16.3 Polar moment of inertia

All of the basic equations of Dynamics concern the motion of mass. In particular, linear momentum, angular momentum, and energy are all expressed as sums over all the bits of mass in a system, with each bit of mass multiplied by some terms concerning position, velocity and acceleration. From the earlier sections in this chapter we know how to find the velocity and acceleration of every bit of mass on a 2-D rigid object as it spins about a fixed axis. So it is just a matter of doing integrals or sums to calculate the various momentum and energy quantities of interest. As an object moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid object one has to evaluate a different integral or sum at every different configuration. But there is a shortcut: for a rotating rigid object a sum (over all atoms, say), or a difficult integral (for example, over the complex region representing a machine part) is reduced to simple multiplication.

The *moment of inertia* I^{cm}①simplifies the expressions for the angular momentum, the rate of change of angular momentum, and the energy of a rigid object. For more general motions the shortcuts need a 3×3 matrix $[\boldsymbol{I}^{\text{cm}}]$ But for 2D mechanics only one component of the matrix $[\boldsymbol{I}^{\text{cm}}]$ is relevant, it is I_{zz}^{cm}, called just I or J for short.

Here are the main results. A flat object spinning with $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ in the xy plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia I^{cm} so that:

$$\vec{\boldsymbol{H}}_{\text{cm}} = I^{\text{cm}}\omega\hat{\boldsymbol{k}} \qquad (16.31)$$

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \qquad (16.32)$$

$$E_{\text{K}/cm} = \frac{1}{2}\omega^2 I^{\text{cm}}. \qquad (16.33)$$

There are two main skills you need to develop associated with I.

- Using the formulas above properly in angular momentum and energy balance. This is covered in sec. 16.4.
- Finding I of an object, as is covered in most freshman calculus texts and is reviewed in this section.

Some facts about moment of inertia are summarized on the inside back cover.

The moments of inertia in 2-D : $[\boldsymbol{I}^{\text{cm}}]$ and $[\boldsymbol{I}^{\text{O}}]$.

The definition of moment of inertia② I^{cm} is

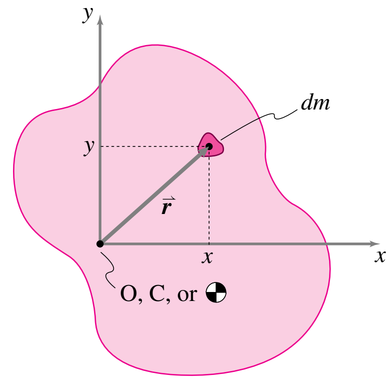


Figure 16.35: A bit of mass dm on a general planar object.

①**The COM is the most common reference point.** The moment of inertia of a given object depends on what reference point is used. Most often when people say 'the' moment of inertia they mean the moment of inertia with respect to the center-of-mass. For clarity, this moment of inertia is often notated as I^{cm} in this book. If a different reference point, say point O is used, the moment of inertia is notated as I^{0}.

②**What do the words 'moment of inertia' mean?** The word inertia, in this case, can be translated to mean mass. So it's 'moment of mass'. The word 'moment' is similar to the word 'moment' as in torque, or (force)·(distance), but slightly more general. The more general word moment is *'the product of a quantity and its distance, raised to a power, from a reference point,'*

$$\text{the n'th moment of a quantity rel to 0} = \int x^n(\text{quantity at } x)\,dx.$$

So, what we used to call 'moment' (meaning torque) we can think of as short for 'the first moment of perpendicular force'. And the moment of inertia is 'the second moment of mass'. The words 'first' and 'second' refer to the exponent n.

$$I^{cm} \equiv \int \underbrace{(x^2 + y^2)}_{r^2} \, dm$$

$$= \int \int r^2 \overbrace{\underbrace{\left(\frac{m_{\text{tot}}}{A}\right)}_{\text{The mass per unit area.}} dA}^{dm} \qquad \text{for a uniform planar object}$$

where $x = x_{/\text{cm}}$, $y = y_{/\text{cm}}$ and $r = r_{/\text{cm}}$ are the distances in the x and y directions of the bits of mass (dm) from the center of mass and $r = r_{/cm}$ is the total distance. Similarly if all distances are measured relative to a point C or O (instead of relative to the center of mass) then similar formulas as above calculate I^C or I^O. See box 16.7 on page 5 for the calculation of the moments of inertia of some simple objects.

