 0. It has center of mass at G. Forces $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ are applied. **b)** The free-body diagram shows all the forces acting on the object, including the reaction force ($\vec{\boldsymbol{F}}_0 = F_{0x}\hat{\boldsymbol{\imath}} + F_{0y}\hat{\boldsymbol{\jmath}}$) at 0 and the gravity force mg at G.

Linear momentum: $\vec{L}$ and $\dot{\vec{L}}$

For any system in any motion we have the general result that

$$\vec{\boldsymbol{L}} = m_{\text{tot}} \vec{\boldsymbol{v}}_{\text{cm}} \quad \text{and} \quad \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}} \vec{\boldsymbol{a}}_{\text{cm}}.$$

For a rigid object, the center-of-mass is a particular point G that is fixed relative to the object. So the velocity and acceleration of that point can be expressed the same way as for any other point. So, for an object in planar rotational motion about 0

$$\vec{\boldsymbol{L}} = m_{\text{tot}}\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{G}/0}$$

and

$$\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\underbrace{\left(\dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\text{G}/0} - \omega^2\vec{\boldsymbol{r}}_{\text{G}/0}\right)}_{\vec{\boldsymbol{a}}_{\text{G}}}.$$

If the center-of-mass is at 0 the momentum and its rate of change are both zero. But if the center-of-mass is off the axis of rotation, there must be a net force on the object with a component parallel to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\omega \neq 0$) and also a component orthogonal to $\vec{\boldsymbol{r}}_{0/\text{G}}$ (if $\dot{\omega} \neq 0$). This net force might be applied at 0 or G or any other place(s) on the object.

Angular momentum: $\vec{\boldsymbol{H}}_{/\text{O}}$ and $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$

The angular momentum itself is easy enough to calculate, using the short hand notation that $\vec{\boldsymbol{r}}$ is the position vector $\vec{\boldsymbol{r}}_{/0}$ of a point relative to hinge point O.

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\text{O}} &= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm && \text{(a)} \\
&= \int \vec{\boldsymbol{r}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})\, dm && \text{(b)} \\
&= \omega\hat{\boldsymbol{k}} \int r^2\, dm && \text{(c)} \\
&= \underbrace{I_{zz}^{O}}_{I_{zz}^{O} \text{ is the 'polar' moment of inertia.}} \omega\hat{\boldsymbol{k}} && \\
\Rightarrow \quad H_0 &= \omega I_{zz}^{O} && \text{(d)}
\end{aligned} \tag{16.35}$$

Here eqn. (16.35)c is a vector equation. But since both sides are in the $\hat{\boldsymbol{k}}$ direction we can dot both sides with $\hat{\boldsymbol{k}}$ to get the scalar moment equation eqn. (16.35)d, taking both M_{net} and ω as positive when counterclockwise.

The moment of inertia shortcut. Note that an integral over all the mass of $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$ has been reduced to a simple multiplication by using that

$$\int r^2\, dm = I_{zz}^{0}.$$

This shortcut is especially useful because $\int r^2\, dm$ does not change as the object rotates about 0.

However, the moment of inertia is never needed for problem solving. One can just evaluate, for example, $\vec{\boldsymbol{H}}_{/\text{O}}$ by evaluating the integral $\int \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}\, dm$ for the specific problem at hand.

To get the all important term for angular momentum balance, namely $\dot{\vec{\boldsymbol{H}}}_{/0}$, for this system we can differentiate eqn. (16.35). We can also use the general expression for $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$ to write the angular momentum balance equation as follows.

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= \text{rate of change of angular momentum}_{/0} && \text{(a)}\\
&= \int_{\text{all mass}} \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}\, dm && \text{(b)}\\
&= \int \vec{\boldsymbol{r}} \times \left(\underbrace{-\omega^2 \vec{\boldsymbol{r}} + \dot{\omega} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}}_{\vec{\boldsymbol{a}}} \right) dm && \text{(c)}\\
&= \int \vec{\boldsymbol{r}} \times (\dot{\omega} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}})\, dm \quad (\text{because } \vec{\boldsymbol{r}} \times \vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}) && \text{(d)}\\
&= \int \vec{\boldsymbol{r}} \times (\dot{\omega} \hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}})\, dm && \text{(e)}\\
\dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= \dot{\omega} \hat{\boldsymbol{k}} \int r^2\, dm \;=\; \dot{\omega} I^0_{zz} \hat{\boldsymbol{k}} && \text{(f)}\\
\Rightarrow \quad \dot{H}_{/0} &= \dot{\omega} \int r^2\, dm \;=\; \dot{\omega} I^0_{zz} && \text{(g)}
\end{aligned}
\qquad (16.36)
$$

We get from eqn. (16.36)e to eqn. (16.36)f by noting that $\vec{\boldsymbol{r}}$ is perpendicular to $\hat{\boldsymbol{k}}$. Thus, using the right hand rule twice we get $\vec{\boldsymbol{r}} \times (\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}) = r^2 \hat{\boldsymbol{k}}$.

Eqn. 16.36f and eqn. (16.36)g are the vector and scalar versions for the rate of change of angular momentum with respect to point 0 for rotation of a planar object about 0. Repeating,

$$
\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/\text{O}} &= \dot{\omega} \hat{\boldsymbol{k}} \int r^2\, dm \;=\; \dot{\omega} I^0_{zz} \hat{\boldsymbol{k}}\\
\text{and} \qquad \dot{H}_{/0} &= \dot{\omega} \int r^2\, dm \;=\; \dot{\omega} I^0_{zz}.
\end{aligned}
\qquad (16.37)
$$

Again, $\dot{\vec{\boldsymbol{H}}}_{/\text{O}}$ can always be evaluated either of two ways: 1) as a sum or integral of $\vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}$ over all the mass or 2) using the shortcut with I^0_{zz}.

Power and Energy

Let's assume that there are a set of point forces applied to an object[2].

[2] Forces distributed on a curve, surface or throughout a volume could be treated similarly with sums replaced by line, surface, or volume integrals.

And, to be contrary, let's assume the mass is continuously distributed (the derivation for rigidly connected point masses would be similar). The power balance equation for one rotating rigid object is (discussed below):

$$
\begin{array}{rcll}
\text{Net power in} & = & \text{rate of change of kinetic energy} & \text{(a)} \\
P & = & \dot{E}_{\mathrm{K}} & \text{(b)} \\
\displaystyle\sum_{\text{all applied forces}} \vec{F}_i \cdot \vec{v}_i & = & \displaystyle\frac{d}{dt}\int_{\text{all mass}} \frac{1}{2} v^2\, dm & \text{(c)} \\
\sum \vec{F}_i \cdot (\vec{\omega} \times \vec{r}_i) & = & \displaystyle\frac{d}{dt}\int \frac{1}{2}(\vec{\omega}\times\vec{r})\cdot(\vec{\omega}\times\vec{r})\, dm & \text{(d)} \\
\sum \vec{\omega} \cdot (\vec{r}_i \times \vec{F}_i) & = & \displaystyle\frac{d}{dt}\int \frac{1}{2}\omega^2 r^2\, dm & \text{(e)} \\
\vec{\omega} \cdot \sum (\vec{r}_i \times \vec{F}_i) & = & \displaystyle\frac{d}{dt}\left(\frac{1}{2}\omega^2\right)\int r^2\, dm & \text{(f)} \\
\vec{\omega} \cdot \sum \vec{M}_i & = & \dot{\omega}\omega \displaystyle\int r^2\, dm & \text{(g)} \\
\vec{\omega} \cdot \vec{M}_{\mathrm{tot}} & = & \dot{\vec{\omega}} \cdot \underbrace{\left(\vec{\omega}\displaystyle\int r^2\, dm\right)}_{\vec{H}_{/0}} & \text{(h)}
\end{array}
\tag{16.38}
$$

When not directly labeled, positions and moments are assumed to be relative to the hinge at 0. Derivation 16.38 is two derivations in one. The left side about power and the right side about kinetic energy. Let's discuss one at a time.

Power. On the left side of eqn. (16.38) we note in (c) that the power of each force is the dot product of the force with the velocity of the point it touches. In (d) we use what we know about the velocities of points on rotating rigid bodies. In (e) we use the vector identity $\vec{A}\cdot\vec{B}\times\vec{C} = \vec{B}\cdot\vec{C}\times\vec{A}$ (see sec. 1.3, 82). In (f) we note that $\vec{\omega}$ is common to all points so factors out of the sum. In (g) we note that $\vec{r}\times\vec{F}_i$ is the moment of the force about pt O. And in (g) we sum the moments of the forces. So the power of a set of forces acting on a rigid object is the product of their net moment (about 0) and the object angular velocity,

$$P = \vec{\omega}\cdot\vec{M}_{\mathrm{tot}}. \tag{16.39}$$

Kinetic energy. On the right side of eqn. (16.38) we note in (c) that the kinetic energy is the sum of the kinetic energy of the mass increments. In (d) we use what we know about the velocities of these bits of mass, given that they are on a common rotating object. In (e) we use that the magnitude of the cross product of orthogonal vectors is the product of the magnitudes ($|\vec{A}\times\vec{B}| = AB$) and that the dot product of a vector with itself is its magnitude squared ($\vec{A}\cdot\vec{A} = A^2$). In (f) we factor out ω^2 because it is common to all the mass increments and note that the remaining integral is constant in time for a rigid object. In (g) we carry out the derivative. In (h) we de-simplify the result from (g) in order to show a more general form that we will find later in 3D mechanics. Eqn. (h) follows from (g) because $\vec{\omega}$ is parallel to $\dot{\vec{\omega}}$ for 2D rotations.

Note that we started here with the basic power balance equation from the front inside cover. Instead, we could have derived power balance from our angular momentum balance expression (see box 16.8 on 5).

Example: Spinning disk

The round flat uniform disk in fig. 16.48 is in the xy plane spinning at the constant rate $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ about its center. It has mass m_{tot} and radius R_0. What force is required to cause this motion? What torque? What power?

From linear momentum balance we have:

$$\sum \vec{\boldsymbol{F}}_i = \dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{cm} = \vec{\boldsymbol{0}},$$

Which we could also have calculated by evaluating the integral $\dot{\vec{\boldsymbol{L}}} \equiv \int \vec{\boldsymbol{a}}\, dm$ instead of using the general result that $\dot{\vec{\boldsymbol{L}}} = m_{\text{tot}}\vec{\boldsymbol{a}}_{cm}$. From angular momentum balance we have:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{i/O} &= \dot{\vec{\boldsymbol{H}}}_{/O} \\
\Rightarrow \quad \vec{\boldsymbol{M}} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{a}}\, dm \\
&= \int_0^{R_0}\int_0^{2\pi} (R\hat{\boldsymbol{e}}_R) \times (-R\omega^2\hat{\boldsymbol{e}}_R) \underbrace{\overbrace{\left(\frac{m_{\text{tot}}}{\pi R_O^2}\right)}^{\rho} \overbrace{R\, d\theta\, dR}^{dA}}_{dm} \\
&= \int\int \vec{\boldsymbol{0}}\, d\theta\, dR \\
&= \vec{\boldsymbol{0}}.
\end{aligned}$$

So the net force and moment needed are $\vec{\boldsymbol{F}} = \vec{\boldsymbol{0}}$ and $\vec{\boldsymbol{M}} = \vec{\boldsymbol{0}}$. Like a particle that moves at constant velocity with no force, a uniform disk rotates at constant rate with no torque (at least in 2D).

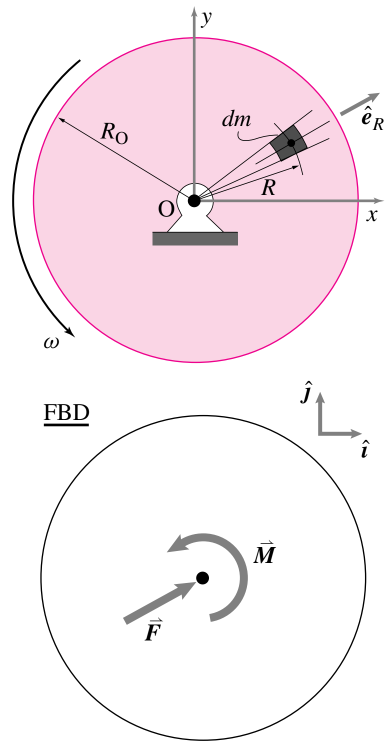

Figure 16.48: A uniform disk turned by a motor at a constant rate. The free-body diagram shows a force and moment equivalent to the force system that acts including gravity, bearing forces, etc. A bit of mass dm occupies the space between the radial lines at θ and $\theta + d\theta$ and between the circles at radii R and $R + dR$.

16.8 The relation between angular momentum balance and power balance

For this system, angular momentum balance can be derived from power balance and vice versa. Thus neither is essentially more fundamental than the other and both are reliable. First we can derive power balance from angular momentum balance as follows:

$$\begin{aligned}
\vec{\boldsymbol{M}}_{\text{net}} &= \dot{\omega}\hat{\boldsymbol{k}} \int r^2\, dm \\
\vec{\omega}\cdot\vec{\boldsymbol{M}}_{\text{net}} &= \vec{\omega}\cdot\left(\dot{\omega}\hat{\boldsymbol{k}}\int r^2\, dm\right). \\
P &= \dot{E}_{\text{K}}
\end{aligned} \qquad (16.40)$$

That is, when we dot both sides of the angular momentum equation with $\vec{\omega}$ we get on the left side a term which we recognize as the power of the forces and on the right side a term which is the rate of change of kinetic energy.

The opposite derivation starts with the power balance fig. 16.38(g)

$$\begin{aligned}
\vec{\omega}\cdot\sum\vec{\boldsymbol{M}}_i &= \dot{\omega}\omega\int r^2\, dm \quad \text{(g)} \\
\Rightarrow \quad \omega\left(\hat{\boldsymbol{k}}\cdot\sum\vec{\boldsymbol{M}}_i\right) &= \dot{\omega}\omega\int r^2\, dm \\
\Rightarrow \quad \left(\hat{\boldsymbol{k}}\cdot\sum\vec{\boldsymbol{M}}_i\right) &= \dot{\omega}\int r^2\, dm
\end{aligned} \qquad (16.41)$$

and, assuming $\omega \neq 0$, divide by ω to get the angular momentum equation for planar rotational motion.

Using moment-of-inertia in 2-D circular motion dynamics

Once one knows the velocity and acceleration of all points in a system one can find all of the motion quantities in the equations of motion by adding or integrating using the defining sums from earlier chapters. This addition or integration is an impractical task for many motions of many objects where the required sums may involve billions and billions of atoms or a difficult integral. As you recall from earlier chapters, the linear momentum and the rate of change of linear momentum can be calculated by just keeping track of the center-of-mass of the system of interest. One wishes for something so simple for the calculation of angular momentum.

It turns out that we are in luck if we are only interested in the two-dimensional motion of two-dimensional rigid bodies. The luck is not so great for 3-D rigid bodies but still there is some simplification. For general motion of non-rigid bodies there is no simplification to be had. The simplification is to use the moment of inertia for the bodies rather than evaluating the momenta and energy quantities as integrals and sums. Of course one may have to do a sum or integral to evaluate $I \equiv I_{zz}^{cm}$ or $[\boldsymbol{I}^{\mathrm{cm}}]$ but once this calculation is done, one need not work with the integrals while worrying about the dynamics. At this point we will assume that you are comfortable calculating and looking-up moments of inertia. We proceed to use it for the purposes of studying mechanics. For constant rate rotation, we can calculate the velocity and acceleration of various points on a rigid body using $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{a}} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}})$. So we can calculate the various motion quantities of interest: linear momentum $\vec{\boldsymbol{L}}$, rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$, angular momentum $\vec{\boldsymbol{H}}$, rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$, and kinetic energy E_{K}.

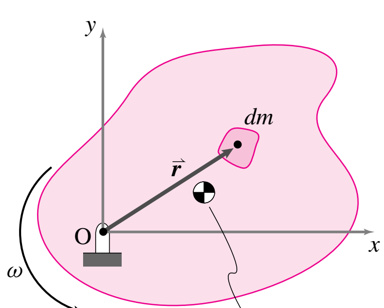

Figure 16.49: A two-dimensional body is rotating around the point O at constant rate ω. A differential bit of mass dm is shown. The center-of-mass is also shown.

Consider a two-dimensional rigid body like that shown in fig. 16.49. Now let us consider the various motion quantities in turn. First the linear momentum $\vec{\boldsymbol{L}}$. The linear momentum of any system in any motion is $\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate ω about point O (using $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$):

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o} m_{tot}.$$

Similarly, for any system, we can calculate the rate of change of linear momentum $\dot{\vec{\boldsymbol{L}}}$ as $\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot}$. So, for a rigid body spinning at constant rate,

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}_{cm} m_{tot} = \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{cm/o}) m_{tot}.$$

That is, the linear momentum is correctly calculated for this special motion, as it is for all motions, by thinking of the body as a point mass at the center-of-mass.

Unlike the calculation of linear momentum, the angular momentum turns out to be something different than would be calculated by using a point mass at the center of mass. You can remember this important fact by looking at the case when the rotation is about the center-of-mass (point

O coincides with the center-of-mass). In this case one can intuitively see that the angular momentum of a rigid body *is not* zero even though the center-of-mass is not moving. Here's the calculation just to be sure:

$$
\begin{aligned}
\vec{\boldsymbol{H}}_{/O} &= \int \vec{\boldsymbol{r}}_{/O} \times \vec{\boldsymbol{v}}\, dm && \text{(by definition of } \vec{\boldsymbol{H}}_{/O}) \\
&= \int \vec{\boldsymbol{r}}_{/O} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{/O})\, dm && \text{(using } \vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}) \\
&= \int (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}}) \times \left[(\omega\hat{\boldsymbol{k}}) \times (x_{/O}\hat{\boldsymbol{\imath}} + y_{/O}\hat{\boldsymbol{\jmath}})\right] dm && \text{(substituting } \vec{\boldsymbol{r}}_{/O} \text{ and } \vec{\boldsymbol{\omega}}) \\
&= \{\int (x^2_{/O} + y^2_{/O})\, dm\}\omega\hat{\boldsymbol{k}} && \text{(doing cross products)} \\
&= \{\int r^2_{/O}\, dm\}\omega\hat{\boldsymbol{k}} \\
&= \underbrace{I^O_{zz}}\ \omega\hat{\boldsymbol{k}}
\end{aligned}
$$

I^O_{zz} is the 'polar' moment of inertia.

We have defined the 'polar' moment of inertia as $I^o_{zz} = \int r^2_{/o}\, dm$. In order to calculate I^o_{zz} for a specific body, assuming uniform mass distribution for example, one must convert the differential quantity of mass dm into a differential of geometric quantities. For a line or curve, $dm = \rho d\ell$; for a plate or surface, $dm = \rho dA$, and for a 3-D region, $dm = \rho dV$. $d\ell$, dA, and dV are differential line, area, and volume elements, respectively. In each case, ρ is the mass density per unit length, per unit area, or per unit volume, respectively. To avoid clutter, we do not define a different symbol for the density in each geometric case. The differential elements must be further defined depending on the coordinate systems chosen for the calculation; *e.g.,* for rectangular coordinates, $dA = dxdy$ or, for polar coordinates, $dA = rdrd\theta$.

