16.3 Polar moment of inertia

All of the basic equations of Dynamics concern the motion of mass. In particular, linear momentum, angular momentum, and energy are all expressed as sums over all the bits of mass in a system, with each bit of mass multiplied by some terms concerning position, velocity and acceleration. From the earlier sections in this chapter we know how to find the velocity and acceleration of every bit of mass on a 2-D rigid object as it spins about a fixed axis. So it is just a matter of doing integrals or sums to calculate the various momentum and energy quantities of interest. As an object moves and rotates the region of integration and the values of the integrands change. So, in principle, in order to analyze a rigid object one has to evaluate a different integral or sum at every different configuration. But there is a shortcut: for a rotating rigid object a sum (over all atoms, say), or a difficult integral (for example, over the complex region representing a machine part) is reduced to simple multiplication.

The *moment of inertia* I^{cm}①simplifies the expressions for the angular momentum, the rate of change of angular momentum, and the energy of a rigid object. For more general motions the shortcuts need a 3×3 matrix $[\boldsymbol{I}^{\text{cm}}]$ But for 2D mechanics only one component of the matrix $[\boldsymbol{I}^{\text{cm}}]$ is relevant, it is I_{zz}^{cm}, called just I or J for short.

Here are the main results. A flat object spinning with $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ in the xy plane has a mass distribution which gives, by means of a calculation which we will discuss shortly, a moment of inertia I^{cm} so that:

$$\vec{\boldsymbol{H}}_{\text{cm}} = I^{\text{cm}}\omega\hat{\boldsymbol{k}} \qquad (16.31)$$

$$\dot{\vec{\boldsymbol{H}}}_{\text{cm}} = I^{\text{cm}}\dot{\omega}\hat{\boldsymbol{k}} \qquad (16.32)$$

$$E_{\text{K}/cm} = \frac{1}{2}\omega^2 I^{\text{cm}}. \qquad (16.33)$$

There are two main skills you need to develop associated with I.

- Using the formulas above properly in angular momentum and energy balance. This is covered in sec. 16.4.
- Finding I of an object, as is covered in most freshman calculus texts and is reviewed in this section.

Some facts about moment of inertia are summarized on the inside back cover.

The moments of inertia in 2-D : $[\boldsymbol{I}^{\text{cm}}]$ and $[\boldsymbol{I}^{\text{O}}]$.

The definition of moment of inertia② I^{cm} is

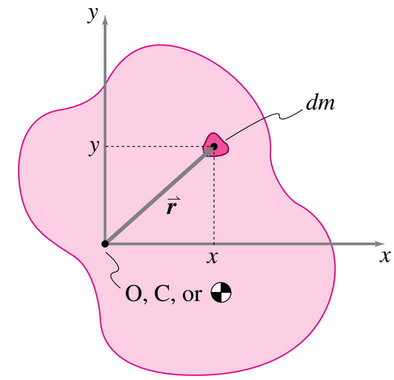


Figure 16.35: A bit of mass dm on a general planar object.

①**The COM is the most common reference point.** The moment of inertia of a given object depends on what reference point is used. Most often when people say 'the' moment of inertia they mean the moment of inertia with respect to the center-of-mass. For clarity, this moment of inertia is often notated as I^{cm} in this book. If a different reference point, say point O is used, the moment of inertia is notated as I^{O}.

②**What do the words 'moment of inertia' mean?** The word inertia, in this case, can be translated to mean mass. So it's 'moment of mass'. The word 'moment' is similar to the word 'moment' as in torque, or (force)·(distance), but slightly more general. The more general word moment is *'the product of a quantity and its distance, raised to a power, from a reference point,'*

$$\text{the n'th moment of a quantity rel to 0} = \int x^n (\text{quantity at } x)\, dx.$$

So, what we used to call 'moment' (meaning torque) we can think of as short for 'the first moment of perpendicular force'. And the moment of inertia is 'the second moment of mass'. The words 'first' and 'second' refer to the exponent n.

$$I^{cm} \equiv \int \underbrace{(x^2+y^2)}_{r^2}\, dm$$

$$= \int\int r^2 \overbrace{\underbrace{\left(\frac{m_{\text{tot}}}{A}\right)}_{\boxed{\text{The mass per unit area.}}} dA}^{dm} \qquad \text{for a uniform planar object}$$

where $x = x_{/\text{cm}}$, $y = y_{/\text{cm}}$ and $r = r_{/\text{cm}}$ are the distances in the x and y directions of the bits of mass (dm) from the center of mass and $r = r_{/cm}$ is the total distance. Similarly if all distances are measured relative to a point C or O (instead of relative to the center of mass) then similar formulas as above calculate I^C or I^O. See box 16.7 on page 5 for the calculation of the moments of inertia of some simple objects.