The term $I^{\text{cm}} = I^{\text{cm}}_{zz}$ is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the I_{xx} and I_{yy}, terms which have little utility in planar dynamics (I_{xx} and I_{yy}, as area inertias rather than mass inertias, are all-important when calculating the bending stiffness or the stress in elastic beams, however).

What, physically, is the polar moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to I in proportion to the square of its distance from the reference point. As we will show in sec. 16.4 on page 791 (see eqn. (16.36) on page 793) I^{cm} is just the quantity we need to do mechanics problems.

Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia I^{cm}, is the *radius of gyration*, r_{gyr} ③. The radius of gyration is defined as:

$$r_{\text{gyr}} \equiv \sqrt{I/m} \quad \Rightarrow \quad r_{\text{gyr}}^2 m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same I and the same mass as the given object. It is an equivalent single radius for the whole mass distribution ④.

Example: Radius of gyration of a hoop
The radius of gyration about its center of a circular hoop of mass m and radius R about its center is $r_{\text{gyr}} = R$.

Other reference points

For the most part it is I^{cm} which is of primary interest. Other reference points are useful ⑤

③ k **for stiffness.** The radius of gyration r_{gyr} is sometimes called k but we save k for stiffness in this book.

④ **Radius of gyration and average distance of mass.** The radius of gyration is an equivalent radius of the collection of mass. But the radius of gyration is not exactly the average radius of the distributed mass. Rather the radius of gyration is the square root of the mean square radius:

$$r_{\text{gyr}} \equiv \sqrt{\frac{\int r^2 \, dm}{m_{\text{tot}}}}.$$

For the special cases where all the mass is at the same radius R (as for a circular hoop or for masses distributed on the vertices of, say, a regular pentagon) the radius of gyration is R. For all other cases the radius of gyration is larger than the average radius of the mass.

⑤ **Notation: simple vs precise.** When just the symbol I is used one assumes $I = I^{\text{cm}}_{zz}$, and if just I^{cm} is used, one assumes $I^{\text{cm}} = I^{\text{cm}}_{zz}$. Similarly I^0 and I^{C} are assumed to mean I^0_{zz} and I^{C}_{zz}, respectively.

1. If the rigid object is hinged at a fixed point O then I^0 can be used in a slight short cut in calculation of angular momentum and energy; and
2. If one wants to calculate the moment of inertia of a composite object about its center-of-mass then it is useful to first find the moment of inertia of each of its parts about that system's center of mass (which is generally not the center-of-mass of any of the separate parts).
3. Sometimes it is easiest to set up the integral for moment of inertia about a special point C that is not the center of mass and then use the parallel axis theorem (eqn. (16.34)) to find the moment of inertia about the center of mass (see, for example the semi-circle example on 4).

The parallel axis theorem

The planar parallel axis theorem is the equation

$$I_{zz}^{C} = I_{zz}^{cm} + m_{\text{tot}}\,\underbrace{r_{cm/C}^{2}}_{d^2}\,. \tag{16.34}$$

In this equation $d = r_{cm/C}$ is the distance from the center-of-mass to a line parallel to the z-axis which passes through point C. See box 16.6 on page 8 for a derivation of the parallel axis theorem for planar objects.

Note that $I_{zz}^{C} \geq I_{zz}^{cm}$, always.

Moment of inertia of complex objects. One can calculate the moment of inertia of a composite object about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of m_i is (x_i, y_i) relative to a fixed origin, and the moment of inertia of that part about its center of mass is I_i. We can then find the moment of inertia of the composite I_{tot} about its center-of-mass (x_{cm}, y_{cm}) by the following sequence of calculations, in order:

$$\begin{array}{llcl}
(1) & m_{\text{tot}} & = & \sum m_i \\
(2) & x_{cm} & = & \left[\sum x_i m_i\right]/m_{\text{tot}} \\
 & y_{cm} & = & \left[\sum y_i m_i\right]/m_{\text{tot}} \\
(3) & d_i^2 & = & (x_i - x_{cm})^2 + (y_i - y_{cm})^2 \\
(4) & I_{tot} & = & \sum\left[I_i^{cm} + m_i d_i^2\right].
\end{array}$$

You can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculations in about the order prescribed above in any case. This sequence of steps lends itself naturally to a computer spread sheet or to any program that deals easily with arrays of numbers. This is laid out for the similar center-of-mass calculation on page 157 [6].