Since $\vec{\boldsymbol{H}}$ and $\vec{\boldsymbol{\omega}}$ always point in the $\hat{\boldsymbol{k}}$ direction for two dimensional problems people often just think of angular momentum as a scalar and write the equation above simply as '$H = I\omega$,' the form usually seen in elementary physics courses.

The derivation above has a feature that one might not notice at first sight. The quantity called I^O_{zz} does not depend on the rotation of the body. That is, the value of the integral does not change with time, so I^O_{zz} is a constant. So, perhaps unsurprisingly, a two-dimensional body spinning about the z-axis through O has constant angular momentum about O if it spins at a constant rate. ③

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{0}}.$$

Now, of course we could find this result about constant rate motion of 2-D bodies somewhat more cumbersomely by plugging in the general formula

③ Note tha
about some
be constant
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for rate of change of angular momentum as follows:

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \int \vec{r}_{/O} \times \vec{a}\, dm \\
&= \int \vec{r}_{/O} \times (\vec{\omega} \times (\vec{\omega} \times \vec{r}_{/O}))\, dm \\
&= \int (x_{/O}\hat{\imath} + y_{/O}\hat{\jmath}) \times \left[\omega\hat{k} \times (\omega\hat{k} \times (x_{/O}\hat{\imath} + y_{/O}\hat{\jmath}))\right] dm \\
&= \vec{0}.
\end{aligned} \quad (16.42)$$

Using moment of inertia about the center of mass. Often it is easier to think of the motion as composed of two parts, motion of the center of mass and motion relative to the center of mass, as explained in box 16.9 on page 9. Thus we have two terms for angular momentum $\vec{H}_{/O}$ and its rate of change $\dot{\vec{H}}_{/O}$:

$$\vec{H}_{/O} = \vec{r}_{G/O} \times m\vec{v}_{cm} + I_{zz}^{cm}\omega\hat{k} \quad (16.43)$$

$$\dot{\vec{H}}_{/O} = \vec{r}_{G/O} \times m\vec{a}_{cm} + I_{zz}^{cm}\alpha\hat{k} \quad (16.44)$$

Kinetic energy. Finally, we can calculate the kinetic energy by adding up $\frac{1}{2}m_i v_i^2$ for all the bits of mass on a 2-D body spinning about the z-axis:

$$E_K = \int \frac{1}{2}v^2\, dm = \int \frac{1}{2}(\omega r)^2\, dm = \frac{1}{2}\omega^2 \int r^2\, dm = \frac{1}{2}I_{zz}^{o}\omega^2 \quad . \quad (16.45)$$

The kinetic energy can also be written as a sum of contributions of motion of the center of mass and motion relative to the center of mass,

$$E_K = \frac{1}{2}mv_{cm}^2 + \frac{1}{2}I_{zz}^{cm}\omega^2 \quad . \quad (16.46)$$

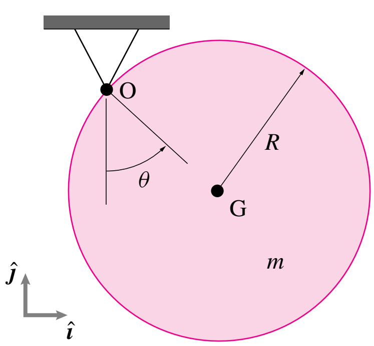


Figure 16.50

Example: Pendulum disk

For the disk shown in fig. 16.50, we can calculate the rate of change of angular momentum about point O as

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \vec{r}_{G/O} \times m\vec{a}_{cm} + I_{zz}^{cm}\alpha\hat{k} \\
&= R^2 m\ddot{\theta}\hat{k} + I_{zz}^{cm}\ddot{\theta}\hat{k} \\
&= (I_{zz}^{cm} + R^2 m)\ddot{\theta}\hat{k}.
\end{aligned}$$

Alternatively, we could calculate directly

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= I_{zz}^{O}\alpha\hat{k} \\
&= \underbrace{(I_{zz}^{cm} + R^2 m)}_{\text{by the parallel axis theorem}}\ddot{\theta}\hat{k}.
\end{aligned}$$

Note that we are using the planarity of the objects and of their motion for our calculations④.

④**2D *vs* 3D.** Beware of falling into the common misconception that the formula $M = I\alpha$ applies in three dimensions by just thinking of the scalars as vectors and matrices. In 3D the formula

$$\dot{\vec{H}}_{/O} = [I^O] \cdot \underbrace{\dot{\vec{\omega}}}_{\vec{\alpha}}$$

is only correct when $\vec{\omega}$ is zero or when $\vec{\omega}$ is an eigen vector of $[I_{/O}]$. That is, the vector equation

$$\sum \text{Moments about O} = [I^O] \cdot \vec{\alpha}$$

is generally wrong in 3D.

The equation for linear momentum balance is the same as always, we just need to calculate the acceleration of the center-of-mass of the spinning body.

$$\dot{\vec{L}} = m_{tot}\vec{a}_{cm} = m_{tot}\left[\vec{\omega}\times(\vec{\omega}\times\vec{r}_{cm/O}) + \dot{\vec{\omega}}\times\vec{r}_{cm/O}\right] \tag{16.47}$$

Finally, the kinetic energy for a planar rigid body rotating in the plane is:

$$E_K = \frac{1}{2}\vec{\omega}\cdot([I^{cm}]\cdot\vec{\omega}) + \frac{1}{2}m\underbrace{v_{cm}^2}_{\boxed{\vec{v}_{cm} = \vec{\omega}\times\vec{r}_{cm/O}}}.$$

16.9 Simplifying $\vec{H}_{/C}$ using the center of mass

The definition of angular momentum relative to a point C is

$$\vec{H}_{/C} = \sum \vec{r}_{i/C}\times m_i\vec{v}_i.$$

If we rewrite $\vec{v}_i$ as

$$\vec{v}_i = (\vec{v}_i - \vec{v}_{cm}) + \vec{v}_{cm} = \vec{v}_{i/cm} + \vec{v}_{cm}$$

and

$$\vec{r}_i = (\vec{r}_i - \vec{r}_{cm}) + \vec{r}_{cm} = \vec{r}_{i/cm} + \vec{r}_{cm}$$

then

$$\begin{aligned}
\vec{H}_{/C} &= \sum\left(\vec{r}_{cm} + \vec{r}_{i/cm}\right)\times\left[\vec{v}_{cm} + \vec{v}_{i/cm}\right]m_i.\\
&= \sum\vec{r}_{cm}\times\vec{v}_{cm}m_i + \sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}m_i\\
&\quad + \sum\vec{r}_{cm}\times\vec{v}_{i/cm}m_i + \sum\vec{r}_{i/cm}\times\vec{v}_{cm}m_i\\
&= \vec{r}_{cm}\times\vec{v}_{cm}m_{tot} + \sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}\,m_i\\
&\quad + \vec{r}_{cm}\times\underbrace{\left[\sum\vec{v}_{i/cm}\,\mathrm{m}_i\right]}_{\vec{0}} + \underbrace{\left[\sum\vec{r}_{i/cm}\,m_i\right]}_{\vec{0}}\times\vec{v}_{cm}
\end{aligned}$$

So,

$$\vec{H}_{/C} = \underbrace{\vec{r}_{cm}\times\vec{v}_{cm}m_{tot}}_{\text{contribution of center of mass motion}} + \underbrace{\sum\vec{r}_{i/cm}\times\vec{v}_{i/cm}\,m_i}_{\text{contribution of motion relative to center-of-mass}}.$$

The reason $\sum\vec{r}_{i/cm}\,m_i = \vec{0}$ is somewhat intuitive. It is what you would calculate if you were looking for the center-of-mass relative to the center of mass. More formally,

$$\begin{aligned}
\sum\vec{r}_{i/cm}\,m_i &= \sum(\vec{r}_i - \vec{r}_{cm})\,m_i\\
&= \underbrace{\sum\vec{r}_i\,m_i}_{m_{tot}\vec{r}_{cm}} - m_{tot}\vec{r}_{cm}\\
&= \vec{0}.
\end{aligned}$$

Similarly, $\sum\vec{v}_{i/cm}\,m_i = \vec{0}$ because it is what you would calculate if you were looking for the velocity of the center-of-mass relative to the center of mass.

The central result of this box is that

> angular momentum of any system is that due to motion *of* the center-of-mass *plus* motion *relative to* the center-of-mass.

SAMPLE 16.13 **A rod in a constant rate circular motion:** A uniform rod of mass m and length ℓ is connected to a motor at end O. A ball of mass m is attached to the rod at end B. The motor turns the rod in the counterclockwise direction at a constant angular speed ω. There is gravity pointing in the $-\hat{\jmath}$ direction. Find the torque applied by the motor (i) at the instant shown and (ii) when $\theta = 0°,\ 90°,\ 180°$. How does the torque change if the angular speed is doubled?

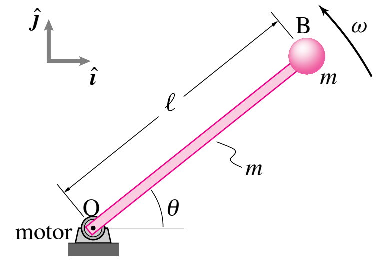


Figure 16.51: A rod goes in circles at a constant rate.

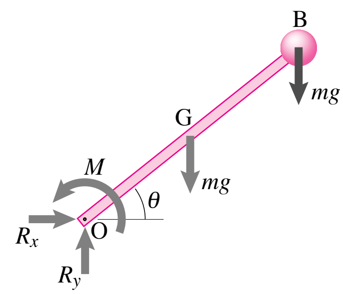
Solution The FBD of the rod and ball system is shown in Fig. 16.52(a). Since the system is undergoing circular motion at a constant speed, the acceleration of the ball as well as every point on the rod is just radial (pointing towards the center of rotation O) and is given by $\vec{a} = -\omega^2 r\hat{\lambda}$ where r is the radial distance from the center O to the point of interest and $\hat{\lambda}$ is a unit vector along OB pointing away from O (Fig. 16.52(b)).

Angular Momentum Balance about point O gives

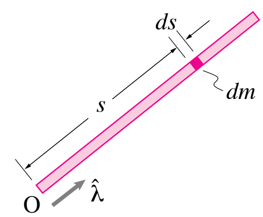
$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

$$\begin{aligned}
\sum \vec{M}_O &= \vec{r}_{G/O} \times (-mg\hat{\jmath}) + \vec{r}_{B/O} \times (-mg\hat{\jmath}) + M\hat{k} \\
&= -\frac{\ell}{2}\cos\theta mg\hat{k} - \ell\cos\theta mg\hat{k} + M\hat{k} \\
&= (M - \frac{3\ell}{2}mg\cos\theta)\hat{k} \qquad (16.48)
\end{aligned}$$

$$\begin{aligned}
\dot{\vec{H}}_{/O} &= \overbrace{\vec{r}_{B/O} \times m\vec{a}_B}^{(\dot{\vec{H}}_{/O})_{\text{ball}}} + \overbrace{\int_m \vec{r}_{dm/O} \times \vec{a}_{dm}\, dm}^{(\dot{\vec{H}}_{/O})_{\text{rod}}} \\
&= \ell\hat{\lambda} \times (-m\omega^2\ell\hat{\lambda}) + \int_m \overbrace{s\hat{\lambda}}^{\vec{r}_{dm/O}} \times \overbrace{(-\omega^2 s\hat{\lambda})}^{\vec{a}_{dm}}\, dm \\
&= \vec{0} \qquad (\text{since } \hat{\lambda} \times \hat{\lambda} = \vec{0}) \qquad (16.49)
\end{aligned}$$



(a) FBD of rod+ball system



(b) Calculation of $\dot{\vec{H}}_O$ of the rod

Figure 16.52: A rod goes in circles at a constant rate.

(i) Equating (16.48) and (16.49) we get

$$M = \frac{3}{2}mg\ell\cos\theta.$$

$$\boxed{M = \frac{3}{2}mg\ell\cos\theta}$$

(ii) Substituting the given values of θ in the above expression we get

$$M(\theta = 0°) = \frac{3}{2}mg\ell, \quad M(\theta = 90°) = 0 \quad M(\theta = 180°) = -\frac{3}{2}mg\ell$$

$$\boxed{M(0°) = \frac{3}{2}mg\ell, \quad M(90°) = 0 \quad M(180°) = -\frac{3}{2}mg\ell}$$

1

It is clear from the expression of the torque that it does not depend on the value of the angular speed ω! Therefore, the torque will not change if the speed is doubled. In fact, as long as the speed remains constant at any value, the only torque required to maintain the motion is the torque to counteract the moments due to gravity at O.

1 The values obtained above make sense (at least qualitatively). To make the rod and the ball go up from the 0° position, the motor has to apply some torque in the counterclockwise direction. In the 90° position no torque is required for the *dynamic balance*. In the 180° position the system is accelerating downwards under gravity; therefore, the motor has to apply a clockwise torque to make the system maintain a uniform speed.

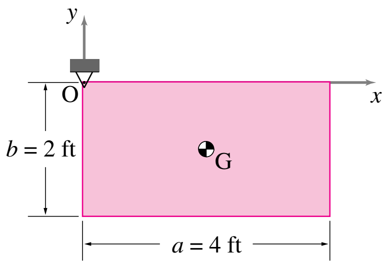


Figure 16.53: A rectangular plate is released from rest from the position shown.

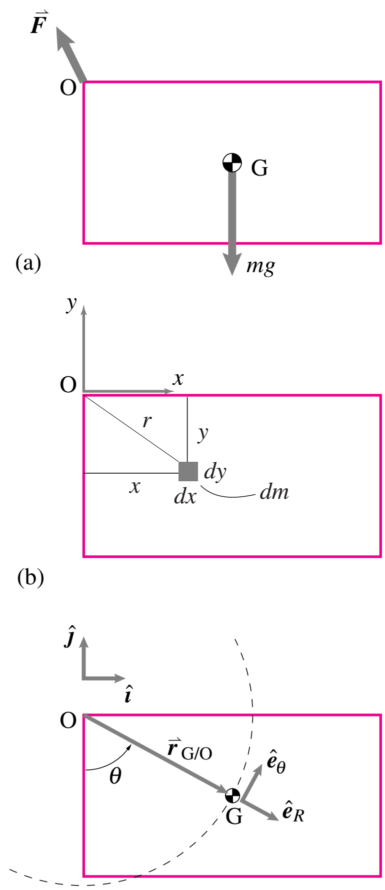


Figure 16.54: (a) The free-body diagram of the plate. (b) Computation of the integral in $\dot{\vec{H}}_{/O} = \dot{\omega}\hat{\boldsymbol{k}}\int_m r^2 dm$. (c) The geometry of motion. From the given dimensions, $\hat{\boldsymbol{e}}_R = \frac{a\hat{\imath}-b\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, $\hat{\boldsymbol{e}}_\theta = \frac{b\hat{\imath}+a\hat{\jmath}}{\sqrt{(a^2+b^2)}}$, and $r_{G/O} = \frac{\sqrt{a^2+b^2}}{2}$.

SAMPLE 16.14 At the onset of circular motion: A $2' \times 4'$ rectangular plate of mass 20 lbm is pivoted at one of its corners as shown in the figure. The plate is released from rest in the position shown. Find the force on the support immediately after release.

Solution The free-body diagram of the plate is shown in Fig. 16.54. The force $\vec{F}$ applied on the plate by the support is unknown.

The linear momentum balance for the plate gives

$$\begin{aligned} \sum \vec{F} &= m\vec{a}_G \\ \vec{F} - mg\hat{\jmath} &= m(\ddot{\theta}\, r_{G/O}\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2 r_{G/O}\hat{\boldsymbol{e}}_R) \\ &= m\,\ddot{\theta}\, r_{G/O}\hat{\boldsymbol{e}}_\theta \quad (\text{since } \dot{\theta} = 0 \text{ at } t = 0). \end{aligned} \tag{16.50}$$

Thus to find $\vec{F}$ we need to find $\ddot{\theta}$.

The angular momentum balance for the plate about the fixed support point O gives

$$\vec{M}_O = \dot{\vec{H}}_O$$

where

$$\begin{aligned} \vec{M}_O &= \vec{r}_{G/O} \times mg(-\hat{\jmath}) \\ &= \underbrace{(\frac{a}{2}\hat{\imath} - \frac{b}{2}\hat{\jmath})}_{\vec{r}_{G/O}} \times mg(-\hat{\jmath}) = -mg\frac{a}{2}\hat{\boldsymbol{k}}, \end{aligned}$$

and

$$\begin{aligned} \dot{\vec{H}}_{/O} &= \dot{\omega}\hat{\boldsymbol{k}} \int_m r^2 dm = \ddot{\theta}\hat{\boldsymbol{k}} \int_0^b \int_0^a \overbrace{(x^2+y^2)}^{r^2} \overbrace{\frac{m}{ab} dxdy}^{dm} \\ &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{\boldsymbol{k}}. \end{aligned}$$

Thus,

$$\begin{aligned} -mg\frac{a}{2}\hat{\boldsymbol{k}} &= \frac{m(a^2+b^2)}{3}\ddot{\theta}\hat{\boldsymbol{k}} \\ \Rightarrow \quad \ddot{\theta} &= -\frac{3ga}{2(a^2+b^2)} \\ &= -\frac{3 \cdot 32.2\,\text{ft/s}^2 \cdot 4\,\text{ft}}{2(16+4)\,\text{ft}^2} = -9.66\,\text{rad/s}^2. \end{aligned}$$

From eqn. (16.50), the support force is now readily calculated:

$$\begin{aligned} \vec{F} &= mg\hat{\jmath} + m\ddot{\theta}\, r_{G/O}\hat{\boldsymbol{e}}_\theta \\ &= mg\hat{\jmath} + m\ddot{\theta}\frac{\sqrt{a^2+b^2}}{2}\frac{b\hat{\imath}+a\hat{\jmath}}{\sqrt{(a^2+b^2)}} \\ &= \frac{1}{2}m\ddot{\theta}b\hat{\imath} + (mg + \frac{1}{2}m\ddot{\theta}a)\hat{\jmath} \end{aligned}$$

Using the given numerical values of m, a, and b, $\ddot{\theta} = -9.66\,\text{rad/s}^2$, and $g = 32.2\,\text{ft/s}^2$, we get

$$\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\text{lbf}.$$

$\vec{F} = (-6\hat{\imath} + 8\hat{\jmath})\,\text{lbf}$

SAMPLE 16.15 **A compound gear train.** When the gear of an input shaft, often called the *driver* or the *pinion*, is directly meshed in with the gear of an output shaft, the motion of the output shaft is opposite to that of the input shaft. To get the output motion in the same direction as that of the input motion, an *idler* gear is used. If the idler shaft has more than one gear in mesh, then the gear train is called a *compound gear train*.