The term $I^{\text{cm}} = I^{\text{cm}}_{zz}$ is sometimes called the *polar moment of inertia*, or *polar mass moment of inertia* to distinguish it from the I_{xx} and I_{yy}, terms which have little utility in planar dynamics (I_{xx} and I_{yy}, as area inertias rather than mass inertias, are all-important when calculating the bending stiffness or the stress in elastic beams, however).

What, physically, is the polar moment of inertia? It is a measure of the extent to which mass is far from the given reference point. Every bit of mass contributes to I in proportion to the square of its distance from the reference point. As we will show in sec. 16.4 on page 791 (see eqn. (16.36) on page 793) I^{cm} is just the quantity we need to do mechanics problems.

Radius of gyration

Another measure of the extent to which mass is spread from the reference point, besides the moment of inertia I^{cm}, is the *radius of gyration*, r_{gyr} ③. The radius of gyration is defined as:

$$r_{\text{gyr}} \equiv \sqrt{I/m} \quad \Rightarrow \quad r^2_{\text{gyr}} m = I.$$

That is, the radius of gyration of an object is the radius of an equivalent ring of mass that has the same I and the same mass as the given object. It is an equivalent single radius for the whole mass distribution ④.

Example: Radius of gyration of a hoop
The radius of gyration about its center of a circular hoop of mass m and radius R about its center is $r_{\text{gyr}} = R$.

Other reference points

For the most part it is I^{cm} which is of primary interest. Other reference points are useful ⑤

③ k **for stiffness.** The radius of gyration r_{gyr} is sometimes called k but we save k for stiffness in this book.

④ **Radius of gyration and average distance of mass.** The radius of gyration is an equivalent radius of the collection of mass. But the radius of gyration is not exactly the average radius of the distributed mass. Rather the radius of gyration is the square root of the mean square radius:

$$r_{\text{gyr}} \equiv \sqrt{\frac{\int r^2\, dm}{m_{\text{tot}}}}.$$

For the special cases where all the mass is at the same radius R (as for a circular hoop or for masses distributed on the vertices of, say, a regular pentagon) the radius of gyration is R. For all other cases the radius of gyration is larger than the average radius of the mass.

⑤ **Notation: simple vs precise.** When just the symbol I is used one assumes $I = I^{\text{cm}}_{zz}$, and if just I^{cm} is used, one assumes $I^{\text{cm}} = I^{\text{cm}}_{zz}$. Similarly I^0 and I^{C} are assumed to mean I^0_{zz} and I^{C}_{zz}, respectively.

1. If the rigid object is hinged at a fixed point O then I^0 can be used in a slight short cut in calculation of angular momentum and energy; and
2. If one wants to calculate the moment of inertia of a composite object about its center-of-mass then it is useful to first find the moment of inertia of each of its parts about that system's center of mass (which is generally not the center-of-mass of any of the separate parts).
3. Sometimes it is easiest to set up the integral for moment of inertia about a special point C that is not the center of mass and then use the parallel axis theorem (eqn. (16.34)) to find the moment of inertia about the center of mass (see, for example the semi-circle example on 4).

The parallel axis theorem

The planar parallel axis theorem is the equation

$$I_{zz}^{C} = I_{zz}^{cm} + m_{\text{tot}}\,\underbrace{r_{cm/C}^{2}}_{d^2}. \tag{16.34}$$

In this equation $d = r_{cm/C}$ is the distance from the center-of-mass to a line parallel to the z-axis which passes through point C. See box 16.6 on page 8 for a derivation of the parallel axis theorem for planar objects.

Note that $I_{zz}^{C} \geq I_{zz}^{cm}$, always.