[6] The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find the polar mass moment of inertia of a strange cutout shape.

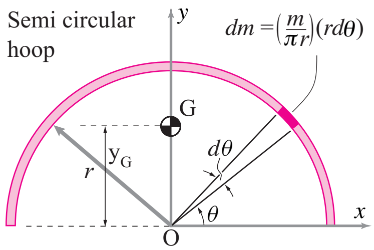


Figure 16.36: The moment of inertia of a semi-circle about the COM is calculated by calculating about point O and then using the parallel axis theorem.

Example: Inertia of a semicircle about its COM.
The moment of inertia of a semicircle with mass m and radius r about O (see fig. 16.36) is

$$I^0 = \int r_{/0}^2 \, dm = mr^2.$$

From the example on page 153 the center of mass is at $y_G = \frac{2}{\pi} r$. By the parallel axis theorem

$$I^0 = I^{cm} + m y_G^2$$

which we can solve for I^{cm} to get

$$I^{cm} = I^0 - m y_G^2 = mr^2 - \frac{4}{\pi^2} mr^2 = \left(1 - \frac{4}{\pi^2}\right) mr^2 \approx 0.6\, mr^2 \, .$$

The radius of gyration is that radius ρ_{gyr} so that $m\rho_{gyr}^2 = I^{cm}$ and is

$$\rho_{gyr} = \sqrt{1 - \frac{4}{\pi^2}} \; r.$$

The perpendicular axis theorem for planar rigid bodies

The perpendicular axis theorem for planar objects is the equation

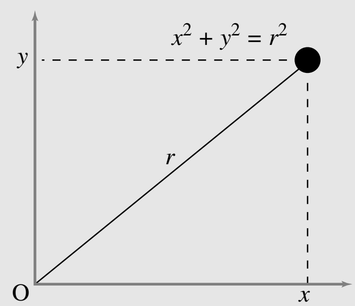
$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 16.6 on page 8. It gives the ‘polar’ inertia I_{zz} in terms of the inertias I_{xx} and I_{yy}. Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

16.7 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I^o_{zz} = \int r^2_{/o}\, dm$$

reduces to the sum

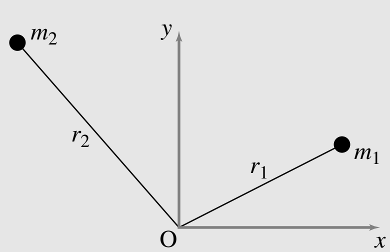
$$I^o_{zz} = \sum r^2_{i/o} m_i$$

which reduces to one term if there is only one mass,

$$I^o_{zz} = r^2 m = (x^2 + y^2)m.$$

So, if $x = 3\,\in$, $y = 4\,\in$, and $m = 0.1\,\text{lbm}$, then $I^o_{zz} = 2.5\,\text{lbm}\,\in^2$. Note that, in this case, $I^{cm}_{zz} = 0$ since the radius from the center-of-mass to the center-of-mass is zero.

Two point masses

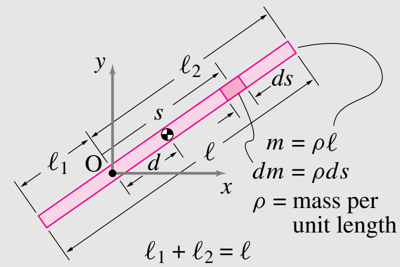


In this case, the sum that defines I^o_{zz} reduces to two terms, so

$$I^o_{zz} = \sum r^2_{i/o} m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if $r_1 = r_2 = r$, then $I^o_{zz} = m_{tot} r^2$.