In the gear train shown in Fig. 16.55, the input shaft is rotating at 2000 rpm and the input torque is 200 N-m. The efficiency (defined as the ratio of output power to input power) of the train is 0.96 and the various radii of the gears are: $R_A = 5\,\mathrm{cm}$, $R_B = 8\,\mathrm{cm}$, $R_C = 4\,\mathrm{cm}$, and $R_D = 10\,\mathrm{cm}$. Find

1. the input power P_{in} and the output power P_{out},
2. the output speed ω_{out}, and
3. the output torque.

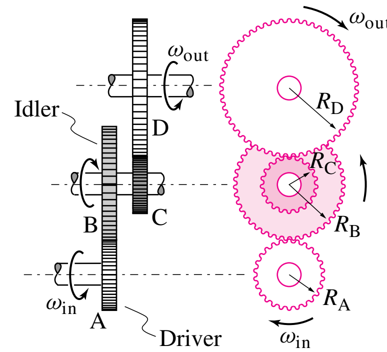


Figure 16.55: A compound gear train.

Solution

1. The power:

$$
\begin{aligned}
P_{\mathrm{in}} &= M_{\mathrm{in}}\omega_{\mathrm{in}} = 200\,\mathrm{N{\cdot}m}\cdot 2000\,\mathrm{rpm}\\
&= 400000\,\mathrm{N{\cdot}m}\cdot\frac{\cancel{\mathrm{rev}}}{\cancel{\mathrm{m}}\mathrm{in}}\cdot\frac{2\pi}{1\,\cancel{\mathrm{rev}}}\cdot\frac{1\,\cancel{\mathrm{m}}\mathrm{in}}{60\,\mathrm{s}}\\
&= 41887.9\,\mathrm{N\,m/s}\approx 42\,\mathrm{kW}.\\
\Rightarrow\quad p_{\mathrm{out}} &= \text{efficiency}\cdot P_{\mathrm{in}} = 0.96\cdot 42\,\mathrm{kW}\approx 40\,\mathrm{kW}
\end{aligned}
$$

$$\boxed{P_{\mathrm{in}} = 42\,\mathrm{kW},\quad p_{\mathrm{out}} = 40\,\mathrm{kW}}$$

2. The angular speed of meshing gears can be easily calculated by realizing that the linear speed of the point of contact has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus,

$$
\begin{aligned}
v_P &= \omega_{\mathrm{in}} R_A = \omega_B R_B\\
\Rightarrow\quad \omega_B &= \omega_{\mathrm{in}}\cdot\frac{R_A}{R_B}\\
\text{and}\quad v_R &= \omega_C R_C = \omega_{\mathrm{out}} R_D\\
\Rightarrow\quad \omega_{\mathrm{out}} &= \omega_C\cdot\frac{R_C}{R_D}\\
\text{But}\quad \omega_C &= \omega_B\\
\Rightarrow\quad \omega_{\mathrm{out}} &= \omega_{\mathrm{in}}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D}\\
&= 2000\,\mathrm{rpm}\cdot\frac{5}{8}\cdot\frac{4}{10} = 500\,\mathrm{rpm}.
\end{aligned}
$$

$$\boxed{\omega_{\mathrm{out}} = 500\,\mathrm{rpm}}$$

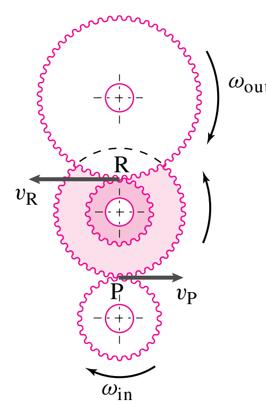


Figure 16.56:

3. The output torque,

$$
\begin{aligned}
M_{\mathrm{out}} = \frac{P_{\mathrm{out}}}{\omega_{\mathrm{out}}} &= \frac{40\,\mathrm{kW}}{500\,\mathrm{rpm}} = \frac{40}{500}\cdot 1000\frac{\mathrm{N{\cdot}m}}{\cancel{\mathrm{s}}}\cdot\frac{\cancel{\mathrm{m}}\mathrm{in}}{\cancel{\mathrm{rev}}}\cdot\frac{1\,\cancel{\mathrm{rev}}}{2\pi}\cdot\frac{60\,\cancel{\mathrm{s}}}{1\,\cancel{\mathrm{m}}\mathrm{in}}\\
&= 764\,\mathrm{N{\cdot}m}.
\end{aligned}
$$

$$\boxed{M_{\mathrm{out}} = 764\,\mathrm{N{\cdot}m}}$$

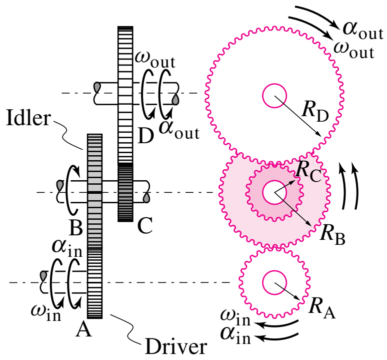


Figure 16.57: An accelerating compound gear train.

SAMPLE 16.16 An accelerating gear train. In the gear train shown in Fig. 16.57, the torque at the input shaft is $M_{in} = 200\,\text{N}{\cdot}\text{m}$ and the angular acceleration is $\alpha_{in} = 50\,\text{rad/s}^2$. The radii of the various gears are: $R_A = 5\,\text{cm}$, $R_B = 8\,\text{cm}$, $R_C = 4\,\text{cm}$, and $R_D = 10\,\text{cm}$ and the moments of inertia about the shaft axis passing through their respective centers are: $I_A = 0.1\,\text{kg m}^2$, $I_{BC} = 5I_A$, $I_D = 4I_A$. Find the output torque M_{out} of the gear train.

Solution Since the difference between the input power and the output power is used in accelerating the gears, we may write

$$P_{in} - P_{out} = \dot{E}_K$$

Let M_{out} be the output torque of the gear train. Then,

$$P_{in} - P_{out} = M_{in}\,\omega_{in} - M_{out}\,\omega_{out}. \tag{16.51}$$

Now,

$$\begin{aligned}
\dot{E}_K &= \frac{d}{dt}(E_K) && (16.52)\\
&= \frac{d}{dt}\left(\frac{1}{2}I_A\,\omega_{in}^2 + \frac{1}{2}I_{BC}\,\omega_{BC}^2 + \frac{1}{2}I_D\,\omega_{out}^2\right)\\
&= I_A\,\omega_{in}\,\dot{\omega}_{in} + I_{BC}\,\omega_{BC}\,\dot{\omega}_{BC} + I_D\,\omega_{out}\,\dot{\omega}_{out}\\
&= I_A\,\omega_{in}\,\alpha_{in} + 5I_A\,\omega_{BC}\,\alpha_{BC} + 4I_A\,\omega_{out}\,\alpha_{out}. && (16.53)
\end{aligned}$$

The different ω's and the α's can be related by realizing that the linear speed or the tangential acceleration of the point of contact between any two meshing gears has to be the same irrespective of which gear's speed and geometry is used to calculate it. Thus, using the linear speed and tangential acceleration calculations for points P and R in Fig. 16.58, we can find ω_B, α_B and ω_{out}, α_{out}. Considering the linear speed of point P, we find

$$\begin{aligned}
v_P &= \omega_{in}\,R_A = \omega_B\,R_B\\
\Rightarrow \quad \omega_B &= \omega_{in}\cdot\frac{R_A}{R_B},
\end{aligned}$$

and considering the tangential acceleration of point P, $(a_P)_\theta$, we find

$$\begin{aligned}
(a_P)_\theta &= \alpha_{in}\,R_A = \alpha_B\,R_B\\
\Rightarrow \quad \alpha_B &= \alpha_{in}\cdot\frac{R_A}{R_B}.
\end{aligned}$$

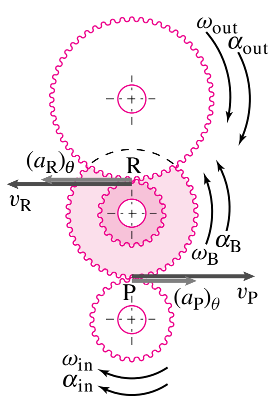


Figure 16.58: The velocity or acceleration of the point of contact between two meshing gears has to be the same irrespective of which meshing gear's geometry and motion is used to compute them.

Similarly,

$$\begin{aligned}
v_R &= \omega_C\,R_C = \omega_{out}\,R_D\\
\Rightarrow \quad \omega_{out} &= \omega_C\cdot\frac{R_C}{R_D},
\end{aligned}$$

and

$$\begin{aligned}
(a_R)_\theta &= \alpha_C\,R_C = \alpha_{out}\,R_D\\
\Rightarrow \quad \alpha_{out} &= \alpha_C\cdot\frac{R_C}{R_D}.
\end{aligned}$$

But

$$\begin{aligned}
\omega_C &= \omega_B = \omega_{BC}\\
\Rightarrow \quad \omega_{out} &= \omega_{in}\cdot\frac{R_A}{R_B}\cdot\frac{R_C}{R_D}
\end{aligned}$$

and

$$\begin{aligned} \alpha_C &= \alpha_B = \alpha_{BC} \\ \Rightarrow \quad \alpha_{\text{out}} &= \alpha_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}. \end{aligned}$$

Substituting these expressions for ω_{out}, α_{out}, ω_{BC} and α_{BC} in equations (16.51) and (16.53), we get

$$\begin{aligned} P_{\text{in}} - P_{\text{out}} &= M_{\text{in}}\,\omega_{\text{in}} - M_{\text{out}}\omega_{\text{in}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D} \\ &= \omega_{\text{in}}\left(M_{\text{in}} - M_{\text{out}} \cdot \frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right). \\ \dot{E}_{\text{K}} &= I_A\left[\omega_{\text{in}}\alpha_{\text{in}} + 5\omega_{\text{in}}\alpha_{\text{in}}\left(\frac{R_A}{R_B}\right)^2 + 4\omega_{\text{in}}\alpha_{\text{in}}\left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right)^2\right] \\ &= I_A\omega_{\text{in}}\left[\alpha_{\text{in}} + 5\alpha_{\text{in}}\left(\frac{R_A}{R_B}\right)^2 + 4\alpha_{\text{in}}\left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right)^2\right]. \end{aligned}$$

Now equating the two quantities, $P_{\text{in}} - P_{\text{out}}$ and $\dot{E}_{\text{K}}$, and canceling ω_{in} from both sides, we obtain

$$\begin{aligned} M_{\text{out}}\frac{R_A}{R_B} \cdot \frac{R_C}{R_D} &= M_{\text{in}} - I_A\alpha_{\text{in}}\left[1 + 5\left(\frac{R_A}{R_B}\right)^2 + 4\left(\frac{R_A}{R_B} \cdot \frac{R_C}{R_D}\right)^2\right] \\ M_{\text{out}}\frac{5}{8} \cdot \frac{4}{10} &= 200\,\text{N}\cdot\text{m} - 5\,\text{kg m}^2 \cdot \text{rad/s}^2\left[1 + 5\left(\frac{5}{8}\right)^2 + 4\left(\frac{5}{8} \cdot \frac{4}{10}\right)^2\right] \\ M_{\text{out}} &= 735.94\,\text{N}\cdot\text{m} \\ &\approx 736\,\text{N}\cdot\text{m}. \end{aligned}$$

$$\boxed{M_{\text{out}} = 736\,\text{N}\cdot\text{m}}$$

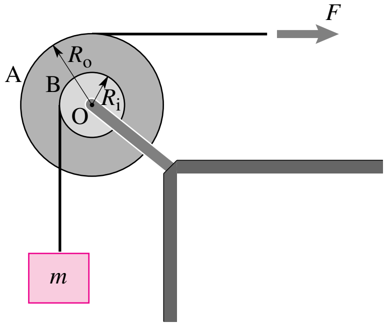


Figure 16.59: Two drums with strings wrapped around are used to pull up a mass m.

SAMPLE 16.17 Drums used as pulleys. Two drums, A and B of radii $R_o = 200$ mm and $R_i = 100$ mm are welded together. The combined mass of the drums is $m_D = 20$ kg and the combined moment of inertia about the z-axis passing through their common center O is $I_{zz/O} = 1.6\,\mathrm{kg\ m^2}$. A string attached to and wrapped around drum B supports a mass $m = 2$ kg. The string wrapped around drum A is pulled with a force $F = 20$ N as shown in Fig. 16.59. Assume there is no slip between the strings and the drums. Find

1. the angular acceleration of the drums,
2. the tension in the string supporting mass m, and
3. the acceleration of mass m.

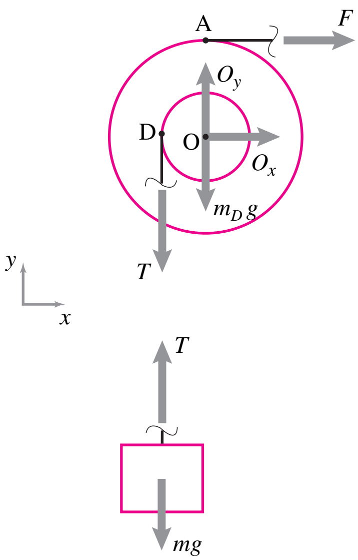


Figure 16.60: Free-body diagram of the drums and the mass m. T is the tension in the string supporting mass m and O_x and O_y are the reactions of the support at O.

Solution The free-body diagram of the drums and the mass are shown in Fig. 16.60 separately where T is the tension in the string supporting mass m and O_x and O_y are the support reactions at O. Since the drums can only rotate about the z-axis, let

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} \quad \text{and} \quad \dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}}.$$

Now, let us do angular momentum balance about the center of rotation O:

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$$

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= TR_i\hat{\boldsymbol{k}} - FR_o\hat{\boldsymbol{k}} \\ &= (TR_i - FR_o)\hat{\boldsymbol{k}}. \end{aligned}$$

Since the motion is restricted to the xy-plane (*i.e.,* 2-D motion), the rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ may be computed as

$$\begin{aligned} \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= I_{zz/cm}\,\dot{\omega}\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_{cm/\mathrm{O}} \times \vec{\boldsymbol{a}}_{cm}\ m_D \\ &= I_{zz/\mathrm{O}}\,\dot{\omega}\hat{\boldsymbol{k}} + \underbrace{\vec{\boldsymbol{r}}_{\mathrm{O}/\mathrm{O}}}_{0} \times \underbrace{\vec{\boldsymbol{a}}_{cm}}_{0}\ m_D \\ &= I_{zz/\mathrm{O}}\,\dot{\omega}\hat{\boldsymbol{k}}. \end{aligned}$$

Setting $\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}$ we get

$$TR_i - FR_o = I_{zz/\mathrm{O}}\,\dot{\omega}. \tag{16.54}$$

Now, let us write linear momentum balance, $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$, for mass m:

$$\underbrace{(T - mg)\hat{\boldsymbol{\jmath}}}_{\sum \vec{F}} = m\vec{\boldsymbol{a}}.$$

Do we know anything about acceleration $\vec{\boldsymbol{a}}$ of the mass? Yes, we know its direction ($\pm\hat{\boldsymbol{\jmath}}$) and we also know that it has to be the same as the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$ of point D on drum B (why?). Thus,

$$\begin{aligned} \vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta \\ &= \dot{\omega}\hat{\boldsymbol{k}} \times (-R_i\hat{\boldsymbol{\imath}}) \\ &= -\dot{\omega}R_i\hat{\boldsymbol{\jmath}}. \end{aligned} \tag{16.55}$$

Therefore,

$$T - mg = -m\dot{\omega}R_i. \tag{16.56}$$

1. **Calculation of $\dot{\omega}$:** We now have two equations, (16.54) and (16.56), and two unknowns, $\dot{\omega}$ and T. Subtracting R_i times Eqn.(16.56) from Eqn. (16.54) we get

$$\begin{aligned}
-FR_o + mgR_i &= (I_{zz/O} + mR_i^2)\dot{\omega} \\
\Rightarrow \qquad \dot{\omega} &= \frac{-FR_o + mgR_i}{(I_{zz/O} + mR_i^2)} \\
&= \frac{-20\,\mathrm{N}\cdot 0.2\,\mathrm{m} + 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 0.1m}{1.6\,\mathrm{kg\ m^2} + 2\,\mathrm{kg}\cdot(0.1\,\mathrm{m})^2} \\
&= \frac{-2.038\,\mathrm{kg\ m^2/\,s^2}}{1.62\,\mathrm{kg\ m^2}} \\
&= -1.258\frac{1}{\mathrm{s^2}}
\end{aligned}$$

$$\boxed{\dot{\vec{\omega}} = -1.26\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}}$$

2. **Calculation of tension T:** From equation (16.56):

$$\begin{aligned}
T &= mg - m\dot{\omega}R_i \\
&= 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2} - 2\,\mathrm{kg}\cdot(-1.26\,\mathrm{s^{-2}})\cdot 0.1\,\mathrm{m} \\
&= 19.87\,\mathrm{N}
\end{aligned}$$

$$\boxed{T = 19.87\,\mathrm{N}}$$

3. **Calculation of acceleration of the mass:** Since the acceleration of the mass is the same as the tangential acceleration of point D on the drum, we get (from eqn. (16.55))

$$\begin{aligned}
\vec{\boldsymbol{a}} &= (\vec{\boldsymbol{a}}_D)_\theta = -\dot{\omega}R_i\hat{\boldsymbol{\jmath}} \\
&= -(-1.26\,\mathrm{s^{-2}})\cdot 0.1\,\mathrm{m} \\
&= 0.126\,\mathrm{m/s^2}\,\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = 0.13\,\mathrm{m/s^2}\,\hat{\boldsymbol{\jmath}}}$$

Comments: It is important to understand why the acceleration of the mass is the same as the tangential acceleration of point D on the drum. We have assumed (as is common practice) that the string is massless and inextensible. Therefore each point of the string supporting the mass must have the same linear displacement, velocity, and acceleration as the mass. Now think about the point on the string which is momentarily in contact with point D of the drum. Since there is no relative slip between the drum and the string, the two points must have the same vertical acceleration. This vertical acceleration for point D on the drum is the tangential acceleration $(\vec{\boldsymbol{a}}_D)_\theta$.

SAMPLE 16.18 Energy method for pulley dynamics: Consider the pulley problem of Sample 16.17 again. Use energy method to

1. find the angular acceleration of the pulley, and
2. the acceleration of the mass.

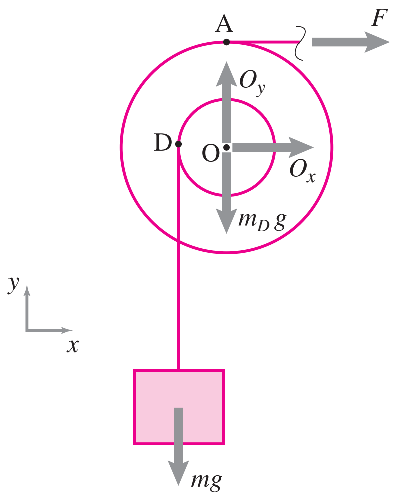


Figure 16.61: Free-body-diagram of the pulley-mass system together. Note that the forces acting at the support point O do no work since point O is stationary.