Moment of inertia of complex objects. One can calculate the moment of inertia of a composite object about its center of mass, in terms of the masses and moments of inertia of the separate parts. Say the position of the center of mass of m_i is (x_i, y_i) relative to a fixed origin, and the moment of inertia of that part about its center of mass is I_i. We can then find the moment of inertia of the composite I_{tot} about its center-of-mass (x_{cm}, y_{cm}) by the following sequence of calculations, in order:

$$\begin{array}{lrcl}
(1) & m_{\text{tot}} & = & \sum m_i \\
(2) & x_{cm} & = & \left[\sum x_i m_i\right]/m_{\text{tot}} \\
 & y_{cm} & = & \left[\sum y_i m_i\right]/m_{\text{tot}} \\
(3) & d_i^2 & = & (x_i - x_{cm})^2 + (y_i - y_{cm})^2 \\
(4) & I_{tot} & = & \sum\left[I_i^{cm} + m_i d_i^2\right].
\end{array}$$

You can reduce this recipe to one grand formula with lots of summation signs. But you would end up doing the calculations in about the order prescribed above in any case. This sequence of steps lends itself naturally to a computer spread sheet or to any program that deals easily with arrays of numbers. This is laid out for the similar center-of-mass calculation on page 157[6].

[6] The tidy recipe just presented is actually more commonly used, with slight modification, in strength of materials than in dynamics. The need for finding area moments of inertia of strange beam cross sections arises more frequently than the need to find the polar mass moment of inertia of a strange cutout shape.

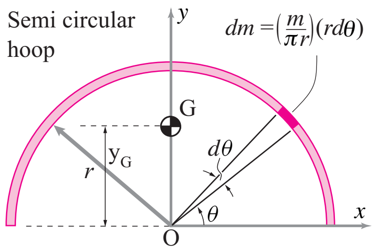


Figure 16.36: The moment of inertia of a semi-circle about the COM is calculated by calculating about point O and then using the parallel axis theorem.

Example: Inertia of a semicircle about its COM.
The moment of inertia of a semicircle with mass m and radius r about O (see fig. 16.36) is

$$I^0 = \int r_{/0}^2\, dm = mr^2.$$

From the example on page 153 the center of mass is at $y_G = \frac{2}{\pi} r$. By the parallel axis theorem

$$I^0 = I^{cm} + my_G^2$$

which we can solve for I^{cm} to get

$$I^{cm} = I^0 - my_G^2 = mr^2 - \frac{4}{\pi^2} mr^2 = \left(1 - \frac{4}{\pi^2}\right) mr^2 \approx 0.6\, mr^2\ .$$

The radius of gyration is that radius ρ_{gyr} so that $m\rho_{gyr}^2 = I^{cm}$ and is

$$\rho_{gyr} = \sqrt{1 - \frac{4}{\pi^2}}\ r.$$

The perpendicular axis theorem for planar rigid bodies

The perpendicular axis theorem for planar objects is the equation

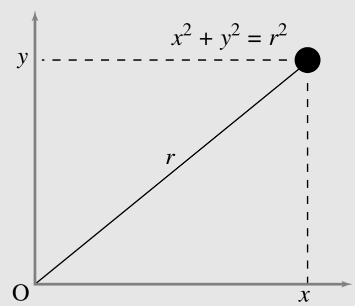
$$I_{zz} = I_{xx} + I_{yy}$$

which is derived in box 16.6 on page 8. It gives the 'polar' inertia I_{zz} in terms of the inertias I_{xx} and I_{yy}. Unlike the parallel axis theorem, the perpendicular axis theorem does *not* have a three-dimensional counterpart. The theorem is of greatest utility when one wants to study the three-dimensional mechanics of a flat object and thus are in need of its full moment of inertia matrix.

16.7 Some examples of 2-D Moment of Inertia

Here, we illustrate some simple moment of inertia calculations for two-dimensional objects. The needed formulas are summarized, in part, by the lower right corner components (that is, the elements in the third column and third row (3,3)) of the matrices in the table on the inside back cover.