A thin uniform rod



Consider a thin rod with uniform mass density, ρ, per unit length, and length ℓ. We calculate I^o_{zz} as

$$\begin{aligned}
I^o_{zz} &= \int r^2 \overbrace{dm}^{\rho ds} \\
&= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\
&= \frac{1}{3}\rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad (\text{since} \quad \rho \equiv const.) \\
&= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3).
\end{aligned}$$

If either $\ell_1 = 0$ or $\ell_2 = 0$, then this expression reduces to $I^o_{zz} = \frac{1}{3}m\ell^2$. If $\ell_1 = \ell_2$, then O is at the center-of-mass and

$$I^o_{zz} = I^{cm}_{zz} = \frac{1}{3}\rho\left(\left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3\right) = \frac{ml^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for I^o_{zz} can be arranged as follows:

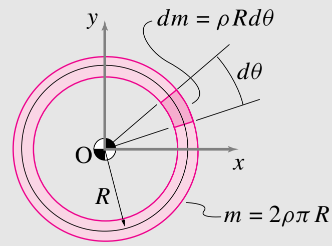
$$\begin{aligned}
I^0_{zz} &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3) \\
&= \underbrace{\rho(\ell_1 + \ell_2)}_{m}\left(\underbrace{\frac{\ell_2 - \ell_1}{2}}_{d}\right)^2 + \underbrace{\rho\frac{(\ell_1 + \ell_2)^3}{12}}_{m\ell^2/12} \\
&= md^2 + m\frac{\ell^2}{12} \\
&= md^2 + I^{cm}_{zz}
\end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center-of-mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

(continued...)

16.7 Some examples of 2-D Moment of Inertia (continued)

A uniform hoop



For a hoop of uniform mass density, ρ, per unit length, we might consider all of the points to have the same radius R. So,

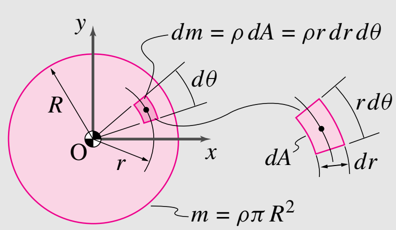
$$I^o_{zz} = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned} I^o_{zz} &= \int r^2 dm \\ &= \int_0^{2\pi} R^2 \rho R d\theta \\ &= \rho R^3 \int_0^{2\pi} d\theta \\ &= 2\pi\rho R^3 = \underbrace{(2\pi\rho R)}_{m} R^2 = mR^2. \end{aligned}$$

This I^o_{zz} is the same as for a single point mass m at a distance R from the origin O. It is also the same as for two point masses if they both are a distance R from the origin. For the hoop, however, O is at the center-of-mass so $I^o_{zz} = I^{cm}_{zz}$ which is not the case for a single point mass.

A uniform disk

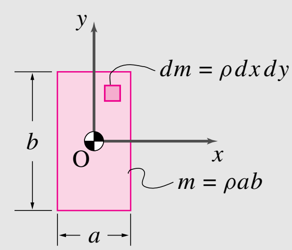


Assume the disk has uniform mass density, ρ, per unit area. For a uniform disk centered at the origin, the center-of-mass is at the origin so

$$\begin{aligned} I^o_{zz} = I^{cm}_{zz} &= \int r^2 dm \\ &= \int_0^R \int_0^{2\pi} r^2 \rho r d\theta dr \\ &= \int_0^R 2\pi\rho r^3 dr \\ &= 2\pi\rho \frac{r^4}{4}\Big|_0^R = \pi\rho\frac{R^4}{2} = (\pi\rho R^2)\frac{R^2}{2} \\ &= m\frac{R^2}{2}. \end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

Uniform rectangular plate



For the special case that the center of the plate is at point O, the center-of-mass is also at O and $I^o_{zz} = I^{cm}_{zz}$.

$$\begin{aligned} I^o_{zz} = I^{cm}_{zz} &= \int r^2 dm \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2+y^2)\overbrace{\rho dx dy}^{dm} \\ &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho\left(\frac{x^3}{3} + xy^2\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\ &= \rho\left(\frac{x^3 y}{3} + \frac{xy^3}{3}\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}}\Bigg|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\ &= \rho\left(\frac{a^3 b}{12} + \frac{ab^3}{12}\right) \\ &= \frac{m}{12}(a^2+b^2). \end{aligned}$$

Note that $\int r^2 dm = \int x^2 dm + \int y^2 dm$ for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle, $\int y^2 dm = \rho\int y^2 dA$. But the integral $y^2 dA$ is just the term often used for I, the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You

(continued...)