Solution In energy method we use speeds, not velocities. Therefore, we have to be careful in our thinking about the direction of motion. In the present problem, let us assume that the pulley rotates and accelerates clockwise. Consequently, the mass moves up against gravity.

1. The energy equation we want to use is

$$P = \dot{E}_K.$$

The power P is given by $P = \sum \vec{\boldsymbol{F}}_i \cdot \vec{\boldsymbol{v}}_i$ where the sum is carried out over all external forces. For the mass and pulley system the external forces that do work are F and mg. The other external forces on the system—the reaction force of the support point O and the weight of the pulley—are acting at point O (see fig. 16.61). But, since point O is stationary, these forces do no work. Therefore,

$$\begin{aligned} P &= \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}_A + m\vec{\boldsymbol{g}} \cdot \vec{\boldsymbol{v}}_m \\ &= F\hat{\boldsymbol{\imath}} \cdot v_A \hat{\boldsymbol{\imath}} + (-mg\hat{\boldsymbol{\jmath}}) \cdot \underbrace{v_D \hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{v}}_m} \\ &= Fv_A - mgv_D. \end{aligned} \tag{16.57}$$

Now we need to calculate the rate of change of kinetic energy $\dot{E}_K$. There are two objects here that have kinetic energy—the hanging mass and the pulley. Hence,

$$\dot{E}_K = (\dot{E}_K)_m + (\dot{E}_K)_{\text{pulley}}.$$

The hanging mass has pure translational motion and hence its kinetic energy is

$$(E_K)_m = \frac{1}{2} m v^2$$

where v is the linear speed of the mass. If we assume the string to be inextensible, then the linear speed v of the mass has to be the same as the tangential speed of point D of the pulley. Thus $v = v_D$ and

$$(E_K)_m = \frac{1}{2} m v_D^2.$$

The pulley, on the other hand, has pure rotational motion about point O, and hence its kinetic energy is given by

$$(E_K)_{\text{pulley}} = \frac{1}{2} I_{zz}^{\text{O}} \omega^2.$$

Summing the two kinetic energies and differentiating with respect to time t, we get

$$\begin{aligned} \dot{E}_K &= \frac{d}{dt}\left(\frac{1}{2}\, m\, v_D^2 + \frac{1}{2} I_{zz}^{\text{O}} \omega^2\right) && (16.58) \\ &= m\, v_D\, \dot{v}_D + I_{zz}^{\text{O}}\, \omega\, \dot{\omega}. && (16.59) \end{aligned}$$

Now equating the power and the rate of change of kinetic energy from eqns. eqn. (16.57) and eqn. (16.59), we get

$$F\, v_A - mg\, v_D = m\, v_D\, \dot{v}_D + I_{zz}^{\text{O}}\, \omega\, \dot{\omega}.$$

From kinematics of circular motion,

$$\begin{aligned} v_A &= \omega R_o, \\ v_D &= \omega R_i \\ \text{and} \qquad \dot{v}_D \equiv (a_D)_\theta &= \dot{\omega} R_i. \end{aligned}$$

Substituting these values in the power balance equation above, we get

$$\begin{aligned} \omega(FR_o - mgR_i) &= \omega\,\dot{\omega}(mR_i^2 + I_{zz}^{\text{O}}) \\ \Rightarrow \quad \dot{\omega} &= \frac{FR_o - mgR_i}{(I_{zz}^{\text{O}} + mR_i^2)} \\ &= \frac{20\,\text{N} \cdot 0.2\,\text{m} - 2\,\text{kg} \cdot 9.81\,\text{m/s}^2 \cdot 0.1m}{1.6\,\text{kg m}^2 + 2\,\text{kg} \cdot (0.1\,\text{m})^2} \\ &= 1.258\frac{1}{\text{s}^2}. \qquad \text{(same as the answer before.)} \end{aligned}$$

Since the sign of $\dot{\omega}$ is positive, our initial assumption of clockwise acceleration of the pulley is correct.

$$\dot{\omega} = 1.26\,\text{rad/s}^2$$

2. From kinematics of circular motion,

$$\begin{aligned} a_m &= (a_D)_\theta \\ &= \dot{\omega} R_i \\ &= 0.126\,\text{m/s}^2. \end{aligned}$$

$$a_m = 0.13\,\text{m/s}^2$$

SAMPLE 16.19 **Energy Accounting:** Consider the pulley problem of Sample 16.17 again.

1. What percentage of the input energy (work done by the applied force F) is used in raising the mass by 1 m?
2. Where does the rest of the energy go? Provide an energy-balance sheet.

Solution

1. Let W_i and W_h be the input energy and the energy used in raising the mass by 1 m, respectively. Then the percentage of energy used in raising the mass is

$$\% \text{ of input energy used} = \frac{W_h}{W_i} \times 100.$$

Thus we need to calculate W_i and W_h to find the answer. W_i is the work done by the force F on the system during the interval in which the mass moves up by 1 m. Let s be the displacement of the force F during this interval. Since the displacement is in the same direction as the force (we know it is from Sample 16.17), the input-energy is

$$W_i = F\,s.$$

So to find W_i we need to find s.
For the mass to move up by 1 m the inner drum B must rotate by an angle θ where

$$1\,\mathrm{m} = \theta\,R_i \qquad \Rightarrow \qquad \theta = \frac{1\,\mathrm{m}}{0.1\,\mathrm{m}} = 10\,\mathrm{rad}.$$

Since the two drums, A and B, are welded together, drum A must rotate by θ as well. Therefore the displacement of force F is

$$s = \theta\,R_o = 10\,\mathrm{rad}\cdot 0.2\,\mathrm{m} = 2\,\mathrm{m},$$

and the energy input is

$$W_i = F\,s = 20\,\mathrm{N}\cdot 2\,\mathrm{m} = 40\mathrm{J}.$$

Now, the work done in raising the mass by 1 m is

$$W_h = mgh = 2\,\mathrm{kg}\cdot 9.81\,\mathrm{m/s^2}\cdot 1\,\mathrm{m} = 19.62\mathrm{J}.$$

Therefore, the percentage of input-energy used in raising the mass

$$= \frac{19.62\,\mathrm{N{\cdot}m}}{40} \times 100 = 49.05\% \approx 49\%.$$

2. The rest of the energy (= 51%) goes in accelerating the mass and the pulley. Let us find out how much energy goes into each of these activities. Since the initial state of the system from which we begin energy accounting is not prescribed (that is, we are not given the height of the mass from which it is to be raised 1 m, nor do we know the velocities of the mass or the pulley at that initial height), let us assume that at the initial state, the angular speed of the pulley is ω_o and the linear speed of the mass is v_o. At the end of raising the mass by 1 m from this state, let the angular speed of the pulley be ω_f and the linear speed of the mass be v_f. Then, the energy

used in accelerating the pulley is

$$
\begin{aligned}
(\Delta E_{\mathrm{K}})_{\text{pulley}} &= \text{final kinetic energy} - \text{initial kinetic energy}\\
&= \frac{1}{2}I\omega_f^2 - \frac{1}{2}I\omega_o^2\\
&= \frac{1}{2}I\underbrace{(\omega_f^2 - \omega_o^2)}
\end{aligned}
$$

assuming constant acceleration, $\omega_f^2 = \omega_o^2 + 2\alpha\theta$, or $\omega_f^2 - \omega_o^2 = 2\alpha\theta$.

$$
\begin{aligned}
&= I\alpha\,\theta \qquad (\text{from Sample 16.18, } \alpha = 1.258\,\mathrm{rad/s^2}.\ )\\
&= 1.6\,\mathrm{kg\,m^2} \cdot 1.258\,\mathrm{rad/s^2} \cdot 10\,\mathrm{rad}\\
&= 20.13\,\mathrm{N{\cdot}m} = 20.13\ \mathrm{J}.
\end{aligned}
$$

Similarly, the energy used in accelerating the mass is

$$
\begin{aligned}
(\Delta E_{\mathrm{K}})_{\text{mass}} &= \text{final kinetic energy} - \text{initial kinetic energy}\\
&= \frac{1}{2}mv_f^2 - \frac{1}{2}mv_o^2\\
&= \frac{1}{2}m\underbrace{(v_f^2 - v_o^2)}_{2ah}\\
&= mah\\
&= 2\,\mathrm{kg} \cdot 0.126\,\mathrm{m/s^2} \cdot 1\,\mathrm{m}\\
&= 0.25\ \mathrm{J}.
\end{aligned}
$$

We can calculate the percentage of input energy used in these activities to get a better idea of energy allocation. Here is the summary table:

Activities	**Energy Spent**	
	in Joule	as % of input energy
In raising the mass by 1 m	19.62	49.05%
In accelerating the mass	0.25	0.62 %
In accelerating the pulley	20.13	50.33 %
Total	40.00	100 %

So, what would you change in the set-up so that more of the input energy is used in raising the mass? Think about what aspects of the motion would change due to your proposed design.

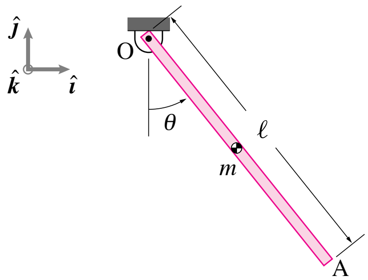


Figure 16.62: A uniform rod swings in the plane about its pinned end O.

SAMPLE 16.20 Equation of motion of a swinging stick: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 16.62).

1. Find the equation of motion using momentum balance.
2. Find the reaction at O as a function of (θ, $\dot{\theta}$, g, m, ℓ).

Solution First we draw a simple sketch of the given problem showing relevant geometry (Fig. 16.62(a)), and then a free-body diagram of the bar (Fig. 16.62(b)).

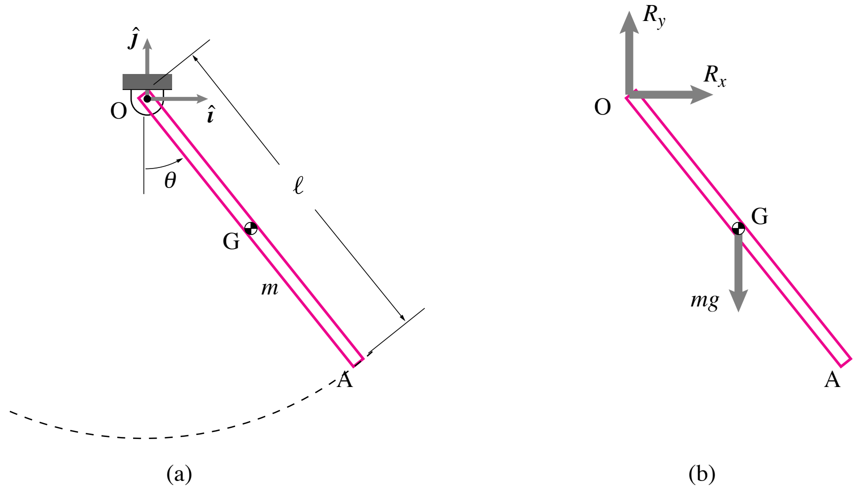


Figure 16.63: (a) A line sketch of the swinging rod and (b) free-body diagram of the rod.

We should note for future reference that

$$\begin{aligned}\vec{\omega} &= \omega\hat{k} \equiv \dot{\theta}\hat{k}\\ \dot{\vec{\omega}} &= \dot{\omega}\hat{k} \equiv \ddot{\theta}\hat{k}\end{aligned}$$

1. **Equation of motion using momentum balance:** We can write angular momentum balance about point O as

$$\sum \vec{M}_{\text{O}} = \dot{\vec{H}}_{/\text{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}\sum \vec{M}_{\text{O}} &= \vec{r}_{G/\text{O}} \times mg(-\hat{\jmath})\\ &= \frac{\ell}{2}(\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) \times mg(-\hat{\jmath})\\ &= -\frac{\ell}{2}mg\,\sin\theta\hat{k}. \qquad (16.60)\\ \dot{\vec{H}}_{/\text{O}} &= \dot{\omega}\hat{k}\int_m r^2 dm\\ &= \ddot{\theta}\hat{k}\int_0^{\ell} s^2\frac{m}{\ell}ds \quad (dm = m/\ell\cdot ds)\\ &= \frac{m\ddot{\theta}}{\ell}\left[\left.\frac{s^3}{3}\right|_0^{\ell}\right] = \frac{m\ell^2}{3}\ddot{\theta}\hat{k} \qquad (16.61)\end{aligned}$$

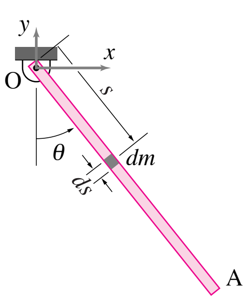


Figure 16.64: Computation of $\dot{\vec{H}}_{/\text{O}}$ by integration over the rod.

Equating (16.60) and (16.61) we get

$$
\begin{aligned}
-\frac{\ell}{2}\cancel{m}\, g\,\sin\theta &= \cancel{m}\frac{\ell^2}{3}\ddot{\theta} \\
\text{or}\qquad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\
\text{or}\qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \qquad (16.62)
\end{aligned}
$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

2. **Reaction at O:** Using linear momentum balance

$$
\begin{aligned}
\sum \vec{\boldsymbol{F}} &= m\,\vec{\boldsymbol{a}}_G, \\
\text{where}\quad \sum \vec{\boldsymbol{F}} &= R_x\hat{\boldsymbol{\imath}} + (R_y - mg)\hat{\boldsymbol{\jmath}}, \\
\text{and}\quad \vec{\boldsymbol{a}}_G &= \frac{\ell}{2}\dot{\omega}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + \frac{\ell}{2}\omega^2(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) \\
&= \frac{\ell}{2}[(\dot{\omega}\cos\theta - \omega^2\sin\theta)\hat{\boldsymbol{\imath}} + (\dot{\omega}\sin\theta + \omega^2\cos\theta)\hat{\boldsymbol{\jmath}}].
\end{aligned}
$$

Dotting both sides of $\sum \vec{\boldsymbol{F}} = m\,\vec{\boldsymbol{a}}_G$ with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ and rearranging, we get

$$
\begin{aligned}
R_x &= m\frac{\ell}{2}(\dot{\omega}\cos\theta - \omega^2\sin\theta) \\
&\equiv m\frac{\ell}{2}(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta), \\
R_y &= mg + m\frac{\ell}{2}(\dot{\omega}\sin\theta + \omega^2\cos\theta) \\
&\equiv mg + m\frac{\ell}{2}(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta).
\end{aligned}
$$

Now substituting the expression for $\ddot{\theta}$ from (16.62) in R_x and R_y, we get

$$
\begin{aligned}
R_x &= -m\sin\theta\left(\frac{3}{4}g\cos\theta + \frac{\ell}{2}\dot{\theta}^2\right), \qquad (16.63) \\
R_y &= mg\left(1 - \frac{3}{4}\sin^2\theta\right) + m\frac{\ell}{2}\dot{\theta}^2\cos\theta. \qquad (16.64)
\end{aligned}
$$

$$\boxed{\vec{\boldsymbol{R}} = -m(\tfrac{3}{4}g\cos\theta + \tfrac{\ell}{2}\dot{\theta}^2)\sin\theta\hat{\boldsymbol{\imath}} + [mg(1 - \tfrac{3}{4}\sin^2\theta) + m\tfrac{\ell}{2}\dot{\theta}^2\cos\theta]\hat{\boldsymbol{\jmath}}}$$

Check: We can check the reaction force in the special case when the rod does not swing but just hangs from point O. The forces on the bar in this case have to satisfy static equilibrium. Therefore, the reaction at O must be equal to mg and directed vertically upwards. Plugging $\theta = 0$ and $\dot{\theta} = 0$ (no motion) in Eqn. (16.63) and (16.64) we get $R_x = 0$ and $R_y = mg$, the values we expect.

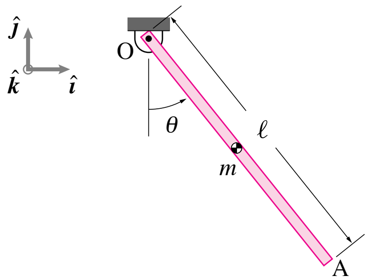


Figure 16.65: A uniform rod swings in the plane about its pinned end O.

SAMPLE 16.21 Swinging stick dynamics using moment of inertia: A uniform bar of mass m and length ℓ is pinned at one of its ends O. The bar is displaced from its vertical position by an angle θ and released (Fig. 16.65). Find the equation of motion of the stick.

Solution We repeat the problem solved in Sample 16.20 here with just one different step of finding the rate of change of angular momentum with the help of moment of inertia formula. As usual, we first draw a free-body diagram of the bar (Fig. 16.66). We assume, $\vec{\omega} = \omega\hat{\boldsymbol{k}} \equiv \dot{\theta}\hat{\boldsymbol{k}}$, and $\dot{\vec{\omega}} = \dot{\omega}\hat{\boldsymbol{k}} \equiv \ddot{\theta}\hat{\boldsymbol{k}}$ We can write angular momentum balance about point O as

$$\sum \vec{\boldsymbol{M}}_{\mathrm{O}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}}.$$

Let us now calculate both sides of this equation:

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \vec{\boldsymbol{r}}_{G/O} \times mg(-\hat{\boldsymbol{\jmath}}) \\
&= \frac{\ell}{2}(\sin\theta\hat{\boldsymbol{\imath}} - \cos\theta\hat{\boldsymbol{\jmath}}) \times mg(-\hat{\boldsymbol{\jmath}}) \\
&= -\frac{\ell}{2}mg\sin\theta\hat{\boldsymbol{k}}. && (16.65) \\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} &= I_{zz/G}\,\dot{\vec{\omega}} + \vec{\boldsymbol{r}}_G \times m\vec{\boldsymbol{a}}_G \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{\boldsymbol{k}} + \vec{\boldsymbol{r}}_G \times m\overbrace{(\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_G - \omega^2\vec{\boldsymbol{r}}_G)}^{\vec{\boldsymbol{a}}_G} \\
&= \frac{m\ell^2}{12}\dot{\omega}\hat{\boldsymbol{k}} + \frac{m\ell^2}{4}\dot{\omega}\hat{\boldsymbol{k}} && (16.66) \\
&= \frac{m\ell^2}{3}\dot{\omega}\hat{\boldsymbol{k}} && (16.67)
\end{aligned}$$

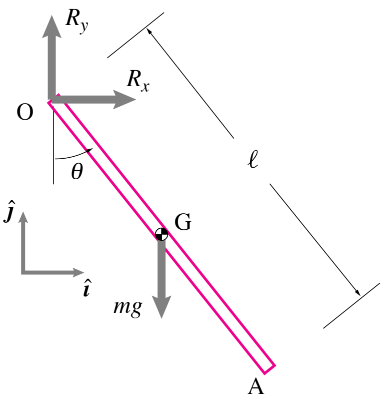


Figure 16.66: The free-body diagram of the rod.

where the last step, $\vec{\boldsymbol{r}}_G \times m\vec{\boldsymbol{a}}_G = \frac{m\ell^2}{4}\dot{\omega}\hat{\boldsymbol{k}}$, should be clear from Fig. 16.67. Equating (16.60) and (16.61) we get

$$\begin{aligned}
-\frac{\ell}{2}\cancel{m}g\sin\theta &= \cancel{m}\frac{\ell^2}{3}\dot{\omega} \\
\text{or} \quad \dot{\omega} + \frac{3g}{2\ell}\sin\theta &= 0 \\
\text{or} \quad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. && (16.68)
\end{aligned}$$

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0$$

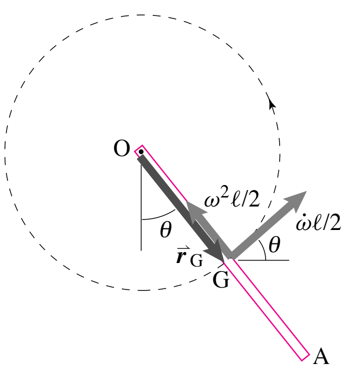


Figure 16.67: Radial and tangential components of $\vec{\boldsymbol{a}}_G$. Since the radial component is parallel to $\vec{\boldsymbol{r}}_G$, $\vec{\boldsymbol{r}}_G \times \vec{\boldsymbol{a}}_G = \frac{\ell^2}{4}\dot{\omega}\hat{\boldsymbol{k}}$.