One point mass



If we assume that all mass is concentrated at one or more points, then the integral

$$I^o_{zz} = \int r^2_{/o}\, dm$$

reduces to the sum

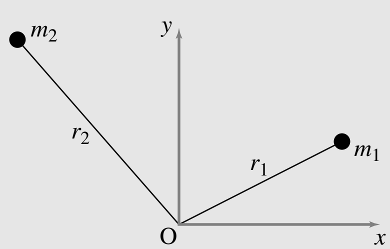
$$I^o_{zz} = \sum r^2_{i/o} m_i$$

which reduces to one term if there is only one mass,

$$I^o_{zz} = r^2 m = (x^2 + y^2)m.$$

So, if $x = 3 \in$, $y = 4 \in$, and $m = 0.1\,\text{lbm}$, then $I^o_{zz} = 2.5\,\text{lbm} \in^2$. Note that, in this case, $I^{cm}_{zz} = 0$ since the radius from the center-of-mass to the center-of-mass is zero.

Two point masses

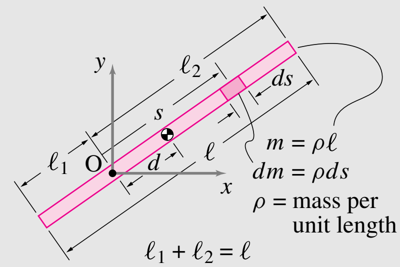


In this case, the sum that defines I^o_{zz} reduces to two terms, so

$$I^o_{zz} = \sum r^2_{i/o} m_i = m_1 r_1^2 + m_2 r_2^2.$$

Note that, if $r_1 = r_2 = r$, then $I^o_{zz} = m_{tot} r^2$.

A thin uniform rod



Consider a thin rod with uniform mass density, ρ, per unit length, and length ℓ. We calculate I^o_{zz} as

$$\begin{aligned}
I^o_{zz} &= \int r^2 \overbrace{dm}^{\rho ds} \\
&= \int_{-\ell_1}^{\ell_2} s^2 \rho ds \quad (s = r) \\
&= \frac{1}{3}\rho s^3 \Big|_{-\ell_1}^{\ell_2} \quad \text{(since} \quad \rho \equiv const.) \\
&= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3).
\end{aligned}$$

If either $\ell_1 = 0$ or $\ell_2 = 0$, then this expression reduces to $I^o_{zz} = \frac{1}{3}m\ell^2$. If $\ell_1 = \ell_2$, then O is at the center-of-mass and

$$I^o_{zz} = I^{cm}_{zz} = \frac{1}{3}\rho\left(\left(\frac{\ell}{2}\right)^3 + \left(\frac{\ell}{2}\right)^3\right) = \frac{ml^2}{12}.$$

We can illustrate one last point. With a little bit of algebraic histrionics of the type that only hindsight can inspire, you can verify that the expression for I^o_{zz} can be arranged as follows:

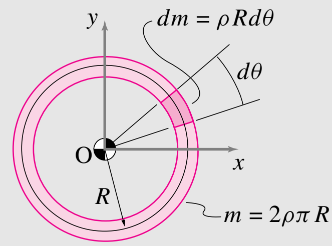
$$\begin{aligned}
I^0_{zz} &= \frac{1}{3}\rho(\ell_1^3 + \ell_2^3) \\
&= \underbrace{\rho(\ell_1 + \ell_2)}_{m}\left(\underbrace{\frac{\ell_2 - \ell_1}{2}}_{d}\right)^2 + \underbrace{\rho\frac{(\ell_1 + \ell_2)^3}{12}}_{m\ell^2/12} \\
&= md^2 + m\frac{\ell^2}{12} \\
&= md^2 + I^{cm}_{zz}
\end{aligned}$$

That is, the moment of inertia about point O is greater than that about the center of mass by an amount equal to the mass times the distance from the center-of-mass to point O squared. This derivation of the *parallel axis theorem* is for one special case, that of a uniform thin rod.

(continued...)

16.7 Some examples of 2-D Moment of Inertia (continued)

A uniform hoop



For a hoop of uniform mass density, ρ, per unit length, we might consider all of the points to have the same radius R. So,

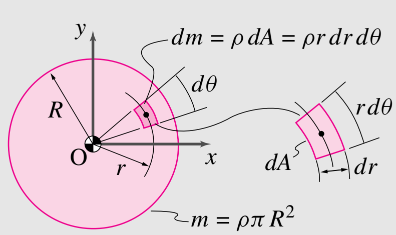
$$I_{zz}^{o} = \int r^2 dm = \int R^2 dm = R^2 \int dm = R^2 m.$$

Or, a little more tediously,

$$\begin{aligned}
I_{zz}^{o} &= \int r^2 dm \\
&= \int_0^{2\pi} R^2 \rho R d\theta \\
&= \rho R^3 \int_0^{2\pi} d\theta \\
&= 2\pi\rho R^3 = \underbrace{(2\pi\rho R)}_{m} R^2 = mR^2.
\end{aligned}$$

This I_{zz}^{o} is the same as for a single point mass m at a distance R from the origin O. It is also the same as for two point masses if they both are a distance R from the origin. For the hoop, however, O is at the center-of-mass so $I_{zz}^{o} = I_{zz}^{cm}$ which is not the case for a single point mass.