16.7 Some examples of 2-D Moment of Inertia (continued)

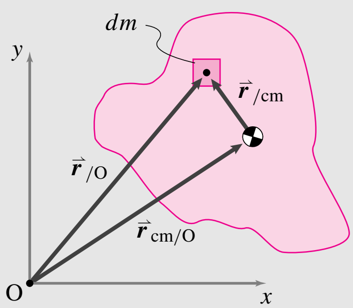
may recall that $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$ for a rectangle. Similarly, $\int x^2 dA = \frac{Aa^2}{12}$. So, the polar moment of inertia $J = I_{zz}^o = m\frac{1}{12}(a^2+b^2)$ can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

16.6 The 2-D parallel axis theorem and the perpendicular axis theorem

Sometimes one wants to know the moment of inertia relative to the center of mass. And, sometimes, if the object is held at a hinge joint at O, relative to that hinge point O There is a simple relation between these two moments of inertia known as *the parallel axis theorem*.

2-D parallel axis theorem

For the two-dimensional mechanics of two-dimensional objects, our only concern is I_{zz}^{o} and I_{zz}^{cm} and *not* the full moment of inertia matrix. In this case, $I_{zz}^{o} = \int r_{/o}^2 \, dm$ and $I_{zz}^{cm} = \int r_{/cm}^2 \, dm$. Now, let's prove the theorem in two dimensions referring to the figure.



$$
\begin{aligned}
I_{zz}^{O} &= \int r_{/O}^2 \, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int [(\underbrace{x_{cm/O} + x_{/cm}}_{x_{/O}})^2 + (\underbrace{y_{cm/O} + y_{/cm}}_{y_{/O}})^2] dm \\
&= \int [(x_{cm/O}^2 + 2x_{cm/O}x_{/cm} + x_{/cm}^2) + \\
&\qquad (y_{cm/O}^2 + 2y_{cm/O}y_{/cm} + y_{/cm}^2)] dm \\
&= (x_{cm/O}^2 + y_{cm/O}^2) \underbrace{\int dm}_{m} + 2x_{cm/O} \underbrace{\int x_{/cm} dm}_{0} + \\
&\qquad 2y_{cm/O} \underbrace{\int y_{/cm} dm}_{0} + \int (x_{/cm}^2 + y_{/cm}^2) dm \\
&= r_{cm/O}^2 m + \underbrace{\int (x_{/cm}^2 + y_{/cm}^2) dm}_{I_{zz}^{cm}} \\
&= I_{zz}^{cm} + \underbrace{r_{cm/O}^2}_{d^2} m
\end{aligned}
$$

The cancellation $\int y_{/\mathrm{cm}} \, dm = \int x_{/\mathrm{cm}} \, dm = 0$ comes from the definition of center of mass.

Sometimes, people write the parallel axis theorem more simply as

$$I^0 = I^{cm} + md^2 \qquad \text{or} \qquad J_O = J_{cm} + md^2$$

using the symbol J to mean I_{zz}. One thing to note about the parallel axis theorem is that the moment of inertia about any point O is always *greater* than the moment of inertia about the center of mass. For a given object, the minimum moment of inertia is about the center-of-mass.

Why the name *parallel axis theorem*? We use the name because the two I's calculated are the moments of inertia about two parallel axes (both in the z direction) through the two points cm and O.

One way to think about the theorem is the following. The moment of inertia of an object about a point O not at the center-of-mass is the same as that of the object about the cm *plus* that of a point mass located at the center-of-mass. If the distance from O to the cm is larger than the outer radius of the object, then the d^2m term is larger than I_{zz}^{cm}. The distance of equality of the two terms is the radius of gyration, r_{gyr}.

Perpendicular axis theorem (applies to planar objects only)

For planar objects,

$$
\begin{aligned}
I_{zz}^{O} &= \int |\vec{r}|^2 \, dm \\
&= \int (x_{/O}^2 + y_{/O}^2) dm \\
&= \int x_{/O}^2 dm + \int y_{/O}^2 dm \\
&= I_{yy}^{O} + I_{xx}^{O}
\end{aligned}
$$

Similarly,

$$I_{zz}^{cm} = I_{xx}^{cm} + I_{yy}^{cm}.$$

That is, the moment of inertia about the z-axis is the sum of the inertias about the two perpendicular axes x and y. Note that the objects must be planar ($z = 0$ everywhere) or the theorem would not be true. For example, $I_{xx}^{o} = \int (y_{/O}^2 + z_{/O}^2) dm \neq \int y_{/O}^2 dm$ for a three-dimensional object.