SAMPLE 16.22 Numerical solution of the swinging stick motion: For the swinging stick considered in Samples 16.20 or 16.23, find the time that the rod takes to fall from $\theta = \pi/2$ to $\theta = 0$ if it is released from rest at $\theta = \pi/2$?

Solution The given initial angle $\pi/2$ is a big value of θ – big in that we cannot assume $\sin\theta \approx \theta$ (obviously $1 \neq 1.5708$). Therefore we may not use the linearized equation (16.71) to solve for t explicitly. We have to solve the full nonlinear equation (16.68) to find the required time. Unfortunately, we cannot get a *closed form* solution of this equation using mathematical skills you have at this level. Therefore, we resort to numerical integration of this equation.

For numerical integration, we need to first write the given differential equation as a set of first order ordinary differential equations. To do so, we introduce ω as a new variable and rewrite eqn. (16.68) as

$$\dot{\theta} = \omega \tag{16.69}$$

$$\dot{\omega} = -\frac{3g}{2\ell}\sin\theta \tag{16.70}$$

Now we need to specify the initial conditions and the time duration for integration, and solve the equations using some ODE solver program. Here is a pseudo-code that lists the steps:

```
g = 9.81,   L = 1      % define constants
ODES = { thetadot = omega
         omegadot = -3*g/(2*L) * sin(theta) }
 ICs = { theta_0 = pi/2
         omega_0 = 0 }
solve ODES with ICs for t = 0 to 4 s
plot theta vs t and plot omega vs t
```

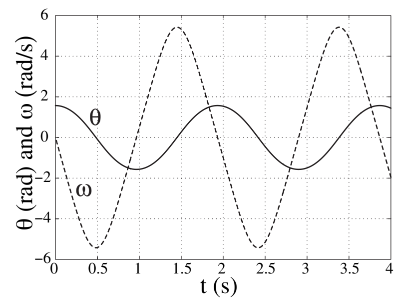


Figure 16.68: Numerical solution is shown by plotting θ and ω against time.

The results obtained from the numerical solution are shown in Fig. 16.68.

The problem of finding the time taken by the bar to fall from $\theta = \pi/2$ to $\theta = 0$ numerically is nontrivial. It is called a *boundary value problem*. We have only illustrated how to solve *initial value problems*. However, we can get a fairly good estimate of the time from the solution obtained.

We first plot θ against time t as shown in fig. 16.69. We find the values of t and the corresponding values of θ that bracket $\theta = 0$. Now, we can use linear interpolation to find the value of t at $\theta = 0$. Proceeding this way, we get $t = 0.48$ (seconds), a little more than we get from the linear ODE in sample 16.23 ($t = 0.41$).

Comments: Additionally, we can also get the value of $\dot{\theta} \equiv \omega$ when $\theta = 0$ using similar interpolation. In fact, from the ω vs t plot, we find that at $t = 0.48\,\mathrm{s}$, $\omega = -5.42\,\mathrm{rad/s}$. How does this result compare with the analytical value of ω from sample 16.23 (which did not depend on the small angle approximation)? Well, we found that

$$\omega = -\sqrt{\frac{3g}{\ell}} = -\sqrt{\frac{3 \cdot 9.81\,\mathrm{m/s^2}}{1\,\mathrm{m}}} = -5.4249\,\mathrm{s^{-1}}.$$

Thus, we get a fairly accurate value from numerical integration.

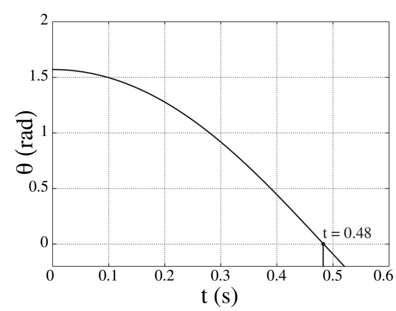


Figure 16.69: Time t corresponding to $\theta = 0$.

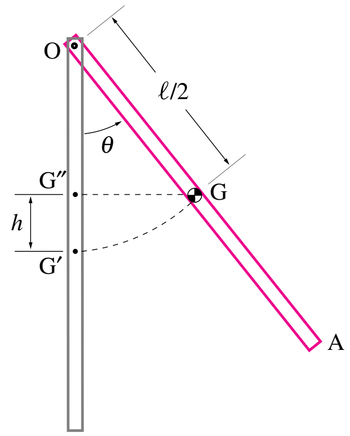


Figure 16.70: Work done by the force of gravity in moving from G′ to G $\int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} = -mg\hat{\boldsymbol{\jmath}} \cdot h\hat{\boldsymbol{\jmath}} = -mgh$.

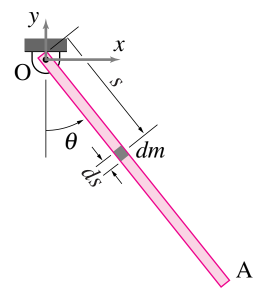


Figure 16.71: The infinitesimal mass dm considered in the calculation of E_{K}.

SAMPLE 16.23 The swinging stick dynamics with energy balance: Consider the same swinging stick as in Sample 16.20. The stick is, again, displaced from its vertical position by an angle θ and released (See Fig. 16.62).

1. Find the equation of motion using energy balance.
2. What is $\dot{\theta}$ at $\theta = 0$ if $\theta(t = 0) = \pi/2$?
3. Find the period of small oscillations about $\theta = 0$.

Solution

1. **Equation of motion using energy balance:** We use the power equation, $\dot{E}_{\mathrm{K}} = P$, to derive the equation of motion of the bar. Now, the kinetic energy is given by

$$E_{\mathrm{K}} = \frac{1}{2}\int_m v^2 dm$$

where v is the speed of the infinitesimal mass element dm. Referring to fig. 16.71, we can write, $dm = (m/\ell)ds$, and $v = \omega s \equiv \dot{\theta}s$. Thus,

$$\begin{aligned} E_{\mathrm{K}} &= \frac{1}{2}\int_0^\ell \dot{\theta}^2 s^2 \frac{m}{\ell} ds \\ &= \frac{m\dot{\theta}^2}{2\ell}\int_0^\ell s^2 ds \\ &= \frac{1}{6} m\ell^2\dot{\theta}^2 \end{aligned}$$

and, therefore,

$$\dot{E}_{\mathrm{K}} = \frac{d}{dt}(\frac{1}{6}m\ell^2\omega^2) = \frac{1}{3}m\ell^2\omega\,\dot{\omega} = \frac{1}{3}m\ell^2\dot{\theta}\ddot{\theta}.$$

Calculation of power (P): There are only two forces acting on the bar, the reaction force, $\vec{\boldsymbol{R}}(= R_x\hat{\boldsymbol{\imath}} + R_y\hat{\boldsymbol{\jmath}})$ and the force due to gravity, $-mg\hat{\boldsymbol{\jmath}}$. Since the support point O does not move, no work is done by $\vec{\boldsymbol{R}}$. Therefore,

$$W = \text{Work done by gravity force in moving from G}' \text{ to G.} = -mgh$$

Note that the negative sign stands for the work done *against* gravity. Now,

$$h = OG' - OG'' = \frac{\ell}{2} - \frac{\ell}{2}\cos\theta = \frac{\ell}{2}(1-\cos\theta).$$

Therefore,

$$\begin{aligned} W &= -mg\frac{\ell}{2}(1-\cos\theta) \\ \text{and} \qquad P &= \dot{W} = \frac{dW}{dt} = -mg\frac{\ell}{2}\sin\theta\dot{\theta}. \end{aligned}$$

Equating $\dot{E}_{\mathrm{K}}$ and P we get

$$\begin{aligned} -\cancel{m}\,g\frac{\ell}{2}\sin\theta\cancel{\dot{\theta}} &= \frac{1}{3}\cancel{m}\,\ell^2\cancel{\dot{\theta}}\ddot{\theta} \\ \text{or} \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta &= 0. \end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0}$$

This equation is, of course, the same as we obtained using balance of angular momentum in Sample 16.20.

2. **Find ω at $\theta = 0$:** We are given that at $t = 0$, $\theta = \pi/2$ and $\dot{\theta} \equiv \omega = 0$ (released from rest). This position is (1) shown in Fig. 16.72. In position (2) $\theta = 0$, i.e., the rod is vertical. Since there are no dissipative forces, the total energy of the system remains constant. Therefore, taking datum for potential energy as shown in Fig. 16.72, we may write

$$\begin{aligned}
\underbrace{E_{K1}}_{0} + V_1 &= E_{K2} + \underbrace{V_2}_{0} \\
\text{or} \qquad mg\frac{\ell}{2} &= \frac{1}{2}\int_m v^2 dm \\
&= \frac{1}{6}m\ell^2\omega^2 \quad \text{(see part (a))} \\
\Rightarrow \qquad \omega &= \pm\sqrt{\frac{3g}{\ell}}
\end{aligned}$$

$$\boxed{\omega = \pm\sqrt{\frac{3g}{\ell}}}$$

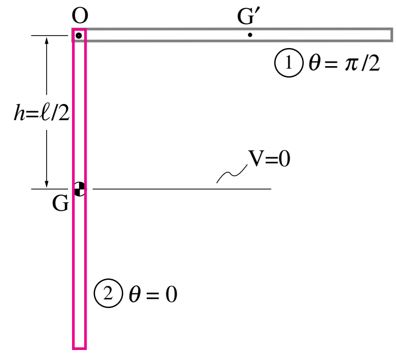


Figure 16.72: The total energy between positions (1) and (2) is constant.

3. **Period of small oscillations:** The equation of motion is

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta = 0.$$

For small θ, $\sin\theta \approx \theta$

$$\Rightarrow \qquad \ddot{\theta} + \frac{3g}{2\ell}\theta = 0 \tag{16.71}$$

$$\text{or} \qquad \ddot{\theta} + \lambda^2\theta = 0$$

$$\text{where} \qquad \lambda^2 = \frac{3g}{2\ell}.$$

Therefore,

$$\begin{aligned}
\text{the circular frequency} &= \lambda = \sqrt{\frac{3g}{2\ell}}, \\
\text{and the time period T} &= \frac{2\pi}{\lambda} = 2\pi\sqrt{\frac{2\ell}{3g}}.
\end{aligned}$$

$$\boxed{T = 2\pi\sqrt{\frac{2\ell}{3g}}}$$

[Say for $g = 9.81\,\mathrm{m/s^2}, \ell = 1\,\mathrm{m}$ we get $\frac{T}{4} = \frac{\pi}{2}\sqrt{\frac{2}{3}\frac{1}{9.81}}\,\mathrm{s} = 0.4097\,\mathrm{s}$]

SAMPLE 16.24 The swinging stick with a destabilizing torque. Consider the swinging stick of Sample 16.20 once again.

1. Find the equation of motion of the stick, if a torque $\vec{M} = M\hat{k}$ is applied at end O and a force $\vec{F} = F\hat{\imath}$ is applied at the other end A.
2. Take $F = 0$ and $M = C\theta$. For $C = 0$ you get the equation of free oscillations obtained in Sample 16.20 or 16.23. For small C, does the period of the pendulum increase or decrease?
3. What happens if C is big?

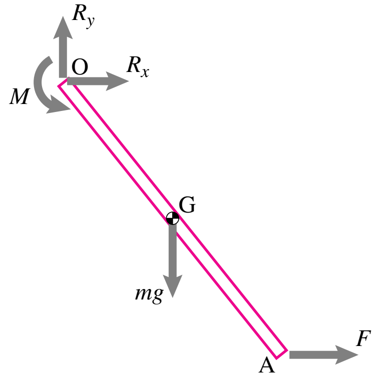


Figure 16.73: Free-body diagram of the bar with applied torque $\vec{M}$ and force $\vec{F}$

Solution

1. A free-body diagram of the bar is shown in Fig. 16.73. Once again, we can use $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$ to derive the equation of motion as in Sample 16.20. We calculated $\sum \vec{M}_O$ and $\dot{\vec{H}}_{/O}$ in Sample 16.20. Calculation of $\dot{\vec{H}}_{/O}$ remains the same in the present problem. We only need to recalculate $\sum \vec{M}_O$.

$$\begin{aligned}\sum \vec{M}_O &= M\hat{k} + \vec{r}_{G/O} \times mg(-\hat{\jmath}) + \vec{r}_{A/O} \times \vec{F} \\ &= M\hat{k} - \frac{\ell}{2} mg \sin\theta \hat{k} + F\ell\cos\theta\hat{k} \\ &= (M + F\ell\cos\theta - \frac{\ell}{2} mg\sin\theta)\hat{k}\end{aligned}$$

and

$$\dot{\vec{H}}_{/O} = m\ddot{\theta}\frac{\ell^2}{3}\hat{k} \quad \text{(see Sample 16.20)}$$

Therefore, from $\sum \vec{M}_O = \dot{\vec{H}}_{/O}$

$$\begin{aligned} M + F\ell\cos\theta - \frac{\ell}{2} mg\sin\theta &= m\ddot{\theta}\frac{\ell^2}{3} \\ \Rightarrow \qquad \ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} &= 0.\end{aligned}$$

$$\boxed{\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3F}{m\ell}\cos\theta - \frac{3M}{m\ell^2} = 0}$$

2. Now, setting $F = 0$ and $M = C\theta$ we get

$$\ddot{\theta} + \frac{3g}{2\ell}\sin\theta - \frac{3C\theta}{m\ell^2} = 0 \qquad (16.72)$$

Numerical Solution: We can numerically integrate (16.72) just as in the previous Sample to find $\theta(t)$. Here is the pseudo-code that can be used for this purpose.

```
g = 9.81, L = 1      % specify parameters
m = 1, C = 4
ODES = { thetadot = omega
         omegadot = -(3*g/(2*L)) * sin(theta)
                  + 3*C/(m*L^2) * theta      }
ICs =  { thetazero = pi/20
         omegazero = 0  }
solve ODES with ICs until t = 10
```

Using this pseudo-code, we find the response of the pendulum. Figure 16.74 shows different responses for various values of C. Note that for $C = 0$, it is the same case as unforced bar pendulum considered above. From Fig. 16.74 it is clear that the bar has periodic motion for small C, with the period of motion increasing with increasing values of C. It makes sense if you look at Eqn. (16.72) carefully. Gravity acts as a restoring force while the applied torque acts as a destabilizing force. Thus, with the resistance of the applied torque, the stick swings more sluggishly making its period of oscillation bigger.

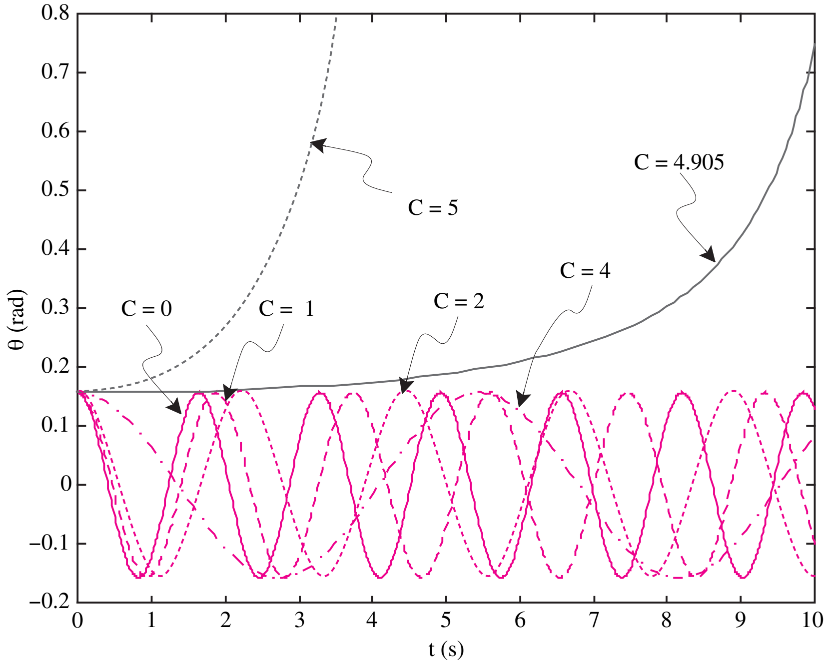


Figure 16.74: $\theta(t)$ with applied torque $M = C\theta$ for $C = 0,\ 1,\ 2,\ 4,\ 4.905,\ 5.$ Note that for small C the motion is periodic but for large C ($C \geq 4.4$) the motion becomes aperiodic.

3. From Fig. 16.74, we see that at about $C \approx 4.9$ the stability of the system changes completely. $\theta(t)$ is not periodic anymore. It keeps on increasing at a faster and faster rate, that is, the bar makes complete loops about point O with ever increasing speed. Does it make physical sense? Yes, it does. As the value of C is increased beyond a certain value (can you guess the value?), the applied torque overcomes any restoring torque due to gravity. Consequently, the bar is forced to rotate continuously in the direction of the applied force.

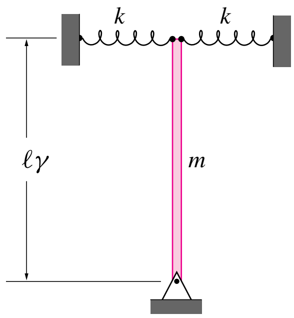
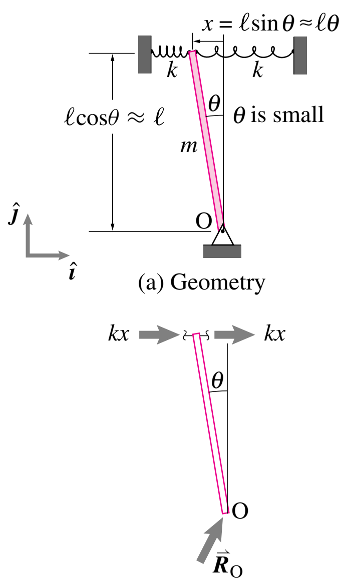


Figure 16.75

SAMPLE 16.25 **A torsional pendulum with linear springs:** A uniform rigid bar of mass $m = 2\,\text{kg}$ and length $\ell = 1\,\text{m}$ is pinned at one end and connected to two springs, each with spring constant k, at the other end. The bar is tweaked slightly from its vertical position. It then oscillates about its original position. The bar is timed for 20 full oscillations which take 12.5 seconds. Ignore gravity.