A uniform disk

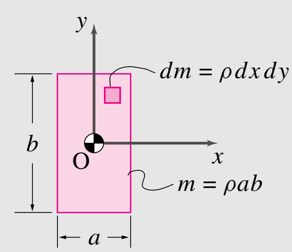


Assume the disk has uniform mass density, ρ, per unit area. For a uniform disk centered at the origin, the center-of-mass is at the origin so

$$\begin{aligned}
I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\
&= \int_0^R \int_0^{2\pi} r^2 \rho r d\theta dr \\
&= \int_0^R 2\pi\rho r^3 dr \\
&= 2\pi\rho \left.\frac{r^4}{4}\right|_0^R = \pi\rho\frac{R^4}{2} = (\pi\rho R^2)\frac{R^2}{2} \\
&= m\frac{R^2}{2}.
\end{aligned}$$

For example, a 1 kg plate of 1 m radius has the same moment of inertia as a 1 kg hoop with a 70.7 cm radius.

Uniform rectangular plate



For the special case that the center of the plate is at point O, the center-of-mass is also at O and $I_{zz}^{o} = I_{zz}^{cm}$.

$$\begin{aligned}
I_{zz}^{o} = I_{zz}^{cm} &= \int r^2 dm \\
&= \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{a}{2}}^{\frac{a}{2}} (x^2 + y^2) \overbrace{\rho dx dy}^{dm} \\
&= \int_{-\frac{b}{2}}^{\frac{b}{2}} \rho \left(\frac{x^3}{3} + xy^2\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}} dy \\
&= \rho\left(\frac{x^3 y}{3} + \frac{xy^3}{3}\right)\Bigg|_{x=-\frac{a}{2}}^{x=\frac{a}{2}}\Bigg|_{y=-\frac{b}{2}}^{y=\frac{b}{2}} \\
&= \rho\left(\frac{a^3 b}{12} + \frac{ab^3}{12}\right) \\
&= \frac{m}{12}(a^2 + b^2).
\end{aligned}$$

Note that $\int r^2 dm = \int x^2 dm + \int y^2 dm$ for all planar objects (the *perpendicular axis theorem*). For a uniform rectangle, $\int y^2 dm = \rho \int y^2 dA$. But the integral $y^2 dA$ is just the term often used for I, the area moment of inertia, in strength of materials calculations for the stresses and stiffnesses of beams in bending. You

(continued...)

16.7 Some examples of 2-D Moment of Inertia (continued)

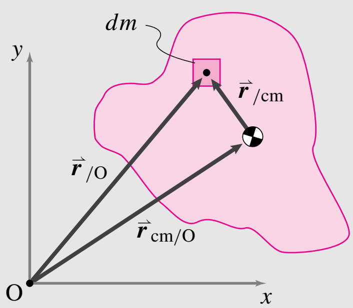
may recall that $\int y^2 dA = \frac{ab^3}{12} = \frac{Ab^2}{12}$ for a rectangle. Similarly, $\int x^2 dA = \frac{Aa^2}{12}$. So, the polar moment of inertia $J = I_{zz}^{o} = m\frac{1}{12}(a^2+b^2)$ can be recalled by remembering the area moment of inertia of a rectangle combined with the perpendicular axis theorem.

16.6 The 2-D parallel axis theorem and the perpendicular axis theorem

Sometimes one wants to know the moment of inertia relative to the center of mass. And, sometimes, if the object is held at a hinge joint at O, relative to that hinge point O There is a simple relation between these two moments of inertia known as *the parallel axis theorem*.

2-D parallel axis theorem

For the two-dimensional mechanics of two-dimensional objects, our only concern is I^o_{zz} and I^{cm}_{zz} and *not* the full moment of inertia matrix. In this case, $I^o_{zz} = \int r^2_{/o}\, dm$ and $I^{cm}_{zz} = \int r^2_{/cm}\, dm$. Now, let's prove the theorem in two dimensions referring to the figure.



$$
\begin{aligned}
I^O_{zz} &= \int r^2_{/O}\, dm \\
&= \int (x^2_{/O} + y^2_{/O}) dm \\
&= \int [(\underbrace{x_{cm/O} + x_{/cm}}_{x_{/O}})^2 + (\underbrace{y_{cm/O} + y_{/cm}}_{y_{/O}})^2] dm \\
&= \int [(x^2_{cm/O} + 2x_{cm/O}x_{/cm} + x^2_{/cm}) + \\
&\qquad (y^2_{cm/O} + 2y_{cm/O}y_{/cm} + y^2_{/cm})] dm \\
&= (x^2_{cm/O} + y^2_{cm/O}) \underbrace{\int dm}_{m} + 2x_{cm/O} \underbrace{\int x_{/cm} dm}_{0} + \\
&\qquad 2y_{cm/O} \underbrace{\int y_{/cm} dm}_{0} + \int (x^2_{/cm} + y^2_{/cm}) dm \\
&= r^2_{cm/O} m + \underbrace{\int (x^2_{/cm} + y^2_{/cm}) dm}_{I^{cm}_{zz}} \\
&= I^{cm}_{zz} + \underbrace{r^2_{cm/O}}_{d^2} m
\end{aligned}
$$

The cancellation $\int y_{/\mathrm{cm}}\, dm = \int x_{/\mathrm{cm}}\, dm = 0$ comes from the definition of center of mass.

Sometimes, people write the parallel axis theorem more simply as

$$I^0 = I^{cm} + md^2 \qquad \text{or} \qquad J_O = J_{cm} + md^2$$

using the symbol J to mean I_{zz}. One thing to note about the parallel axis theorem is that the moment of inertia about any point O is always *greater* than the moment of inertia about the center of mass. For a given object, the minimum moment of inertia is about the center-of-mass.

Why the name *parallel axis theorem*? We use the name because the two I's calculated are the moments of inertia about two parallel axes (both in the z direction) through the two points *cm* and O.

One way to think about the theorem is the following. The moment of inertia of an object about a point O not at the center-of-mass is the same as that of the object about the *cm plus* that of a point mass located at the center-of-mass. If the distance from O to the *cm* is larger than the outer radius of the object, then the d^2m term is larger than I^{cm}_{zz}. The distance of equality of the two terms is the radius of gyration, r_{gyr}.

Perpendicular axis theorem (applies to planar objects only)

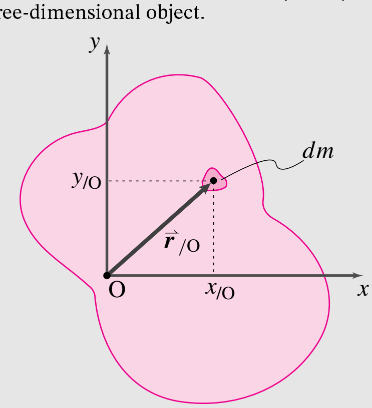
For planar objects,

$$
\begin{aligned}
I^O_{zz} &= \int |\vec{r}|^2\, dm \\
&= \int (x^2_{/O} + y^2_{/O}) dm \\
&= \int x^2_{/O} dm + \int y^2_{/O} dm \\
&= I^O_{yy} + I^O_{xx}
\end{aligned}
$$

Similarly,

$$I^{cm}_{zz} = I^{cm}_{xx} + I^{cm}_{yy}.$$

That is, the moment of inertia about the z-axis is the sum of the inertias about the two perpendicular axes x and y. Note that the objects must be planar ($z = 0$ everywhere) or the theorem would not be true. For example, $I^o_{xx} = \int (y^2_{/O} + z^2_{/O}) dm \neq \int y^2_{/O} dm$ for a three-dimensional object.



SAMPLE 16.9 **Moment of inertia of point masses:** A pendulum is made up of two unequal point masses m and $2m$ connected by a massless rigid rod of length $4r$. The pendulum is pivoted at distance r along the rod from the small mass.