1. Find the equation of motion of the rod.
2. Find the spring constant k.
3. What should be the spring constant of a torsional spring if the bar is attached to one at the bottom and has the same oscillating motion characteristics?



(a) Geometry

(b) Free body diagram

Figure 16.76

Solution

1. Refer to the free-body diagram in figure 16.76. Angular momentum balance for the rod about point O gives

$$\sum \vec{M}_O = \dot{\vec{H}}_{/O}$$

$$\begin{aligned} \text{where} \quad \vec{M}_O &= -2k \overbrace{x}^{\ell\sin\theta} \cdot \ell\cos\theta\hat{k} \\ &= -2k\ell^2\sin\theta\cos\theta\hat{k}, \\ \text{and} \quad \dot{\vec{H}}_{/O} &= I_{zz}^O\ddot{\theta}\hat{k} = \underbrace{\frac{1}{3}m\ell^2}_{I_{zz}^O}\ddot{\theta}\hat{k}. \end{aligned}$$

Thus

$$\frac{1}{3}m\ell^2\ddot{\theta} = -2k\ell^2\sin\theta\cos\theta.$$

However, for small θ, $\cos\theta \approx 1$ and $\sin\theta \approx \theta$,

$$\Rightarrow \quad \ddot{\theta} + \frac{6k\ell^2}{m\ell^2}\theta = 0. \qquad (16.73)$$

$$\boxed{\ddot{\theta} + \frac{6k}{m}\theta = 0}$$

2. Comparing Eqn. (16.73) with the standard harmonic oscillator equation $\ddot{x} + \lambda^2 x = 0$, we get

$$\begin{aligned} \text{angular frequency} \quad \lambda &= \sqrt{\frac{6k}{m}}, \\ \text{and the time period} \quad T &= \frac{2\pi}{\lambda} \\ &= 2\pi\sqrt{\frac{m}{6k}}. \end{aligned}$$

From the measured time for 20 oscillations, the time period (time for one oscillation) is

$$T = \frac{12.5}{20}\,\text{s} = 0.625\,\text{s}$$

Now equating the measured T with the derived expression for T we get

$$\begin{aligned} 2\pi\sqrt{\frac{m}{6k}} &= 0.625\,\mathrm{s} \\ \Rightarrow \qquad k &= 4\pi^2 \cdot \frac{m}{6(0.625\,\mathrm{s})^2} \\ &= \frac{4\pi^2 \cdot 2\,\mathrm{kg}}{6(0.625\,\mathrm{s})^2} \\ &= 33.7\,\mathrm{N/\,m}. \end{aligned}$$

$$\boxed{k = 33.7\,\mathrm{N/\,m}}$$

3. If the two linear springs are to be replaced by a torsional spring at the bottom, we can find the spring constant of the torsional spring by comparison. Let k_{tor} be the spring constant of the torsional spring. Then, as shown in the free-body diagram (see figure 16.77), the restoring torque applied by the spring at an angular displacement θ is $k_{\mathrm{tor}}\theta$. Now, writing the angular momentum balance about point O, we get

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{\mathrm{O}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} \\ -k_{\mathrm{tor}}\theta\hat{\boldsymbol{k}} &= I_{zz}^{O}(\ddot{\theta}\hat{\boldsymbol{k}}) \\ \Rightarrow \qquad \ddot{\theta} + \frac{k_{\mathrm{tor}}}{I_{zz}^{O}}\theta &= 0. \end{aligned}$$

Comparing with the standard harmonic equation, we find the angular frequency

$$\lambda = \sqrt{\frac{k_{\mathrm{tor}}}{I_{zz}^{O}}} = \sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}}.$$

If this system has to have the same period of oscillation as the first system, the two angular frequencies must be equal, *i.e.,*

$$\begin{aligned} \sqrt{\frac{k_{\mathrm{tor}}}{\frac{1}{3}m\ell^2}} &= \sqrt{\frac{6k}{m}} \\ \Rightarrow \qquad k_{\mathrm{tor}} &= 6k \cdot \frac{1}{3}\ell^2 = 2k\ell^2 \\ &= 2\cdot(33.7\,\mathrm{N/\,m})\cdot(1\,\mathrm{m})^2 \\ &= 67.4\,\mathrm{N{\cdot}m}. \end{aligned}$$

$$\boxed{k_{\mathrm{tor}} = 67.4\,\mathrm{N{\cdot}m}}$$

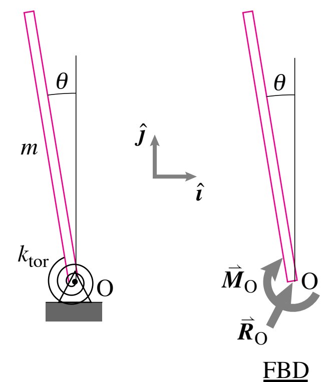


Figure 16.77

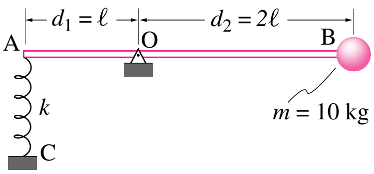
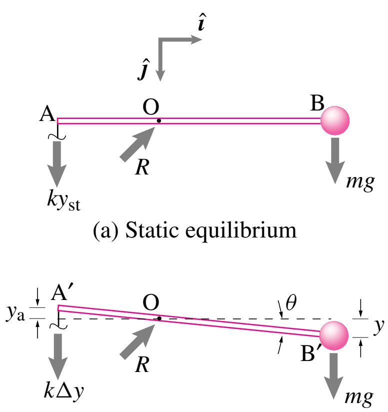


Figure 16.78

SAMPLE 16.26 A spring loaded seesaw: A kid, modelled as a point mass with $m = 10\,\mathrm{kg}$, is sitting at end B of a rigid rod AB of negligible mass. The rod is supported by a spring at end A and a pin at point O. The system is in static equilibrium when the rod is horizontal. Someone pushes the kid vertically downwards by a small distance y and lets go. Given that $AB = 3\,\mathrm{m}$, $AC = 0.5\,\mathrm{m}$, $k = 1\,\mathrm{kN/m}$; find

1. the unstretched (relaxed) length of the spring,
2. the equation of motion (a differential equation relating the position of the mass to its acceleration) of the system, and
3. the natural frequency of the system.

If the rod is pinned at the midpoint instead of at O, what is the natural frequency of the system? How does the new natural frequency compare with that of a mass m simply suspended by a spring with the same spring constant?



(a) Static equilibrium

(b) Mass m is displaced downwards

Figure 16.79: Free-body diagrams

Solution

1. **Static Equilibrium:** The FBD of the (rod + mass) system is shown in Fig. 16.79. Let the stretch in the spring in this position be y_{st} and the relaxed length of the spring be ℓ_0. The balance of angular momentum about point O gives:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/\mathrm{o}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{o}} &= \vec{\mathbf{0}} \qquad \text{(no motion)} \\
\Rightarrow \quad (ky_{st})d_1 - (mg)d_2 &= 0 \\
\Rightarrow \quad y_{st} &= \frac{mg}{k}\cdot\frac{d_2}{d_1} \\
&= \frac{10\,\mathrm{kg}\cdot 9.8\,\mathrm{m/s^2}\cdot 2\ell}{1000\,\mathrm{N/m}\cdot\ell} = 0.196\,\mathrm{m} \\
\text{Therefore,} \quad \ell_0 &= AC - y_{st} \\
&= 0.5\,\mathrm{m} - 0.196\,\mathrm{m} = 0.304\,\mathrm{m}.
\end{aligned}
$$

$$\boxed{\ell_0 = 30.4\,\mathrm{cm}}$$

2. **Equation of motion:** As point B gets displaced downwards by a distance y, point A moves up by a proportionate distance y_a. From geometry, ②

$$
\begin{aligned}
y &\approx d_2\theta \quad \Rightarrow \quad \theta = \frac{y}{d_2} \\
y_a &\approx d_1\theta = \frac{d_1}{d_2}y
\end{aligned}
$$

Therefore, the total stretch in the spring, in this position,

$$\Delta y = y_a + y_{st} = \frac{d_1}{d_2}y + \frac{d_2}{d_1}\frac{mg}{k}$$

Now, Angular Momentum Balance about point O gives:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/\mathrm{o}} &= \dot{\vec{\boldsymbol{H}}}_{/\mathrm{o}} \\
\sum \vec{\boldsymbol{M}}_{/\mathrm{o}} &= \vec{\boldsymbol{r}}_B \times mg\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{r}}_A \times k\Delta y\,\hat{\boldsymbol{\jmath}} \\
&= (d_2 mg - d_1 k\Delta y)\hat{\boldsymbol{k}} && (16.74) \\
\dot{\vec{\boldsymbol{H}}}_{/\mathrm{o}} &= \vec{\boldsymbol{r}}_B \times m\vec{\boldsymbol{a}} = \vec{\boldsymbol{r}}_B \times m\ddot{y}\,\hat{\boldsymbol{\jmath}} && (16.75) \\
&= d_2 m\ddot{y}\,\hat{\boldsymbol{k}} && (16.76)
\end{aligned}
$$

② Here, we are considering a very small y so that we can ignore the arc the point mass B moves on and take its motion to be just vertical (i.e., $\sin\theta \approx \theta$ for small θ).

Equating (16.74) and (16.76) we get

$$
\begin{aligned}
d_2 mg - d_1 k\Delta y &= d_2 m\ddot{y} \\
\text{or} \qquad d_2 mg - d_1 k\left(\frac{d_1}{d_2}y + \frac{d_2 mg}{d_1 k}\right) &= d_2 m\ddot{y} \\
\text{or} \qquad d_2 \cancel{m}g - k\frac{d_1^2}{d_2}y - d_2\cancel{m}g &= d_2 m\ddot{y} \\
\text{or} \qquad \ddot{y} + \frac{k}{m}\left(\frac{d_1}{d_2}\right)^2 y &= 0
\end{aligned}
$$

$$\boxed{\ddot{y} + \frac{k}{m}\left(\frac{d_1}{d_2}\right)^2 y = 0}$$

3. **The natural frequency of the system:** We may also write the previous equation as

$$\ddot{y} + \lambda^2 y = 0 \qquad \text{where} \qquad \lambda^2 = \frac{k}{m}\frac{d_1^2}{d_2^2}. \tag{16.77}$$

Substituting $d_1 = \ell$ and $d_2 = 2\ell$ in the expression for λ we get the natural frequency of the system

$$\lambda = \frac{1}{2}\sqrt{\frac{k}{m}} = \frac{1}{2}\sqrt{\frac{1000\,\mathrm{N/m}}{10\,\mathrm{kg}}} = 5\,\mathrm{s}^{-1}$$

$$\boxed{\lambda = 5\,\mathrm{s}^{-1}}$$

4. **Comparison with a simple spring mass system:**

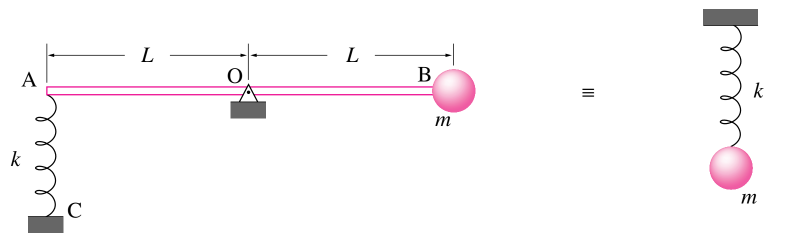


Figure 16.80:

When $d_1 = d_2$, the equation of motion (16.77) becomes

$$\ddot{y} + \frac{k}{m}y = 0$$

and the natural frequency of the system is simply

$$\lambda = \sqrt{\frac{k}{m}}$$

which corresponds to the natural frequency of a simple spring mass system shown in Fig. 16.80.

In our system (with $d_1 = d_2$) any vertical displacement of the mass at B induces an equal amount of stretch or compression in the spring which is exactly the case in the simple spring-mass system. Therefore, the two systems are mechanically equivalent. Such equivalences are widely used in modeling complex physical systems with simpler mechanical models.

16.4 Dynamics

Preparatory Problems

16.4.1 An object consists of a massless bar with two attached masses m_1 and m_2. The object is hinged at O.

a) What is the moment of inertia of the object about point O (I_{zz}^{O})? *

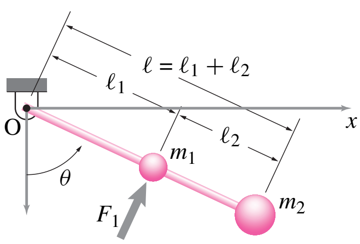
b) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\vec{\boldsymbol{H}}_{/O}$, the angular momentum about point O? *

c) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is $\dot{\vec{\boldsymbol{H}}}_{/O}$, the rate of change of angular momentum about point O? *

d) Given θ, $\dot{\theta}$, and $\ddot{\theta}$, what is T, the total kinetic energy? *

e) Assume that you don't know θ, $\dot{\theta}$ or $\ddot{\theta}$ but you do know that F_1 is applied to the rod, perpendicular to the rod at m_1. What is $\ddot{\theta}$? (Neglect gravity.) *

f) If F_1 were applied to m_2 instead of m_1, would $\ddot{\theta}$ be bigger or smaller? *

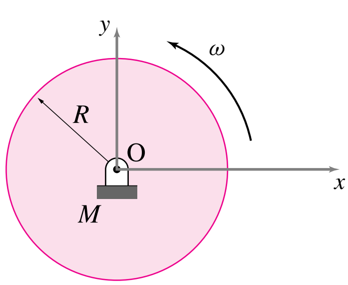


Problem 16.4.1

16.4.2 A uniform circular disc rotates at constant angular speed ω about the origin, which is also the center of the disc. It's radius is R. It's total mass is M.

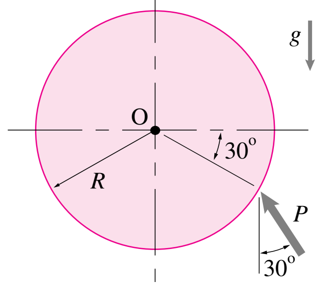
a) What is the total force and moment required to hold it in place (use the origin as the reference point of angular momentum and torque). *

b) What is the total kinetic energy of the disk? *



Problem 16.4.2

16.4.3 The hinged disk of mass m (uniformly distributed) is acted upon by a force P shown in the figure. Determine the *initial* angular acceleration and the reaction forces at the pin O. *



Problem 16.4.3

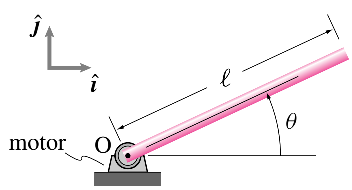
16.4.4 A motor turns a bar. A uniform bar of length ℓ and mass m is turned by a motor whose shaft is attached to the end of the bar at O. The angle that the bar makes (measured counterclockwise) from the positive x axis is $\theta = 2\pi\ t^2/\mathrm{s}^2$. Neglect gravity.

a) Draw a free-body diagram of the bar.

b) Find the force acting on the bar from the motor and hinge at $t = 1\,\mathrm{s}$. *

c) Find the torque applied to the bar from the motor at $t = 1\,\mathrm{s}$. *

d) What is the power produced by the motor at $t = 1\,\mathrm{s}$? *



Problem 16.4.4

16.4.5 A physical pendulum. A swinging stick is sometimes called a 'physical' pendulum. Take the 'body', the system of interest, to be the whole stick.

a) Draw a free-body diagram of the system.

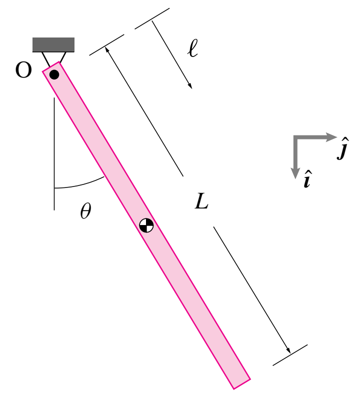
b) Write the equation of angular momentum balance for this system about point O.

c) Evaluate the left-hand-side as explicitly as possible in terms of the forces showing on your Free-Body Diagram.

d) Evaluate the right hand side as completely as possible. You may use the following facts:

$$\begin{aligned}
\vec{\boldsymbol{v}} &= l\dot{\theta}\cos\theta\,\hat{\boldsymbol{\jmath}} + -l\dot{\theta}\sin\theta\,\hat{\boldsymbol{\imath}} \\
\vec{\boldsymbol{a}} &= -l\dot{\theta}^2\left[\cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}\right] \\
&\quad + l\ddot{\theta}\left[\cos\theta\,\hat{\boldsymbol{\jmath}} - \sin\theta\,\hat{\boldsymbol{\imath}}\right]
\end{aligned}$$

where ℓ is the distance along the pendulum from the top, θ is the angle by which the pendulum is displaced counter-clockwise from the vertically down position, $\hat{\boldsymbol{\imath}}$ is vertically down, and $\hat{\boldsymbol{\jmath}}$ is to the right. You will have to set up and evaluate an integral. *



Problem 16.4.5

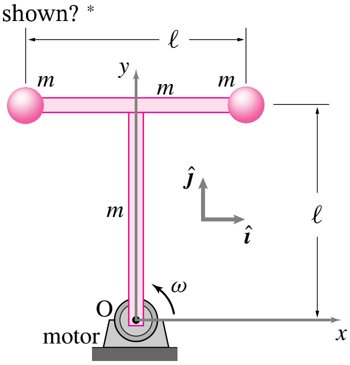
16.4.6 A uniform one meter bar is hung from a hinge that is at the end. It is allowed to swing freely. $g = 10\,\mathrm{m/s^2}$.

a) What is the period of small oscillations for this pendulum? *

b) Suppose the rod is hung 0.4 m from one end. What is the period of small oscillations for this pendulum? Can you explain why it is longer or shorter than when it is hung by its end? *

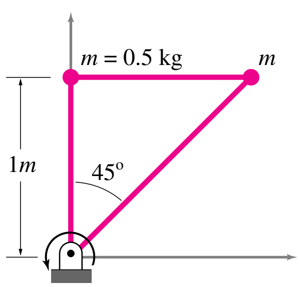
More-Involved Problems

16.4.7 Motor turns a dumbbell. Two uniform bars of length ℓ and mass m are welded at right angles. At the ends of the horizontal bar are two more masses m. The bottom end of the vertical rod is attached to a hinge at O where a motor keeps the structure rotating at constant rate ω (counter-clockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown? *



Problem 16.4.7

16.4.8 The structure shown in the figure consists of two point masses connected by three rigid, massless rods such that the whole structure behaves like a rigid body. The structure rotates counterclockwise at a constant rate of 60 rpm. At the instant shown, find the force in each rod. *

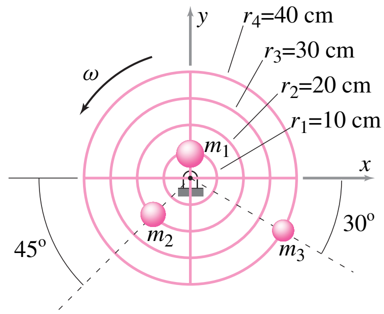


Problem 16.4.8

16.4.9 Balancing a system of rotating particles. A wire frame structure is made of four concentric loops of massless and rigid wires, connected to each other by four rigid wires presently coincident with the x and y axes. Three masses, $m_1 = 200$ grams, $m_2 = 150$ grams and $m_3 = 100$ grams, are glued to the structure as shown in the figure. The structure rotates counterclockwise at a constant rate $\dot{\theta} = 5\,\text{rad/s}$. There is no gravity.

a) Find the net force exerted by the structure on the support at the instant shown. *

b) You are to put a mass m at an appropriate location on the third loop so that the net force on the support is zero. Find the appropriate mass and the location on the loop. *

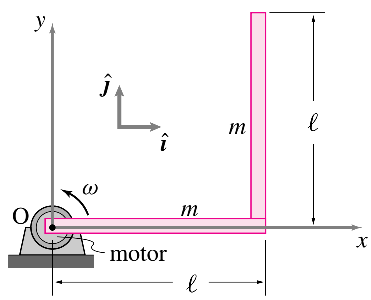


Problem 16.4.9

16.4.10 Motor turns a bent bar. Two uniform bars of length ℓ and uniform mass m are welded at right angles. One end is attached to a hinge at O where a motor keeps the structure rotating at a constant rate ω (counterclockwise). What is the net force and moment that the motor and hinge cause on the structure at the instant shown.

a) neglecting gravity *

b) including gravity. *



Problem 16.4.10: A bent bar is rotated by a motor.

16.4.11 A uniform disk of mass M and radius R rotates about a hinge O in the xy-plane. A point mass m is fixed to the disk at a distance $R/2$ from the hinge. A motor at the hinge drives the disk/point mass assembly with constant angular acceleration α. What torque at the hinge does the motor supply to the system? *

16.4.12 A rigid rod of length ℓ and total mass m is held fixed at one end and whirled around in circular motion at a constant rate ω in the horizontal plane. Ignore gravity.

a) Find the tension in the rod as a function of r, the radial distance from the center of rotation to any desired location on the rod. *

b) Where does the maximum tension occur in the rod? *.

c) At what distance from the center of rotation does the tension drop to half its maximum value? *.

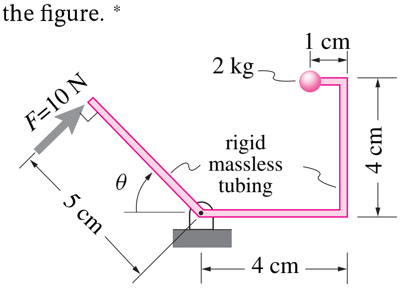
16.4.13 A thin uniform circular disc of mass M and radius R rotates in the xy plane about its center of mass point O. Driven by a motor, it has rate of change of angular speed proportional to angular position, $\alpha = k\theta^{5/2}$. The disc starts from rest at $\theta = 0$.

a) What is the rate of change of angular momentum about the origin at $\theta = \frac{\pi}{3}$ rad? *

b) What is the torque of the motor at $\theta = \frac{\pi}{3}$ rad? *

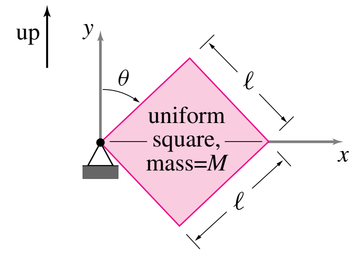
c) What is the total kinetic energy of the disk at $\theta = \frac{\pi}{3}$ rad? *

16.4.14 Neglecting gravity, calculate $\alpha = \dot{\omega} = \ddot{\theta}$ at the instant shown for the system in the figure. *



Problem 16.4.14

16.4.15 The uniform square shown is released from rest at $t = 0$. What is $\alpha = \dot{\omega} = \ddot{\theta}$ immediately after release? *

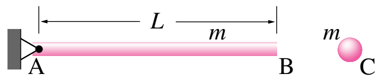


Problem 16.4.15

16.4.16 Acceleration of a trap door. A uniform bar AB of mass m and a ball of the same mass are released from rest from the same horizontal position. The bar is hinged at end A. There is gravity.

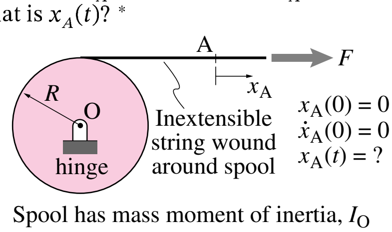
a) Which point on the rod has the same acceleration as the ball, immediately after release. *

b) What is the reaction force on the bar at end A just after release? *



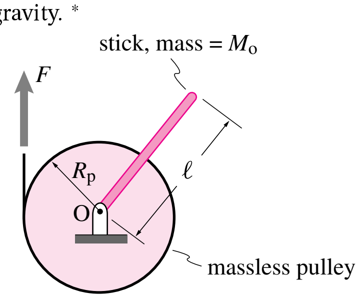
Problem 16.4.16

16.4.17 A disk with radius R has a string wrapped around it which is pulled with a force F. The disk is free to rotate about the axis through O normal to the page. The moment of inertia of the disk about O is I_o. A point A is marked on the string. Given that $x_A(0) = 0$ and that $\dot{x}_A(0) = 0$, what is $x_A(t)$? *



Problem 16.4.17

16.4.18 A uniform stick with length ℓ and mass M_o is welded to a pulley hinged at the center O. The pulley has negligible mass and radius R_p. A string is wrapped many times around the pulley. At time $t = 0$, the pulley, stick, and string are at rest and a force F is suddenly applied to the string. How long does it take for the pulley to make one full revolution? Ignore gravity. *



Problem 16.4.18: String wraps around a pulley with a stick glued to it.

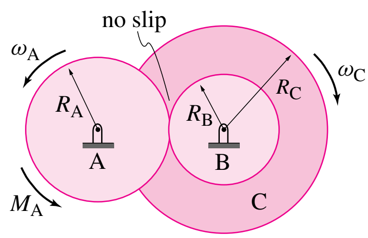
Gears, Belt Drives, and Gear Trains

16.4.19 Constant speed gear train. Gear A is connected to a motor (not shown) and gear B, which is welded to gear C, is connected to a taffy-pulling mechanism. Assume you know the torque $M_{\text{input}} = M_A$ and angular velocity $\omega_{\text{input}} = \omega_A$ of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power? *

b) What is the output power? *

c) What is the output torque $M_{\text{output}} = M_C$, the torque that gear C applies to its surroundings in the clockwise direction? *



Problem 16.4.19

16.4.20 At the input to a gear box a 100 lbf force is applied to gear A. At the output, the machinery (not shown) applies a force of F_B to the output gear. Gear A rotates at constant angular rate $\omega = 2\,\text{rad/s}$, clockwise.

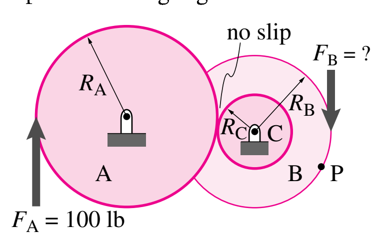
a) What is the angular speed of the right gear? *

b) What is the velocity of point P? *

c) What is F_B? *

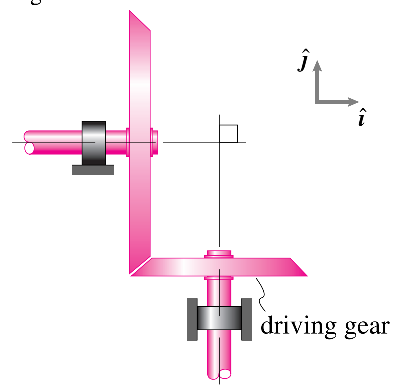
d) If the gear bearings had friction, would F_B have to be larger or smaller in order to achieve the same constant velocity? *

e) If instead of applying a 100 lbf to the left gear it is driven by a motor (not shown) at constant angular speed ω, what is the angular speed of the right gear? *



Problem 16.4.20: Two gears with end loads.

16.4.21 A bevel gear system. A bevel-type gear system, shown in the figure, is used to transmit power between two shafts that are perpendicular to each other. The driving gear has a mean radius of 50 mm and rotates at a constant speed $\omega = 150\,\text{rpm}$. The mean radius of the driven gear is 80 mm and the driven shaft is expected to deliver a torque of $M_{out} = 25\,\text{N·m}$. Assuming no power loss, find the input torque supplied by the driving shaft. *



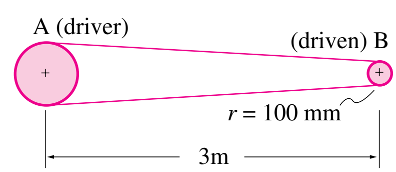
Problem 16.4.21: A bevel gear

16.4.22 *Belt drives* are used to transmit power between parallel shafts. Two parallel shafts, 3 m apart, are connected by a belt passing over the pulleys A and B fixed to the two shafts. The driver pulley A rotates at a constant 200 rpm. The speed ratio between the pulleys A and B is 1:2.5. The input torque is 350 N·m. Assume no loss of power between the two shafts.

a) Find the input power. *

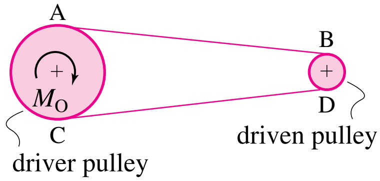
b) Find the rotational speed of the driven pulley B *.

c) Find the output torque at B. *



Problem 16.4.22

16.4.23 In the belt drive system shown, assume that the driver pulley rotates at a constant angular speed ω. If the motor applies a constant torque M_O on the driver pulley, show that the tensions in the two parts, AB and CD, of the belt must be different. Which part has a greater tension? Does your conclusion about unequal tension depend on whether the pulley is massless or not? Assume any dimensions you need. *



Problem 16.4.23

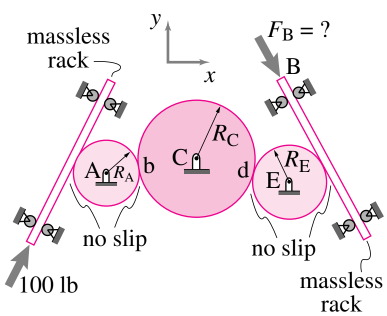
16.4.24 Two racks connected by three constant rate gears. A 100 lbf force is applied to one rack. At the output, the machinery (not shown) applies a force of F_B to the other rack.

a) Assume the gear-train is spinning at constant rate and is frictionless. What is F_B? *

b) If the gear bearings had friction would that increase or decrease F_B to achieve the same constant rate?

c) If instead of applying a 100 lbf to the left rack it is driven by a motor (not shown) at constant speed v, what is the speed of the right rack? *

d) If the angular velocity of the gear is increasing at rate α does this increase or decrease F_B at the given ω.



Problem 16.4.24: Two racks connected by three gears.

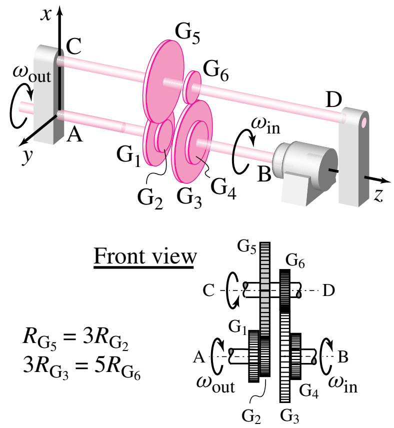
16.4.25 -3-D accelerating gear train. This is really a 2-D problem; each gear turns in a different parallel plane. Shaft B is rigidly connected to gears G_4 and G_5. G_3 meshes with gear G_6. Gears G_6 and G_5 are both rigidly attached to shaft AD. Gear G_5 meshes with G_2 which is welded to shaft A. Shaft A and shaft B spin independently. Assume you know the torque M_{input}, angular velocity ω_{input} and the angular acceleration α_{input} of the input shaft. Assume the bearings and contacts are frictionless.

a) What is the input power? *

b) What is the output power? *

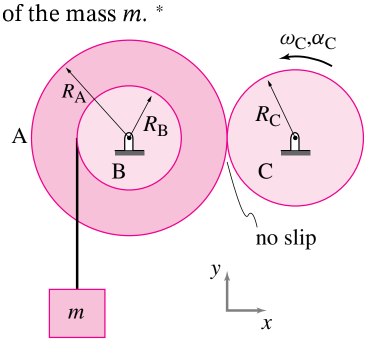
c) What is the angular velocity ω_{output} of the output shaft? *

d) What is the output torque M_{output}? *



Problem 16.4.25: A 3-D gear train.

16.4.26 Gear A with radius R_A = 400 mm is rigidly connected to a drum B with radius R_B = 200 mm. The combined moment of inertia of the gear and the drum about the axis of rotation is I_{zz} = 0.5 kg · m^2. Gear A is driven by gear C which has radius R_C = 300 mm. As the drum rotates, a 5 kg mass m is pulled up by a string wrapped around the drum. At the instant of interest, The angular speed and angular acceleration of the driving gear are 60 rpm and 12 rpm/s, respectively. Find the acceleration of the mass m. *

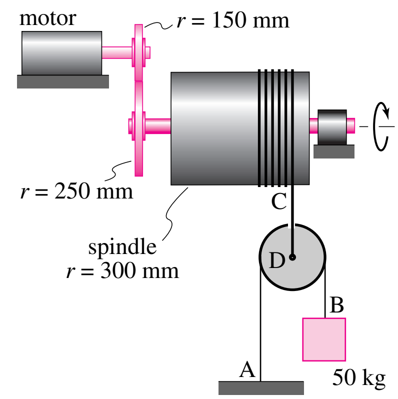


Problem 16.4.26

16.4.27 A spindle and pulley arrangement is used to hoist a 50 kg mass as shown in the figure. Assume that the pulley is to be of negligible mass. When the motor is running at a constant 100 rpm,

a) Find the velocity of the mass at B. *

b) Find the tension in strings AB and CD. *



Problem 16.4.27

16.4.28 Accelerating rack and pinion. The two gears shown are welded together and spin on a frictionless bearing. The inner gear has radius 0.5 m and negligible mass. The outer disk has 1 m radius and a uniformly distributed mass of 0.2 kg. They are loaded as shown with the force F = 20 N on the massless rack which is held in place by massless frictionless rollers. At the time of interest the angular velocity is ω = 2 rad/s (though ω is not constant). The point P is on the disk a distance 1 m from the center. At the time of interest, point P is on the positive y axis.

a) What is the speed of point P? *

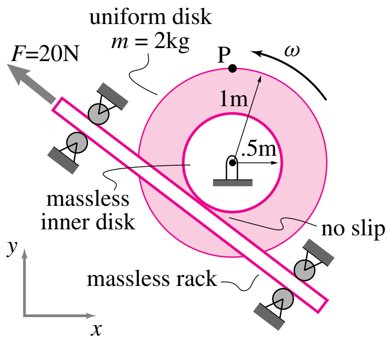
b) What is the velocity of point P? *

c) What is the angular acceleration α of the gear? *

d) What is the acceleration of point P? *

e) What is the magnitude of the acceleration of point P? *

f) What is the rate of increase of the speed of point P? *



Problem 16.4.28: Accelerating rack and pinion

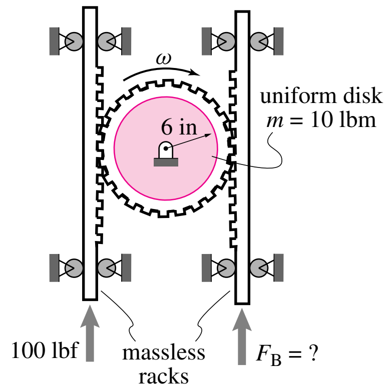
16.4.29 Two racks connected by a gear. A 100 lbf force is applied to one rack. At the output the machinery (not shown) applies a constant force F_B to the other rack.

a) Assume the gear is spinning at constant rate and is frictionless. What is F_B? *

b) If the gear bearing had friction, would that increase or decrease F_B to achieve the same constant rate? *

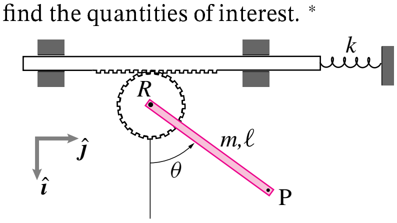
c) If the angular velocity of the gear is increasing at rate α, does this increase or decrease F_B at the given ω. *

d) If the output load F_B is given then the motion of the machine can be found, assuming friction is negligible. Assume that the machine starts from rest with a given output load. So long as rack B moves down and the output force F_B is positive, the output power is positive. The machine starts from rest and runs for a fixed amount of time T. For what value of F_B is the output work maximum? Once you find an answer, can you find an intuitive explanation for the answer? *



Problem 16.4.29: Two racks connected by a gear.

16.4.30 A rack and pinion constrained by a linear spring. Neglect gravity. The spring is relaxed when the angle $\theta = 0$. Assume the system is released from rest at $\theta = \theta_0$. What is the acceleration of the point P at the end of the stick when $\theta = 0$? Answer in terms of any or all of $m, R, \ell, \theta_0, k, \hat{\imath}$, and $\hat{\jmath}$. [Hint: There are several steps of reasoning required. You might want to draw FBD(s), use angular momentum balance, set up a differential equation, solve it, plug values into this solution, and use the result to find the quantities of interest. *



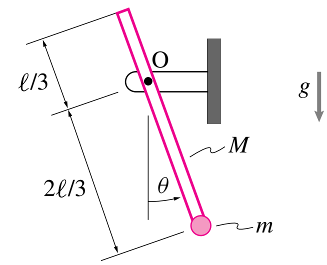
Problem 16.4.30

Pendulum Problems

16.4.31 A small particle of mass m is attached to the end of a thin rod of mass M (uniformly distributed), which is pinned at hinge O, as depicted in the figure.

a) Obtain the equation of motion governing the rotation θ of the rod. *

b) What is the natural frequency of the system for small oscillations θ? *



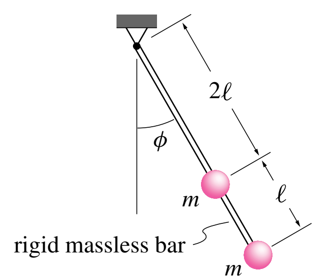
Problem 16.4.31

16.4.32 A rigid massless rod has two equal masses m_B and m_C ($m_B = m_C = m$) attached to it at distances 2ℓ and 3ℓ, respectively, measured along the rod from a frictionless hinge located at a point A. The rod swings freely from the hinge. There is gravity. Let ϕ denote the angle of the rod measured from the vertical. Assume that ϕ and $\dot{\phi}$ are known at the instant of interest.

a) What is $\ddot{\phi}$? Find $\ddot{\phi}$ in terms of m, ℓ, g, ϕ and $\dot{\phi}$. *

b) What is the force of the hinge on the rod? Solve in terms of m, ℓ, g, ϕ, $\dot{\phi}$, $\ddot{\phi}$ and any unit vectors you may need to define. *

c) Would you get the same answers if you put a mass $2m$ at 2.5ℓ? Why or why not? *



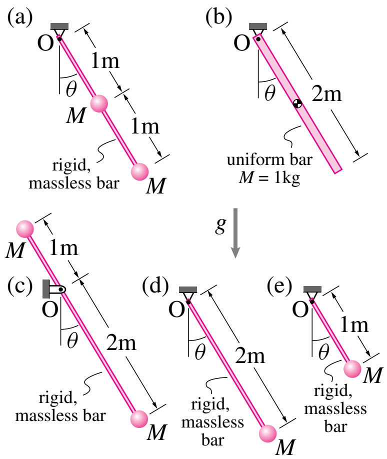
Problem 16.4.32

16.4.33 For the pendula in the figure :

a) Without doing any calculations, try to figure out the relative durations of the periods of oscillation for the five pendula (i.e. the order, slowest to fastest) Assume small angles of oscillation.

b) Calculate the period of small oscillations. [Hint: use balance of angular momentum about the point 0]. *

c) Rank the relative duration of oscillations and compare to your intuitive solution in part (a), and explain in words why things work the way they do.