1. Find the moment of inertia I_{zz}^{cm} of the pendulum.
2. Find the moment of inertia I_{zz}^{O} of the pendulum.
3. Find the radius of gyration of the pendulum.

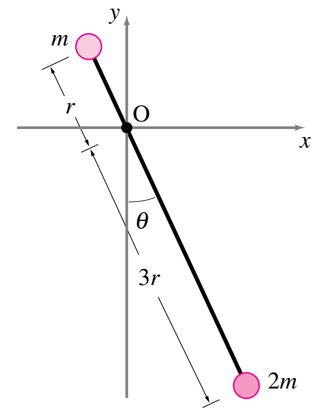


Figure 16.37

Solution

1. First we need to find the center of mass of the system. Let the center of mass C be located at distance r_{cm} from the origin O. An easy way to find the location of C will be to consider the pendulum to lie along the x-axis with the origin at O (see fig. 16.38). Then

$$
\begin{aligned}
x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} \\
\Rightarrow \quad r_{cm} &= \frac{m(-r) + 2m(3r)}{m + 2m} = \frac{5}{3}r.
\end{aligned}
$$

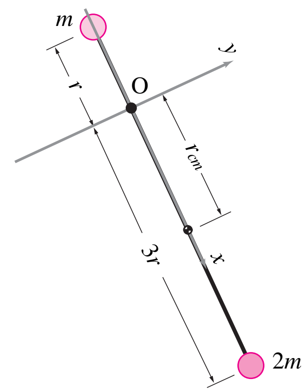


Figure 16.38

So, the moment of inertia about the center of mass is

$$
I_{zz}^{\mathrm{cm}} = \sum m_i r_{i/cm}^2 = m\left(r + \frac{5}{3}r\right)^2 + 2m\left(3r - \frac{5}{3}r\right)^2 = \frac{32}{3}mr^2.
$$

$$\boxed{I_{zz}^{\mathrm{cm}} = 10.67mr^2}$$

2. We can calculate I_{zz}^{O} in two different ways, one directly by summing the contributions of each mass about point O, and the other by using I_{zz}^{cm} and the parallel axis theorem.

$$
\begin{aligned}
I_{zz}^{\mathrm{O}} &= mr^2 + 2m(3r)^2 = 19mr^2 \\
\text{or} \qquad I_{zz}^{\mathrm{O}} &= I_{zz}^{\mathrm{cm}} + m_{tot} r_{cm/O}^2 \\
&= \frac{32}{3}mr^2 + 3m\left(\frac{5}{3}r\right)^2 = 19mr^2.
\end{aligned}
$$

$$\boxed{I_{zz}^{O} = 19mr^2}$$

3. The radius of gyration, by definition, is the distance that gives the desired moment of inertia if the entire mass of the system is concentrated there. Thus, if r_{gyr} is the radius of gyration for the moment of inertia about point O, then

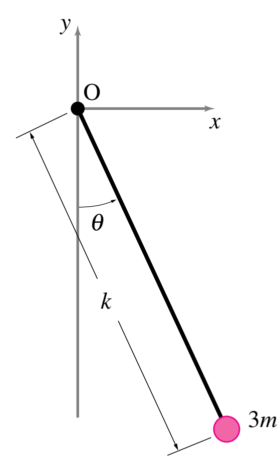


Figure 16.39

$$
\begin{aligned}
I_{zz}^{\mathrm{O}} &= (3m) \cdot r_{gyr}^2 \\
\Rightarrow \quad 19\cancel{m}r^2 &= 3\cancel{m}r_{gyr}^2 \\
\Rightarrow \quad r_{gyr} &= \sqrt{\frac{19}{3}}r = 2.52r.
\end{aligned}
$$

Thus the radius of gyration r_{gyr} of the given pendulum is $r_{gyr} = 2.52r$.