Problem 16.4.33

16.4.34 A pegged compound pendulum. A uniform bar of mass m and length ℓ hangs from a peg at point C and swings in the vertical plane about an axis passing through the peg. The distance d from the center of mass of the rod to the peg can be changed by putting the peg at some other point along the length of the rod.

a) Find the angular momentum of the rod about point C. *

b) Find the rate of change of angular momentum of the rod about C. *

c) How does the period of the pendulum vary with d? Show the variation by plotting the period against $\frac{d}{\ell}$. [Hint, you must first find the equations of motion, linearize for small θ, and then solve.] *

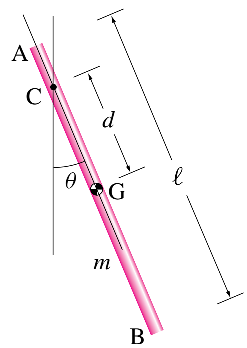
d) Find the total energy of the rod (using point C as a datum for potential energy). *

e) Find $\ddot{\theta}$ when $\theta = \pi/6$. *

f) Find the reaction force on the rod at C, as a function of m, d, ℓ, θ, and $\dot{\theta}$. *

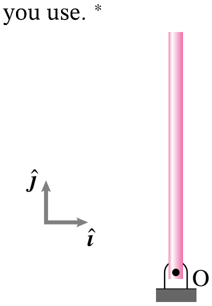
g) For the given rod, what should be the value of d (in terms of ℓ) in order to have the fastest pendulum? *

h) **Test of Schuler's pendulum.** The pendulum with the value of d obtained in (g) is called the Schuler's pendulum. It is not only the fastest pendulum but also the "most accurate pendulum". The claim is that even if d changes slightly over time due to wear at the support point, the period of the pendulum does not change much. Verify this claim by calculating the percent error in the time period of a pendulum of length $\ell = 1\,\text{m}$ under the following three conditions: (i) initial $d = 0.15\,\text{m}$ and after some wear $d = 0.16\,\text{m}$, (ii) initial $d = 0.29\,\text{m}$ and after some wear $d = 0.30\,\text{m}$, and (iii) initial $d = 0.45\,\text{m}$ and after some wear $d = 0.46\,\text{m}$. Which pendulum shows the least error in its time period? What is the connection between this result and the plot obtained in (c)? *



Problem 16.4.34

16.4.35 A uniform stick of length ℓ and mass m is a hair away from vertically up position when it is released with no angular velocity (a 'hair' is a technical word that means 'very small amount, zero for some purposes'). It falls to the right. What is the force on the stick at point O when the stick is horizontal. Solve in terms of ℓ, m, g, $\hat{\imath}$, and $\hat{\jmath}$. Carefully define any coordinates, base vectors, or angles that you use. *



Problem 16.4.35

16.4.36 A massless 10 meter long bar is supported by a frictionless hinge at one end and has a 3.759 kg point mass at the other end. It is released at $t = 0$ from a tip angle of $\phi = .02$ radians measured from vertically upright position (hinge at the bottom). Use $g = 10\,\text{m}\,\text{s}^{-2}$.

a) Using a small angle approximation and the solution to the resulting linear differential equation, find the angle of tip at $t = 1\text{s}$ and $t = 7\text{s}$. Use a calculator, not a numerical integrator. *

b) Using numerical integration of the non-linear differential equation for an inverted pendulum find ϕ at $t = 1\text{s}$ and $t = 7\text{s}$.

c) Make a plot of the angle versus time for your numerical solution. Include on the same plot the angle versus time from the approximate linear solution from part (a).

d) Comment on the similarities and differences in your plots.

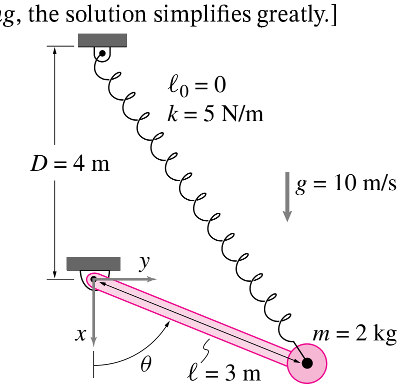
16.4.37 A zero length spring (relaxed length $\ell_0 = 0$) with stiffness $k = 5\,\text{N/m}$ supports the pendulum shown.

a) Find $\ddot{\theta}$ assuming $\dot{\theta} = 2\,\text{rad/s}$, $\theta = \pi/2$. *

b) Find $\ddot{\theta}$ as a function of $\dot{\theta}$ and θ (and k, ℓ, m, and g.) *

[Hint: use vectors (otherwise it's hard)]

[Hint: For the special case, $kD = mg$, the solution simplifies greatly.]



Problem 16.4.37

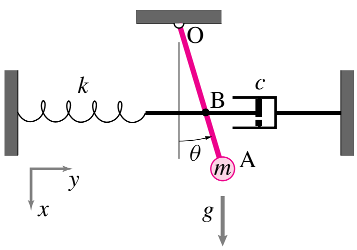
16.4.38 A spring-mass-damper system is depicted in the figure. The horizontal damping force applied at B is given by $F_D = -c\dot{y}_B$
The dimensions and parameters are as follows:

$$\begin{aligned} r_{B/0} &= 2ft \\ r_{A/0} &= \ell = 3ft \\ k &= 2\,\text{lbf}/\text{ft} \\ c &= 0.3\,\text{lbf}\cdot\text{s}/\text{ft} \end{aligned}$$

For small θ, assume that $\sin(\theta) \approx \theta$ and $\cos(\theta) \approx 1$.

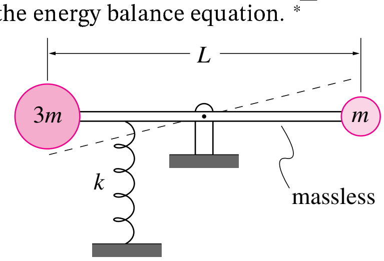
a) Determine the natural circular frequency of small oscillations about equilibrium for the pendulum shown. The static equilibrium position is $\theta = 0$ (pendulum hanging vertically), so the spring is at its rest point in this position. Idealize the pendulum as a point mass attached to a rigid massless rod of length 1, so $I^O = m\ell^2$. Also use the "small angle approximation" where appropriate. *

b) Sketch a graph of θ as a function of $t (t \geq 0)$ if the pendulum is released from rest at position $\theta = 0.2\,\text{rad}$ when $t = 0$. Your graph should show the correct qualitative behavior, but calculations are not necessary.



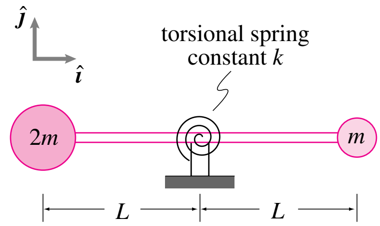
Problem 16.4.38

16.4.39 The asymmetric dumbbell shown in the figure is pivoted in the center and also attached to a spring at one quarter of its length from the bigger mass. When the bar is horizontal, the compression in the spring is y_s. At the instant of interest, the bar is at an angle θ from the horizontal; θ is small enough so that $y \approx \frac{L}{2}\theta$. If, at this position, the velocity of mass 'm' is $v\,\hat{\boldsymbol{\jmath}}$ and that of mass $3m$ is $-v\,\hat{\boldsymbol{\jmath}}$, evaluate the power term $(\sum \vec{F} \cdot \vec{v})$ in the energy balance equation. *



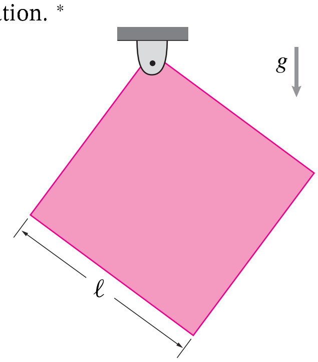
Problem 16.4.39

16.4.40 The dumbbell shown in the figure has a torsional spring with spring constant k (torsional stiffness units are $\frac{\text{kN}\cdot\text{m}}{\text{rad}}$). The dumbbell oscillates about the horizontal position with small amplitude θ. At an instant when the angular velocity of the bar is $\dot{\theta}\hat{\boldsymbol{k}}$, the velocity of the left mass is $-L\dot{\theta}\,\hat{\boldsymbol{\jmath}}$ and that of the right mass is $L\dot{\theta}\,\hat{\boldsymbol{\jmath}}$. Find the expression for the power P of the spring on the dumbbell at the instant of interest. *



Problem 16.4.40

16.4.41 A square plate with side ℓ and mass m is hinged at one corner in a gravitational field g. Find the period of small oscillation. *



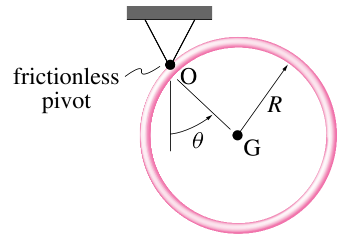
Problem 16.4.41

16.4.42 A thin hoop of radius R and mass M is hung from a point on its edge and swings in its plane. Assuming it swings near to the position where its center of mass G is below the hinge:

a) What is the period of its swinging oscillations? *

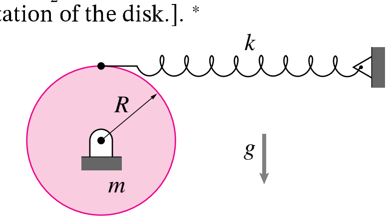
b) If, instead, the hoop was set to swinging in and out of the plane would the period of oscillations be greater or less? *

c) What length ℓ of simple pendulum (a massless rod with a point mass at the end) has the same period of oscillation of this hoop, swinging in its plane? *



Problem 16.4.42

16.4.43 Oscillating disk. A uniform disk with mass m and radius R pivots around a frictionless hinge at its center. It is attached to a massless spring which is horizontal and relaxed when the attachment point is directly above the center of the disk. Assume small rotations and the consequent geometrical simplifications. Assume the spring can carry compression. What is the period of oscillation of the disk if it is disturbed from its equilibrium configuration? [You may use the fact that, for the disk shown, $\dot{\vec{H}}_{/O} = \frac{1}{2}mR^2\ddot{\theta}\hat{\boldsymbol{k}}$, where θ is the angle of rotation of the disk.]. *



Problem 16.4.43

16.4.44 Robotics problem: Simplest balancing of an inverted pendulum. You are holding a stick upside down, one end is in your hand, the other end sticking up. To simplify things, think of the stick as massless but with a point mass at the upper end. Also, imagine that it is only a two-dimensional problem (either you can ignore one direction of falling for simplicity or imagine wire guides that keep the stick from moving in and out of the plane of the paper on which you draw the problem).

You note that if you model your holding the stick as just having a stationary hinge then you get $\ddot{\phi} = \frac{g}{l}\sin\phi$. Assuming small angles, this hinge leads to exponentially growing solutions. Upside-down sticks fall over. How can you prevent this falling?

One way to do keep the stick from falling over is to firmly grab it with your hand, and if the stick tips, apply a torque in order to right it. This corrective torque is (roughly) how your ankles keep you balanced when you stand upright. Your task in this assignment is to design a robot that keeps an inverted pendulum balanced by applying appropriate torque.

Your model is: Inverted pendulum, length ℓ, point mass m, and a hinge at the bottom with a motor that can apply a torque T_m. The stick might be tipped an angle ϕ from the vertical. A *horizontal* disturbing force $F(t)$ is applied to the mass (representing wind, annoying friends, etc).

a) Draw a picture and a FBD

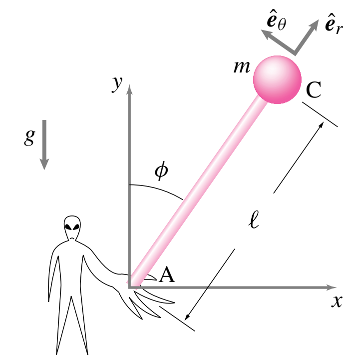
b) Write the equation for angular momentum balance about the hinge point. *

c) Imagine that your robot can sense the angle of tip ϕ and its rate of change $\dot{\phi}$ and can apply a torque in response to that sensing. That is you can make T_m any function of ϕ and $\dot{\phi}$ that you want. Can you find a function that will make the pendulum stay upright? Make a guess (you will test it below).

d) Test your guess the following way: plug it into the equation of motion from part (b), linearize the equation, assume the disturbing force is zero, and see if the solution of the differential equation has exponentially growing (i.e. unstable) solutions. Go back to (c) if it does and find a control strategy that works.

e) Pick numbers and model your system on a computer using the full non-linear equations. Use initial conditions both close to and far from the upright position and plot ϕ versus time.

f) If you are ambitious, pick a non-zero forcing function $F(t)$ (say a sine wave of some frequency and amplitude) and see how that affects the stability of the solution in your simulations.

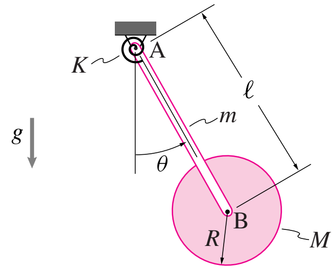


Problem 16.4.44

16.4.45 A thin rod of mass m and length ℓ is hinged with a torsional spring of stiffness K at A, and is connected to a thin disk of mass M and radius R at B. The spring is uncoiled when $\theta = 0$. Determine the natural frequency ω_n of the system for small oscillations θ, assuming that the disk is:

a) welded to the rod, and *

b) pinned frictionlessly to the rod. *



Problem 16.4.45

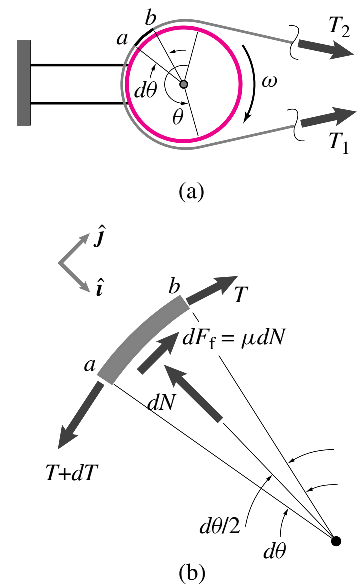
Mixed Problems

16.4.46 Assume that the pulley shown in figure(a) rotates at a constant speed ω. Let the angle of contact between the belt and pulley surface be θ. Assume that the belt is massless and that the condition of impending slip exists between the pulley and the belt. The free-body diagram of an infinitesimal section ab of the belt is shown in figure(b).

a) Write the equations of linear momentum balance for section ab of the belt in the $\hat{\imath}$ and $\hat{\jmath}$ directions.

b) Eliminate the normal force N from the two equations in part (a) and get a differential equation for the tension T in terms of the coefficient of friction μ and The contact angle θ. *

c) Show that the solution to the equation in part (b) satisfies $\frac{T_1}{T_2} = e^{\mu\theta}$, where T_1 and T_2 are the tensions in the lower and the upper segments of the belt, respectively.



Problem 16.4.46

16.4.47 A belt drive is required to transmit 15 kW power from a 750 mm diameter pulley rotating at a constant 300 rpm to a 500 mm diameter pulley. The centers of the pulleys are located 2.5 m apart. The coefficient of friction between the belt and pulleys is $\mu = 0.2$.

a) (See problem 16.4.46.) Draw a neat diagram of the pulleys and the belt-drive system and find the *angle of lap*, the contact angle θ, of the belt on the driver pulley.

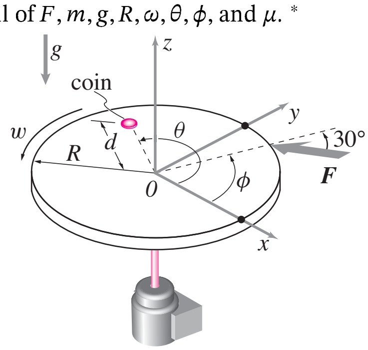
b) Find the rotational speed of the driven pulley. *

c) (See the figure in problem 16.4.46.) The power transmitted by the belt is given by *power* = *net tension* × *belt speed*, i.e., $P = (T_1 - T_2)v$, where v is the linear speed of the belt. Find the maximum tension in the belt. [Hint: $\frac{T_1}{T_2} = e^{\mu\theta}$ (see problem 16.4.46).] *

d) The belt in use has a 15 mm × 5 mm rectangular cross-section. Find the maximum tensile stress in the belt. *

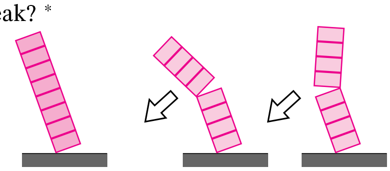
16.4.48 Slippery money A round uniform flat horizontal platform with radius R and mass m is mounted on frictionless bearings with a vertical axis at 0. At the instant of interest it is rotating counter clockwise (looking down) with angular velocity $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. A force in the xy plane with magnitude F is applied at the perimeter at an angle of 30° from the radial direction. The force is applied at a location that is ϕ from the fixed positive x axis. At the instant of interest a small coin sits on a radial line that is an angle θ from the fixed positive x axis (with mass much much smaller than m). Gravity presses it down, the platform holds it up, and friction (coefficient=μ) keeps it from sliding.

Find the biggest value of d for which the coin does not slide in terms of some or all of $F, m, g, R, \omega, \theta, \phi$, and μ. *



Problem 16.4.48

16.4.49 Frequently parents will build a tower of blocks for their children. Just as frequently, kids knock them down. In falling (even when they start to topple aligned), these towers invariably break in two (or more) pieces at some point along their length. Why does this breaking occur? What condition is satisfied at the point of the break? Will the stack bend towards or away from the floor after the break? *



Problem 16.4.49