$$\boxed{r_{gyr} = 2.52\,r}$$

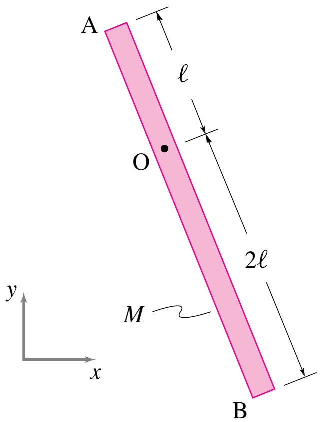


Figure 16.40

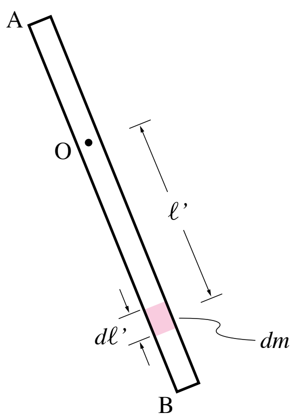


Figure 16.41

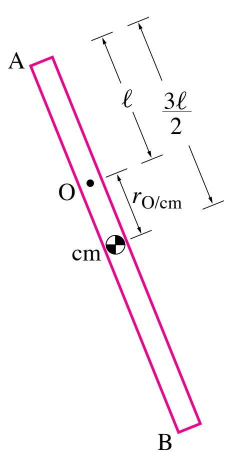


Figure 16.42

SAMPLE 16.10 **Moment of inertia of a rod:** A uniform rigid rod AB of mass $M = 2\,\text{kg}$ and length $3\ell = 1.5\,\text{m}$ swings about the z-axis passing through the pivot point O.

1. Find the moment of inertia I_{zz}^{O} of the bar using the fundamental definition $I_{zz}^{O} = \int_m r_{/O}^2 dm$.

2. Find I_{zz}^{O} using the parallel axis theorem given that $I_{zz}^{\text{cm}} = \frac{1}{12}m\ell^2$ where m = total mass, and ℓ = total length of the rod. (You can find I_{zz}^{cm} for many commonly encountered objects in the table on the inside backcover of the text).

Solution

1. Since we need to carry out the integral, $I_{zz}^{O} = \int_m r_{/O}^2 dm$, to find I_{zz}^{O}, let us consider an infinitesimal length segment $d\ell'$ of the bar at distance ℓ' from the pivot point O. (see Figure 16.41). Let the mass of the infinitesimal segment be dm.
   Now the mass of the segment may be written as

$$\begin{aligned} dm &= (\text{mass per unit length of the bar}) \cdot (\text{length of the segment}) \\ &= \frac{M}{3\ell}\, d\ell' \qquad \left(\text{Note: } \frac{\text{mass}}{\text{unit length}} = \frac{\text{total mass}}{\text{total length}}\right). \end{aligned}$$

   We also note that the distance of the segment from point O, $r_{/O} = \ell$. Substituting the values found above for $r_{/O}$ and dm in the formula we get

$$\begin{aligned} I_{zz}^{O} &= \int_{-\ell}^{2\ell} \underbrace{(\ell')^2}_{r_{/O}^2} \underbrace{\frac{M}{3\ell} d\ell'}_{dm} \\ &= \frac{M}{3\ell}\int_{-\ell}^{2\ell} (\ell')^2 d\ell' = \frac{M}{3\ell}\left[\frac{\ell'^3}{3}\right]_{-\ell}^{2\ell} \\ &= \frac{M}{3\ell}\left[\frac{8\ell^3}{3} - \left(-\frac{\ell^3}{3}\right)\right] = M\ell^2 \\ &= 2\,\text{kg}\cdot(0.5\,\text{m})^2 = 0.5\,\text{kg}\cdot\text{m}^2. \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg}\cdot\text{m}^2}$$

2. The parallel axis theorem states that

$$I_{zz}^{O} = I_{zz}^{\text{cm}} + M r_{O/\text{cm}}^2.$$

   Since the rod is uniform, its center-of-mass is at its geometric center, *i.e.,* at distance $\frac{3\ell}{2}$ from either end. From the Fig 16.42 we can see that

$$r_{O/\text{cm}} = AG - AO = \frac{3\ell}{2} - \ell = \frac{\ell}{2}$$

$$\begin{aligned} \text{Therefore,} \quad I_{zz}^{O} &= \underbrace{\frac{1}{12}M(3\ell)^2}_{I_{zz}^{\text{cm}}} + M\left(\frac{\ell}{2}\right)^2 \\ &= \frac{9}{12}M\ell^2 + M\frac{\ell^2}{4} = M\ell^2 \\ &= 0.5\,\text{kg}\cdot\text{m}^2 \quad (\text{same as in (a), of course}) \end{aligned}$$

$$\boxed{I_{zz}^{O} = 0.5\,\text{kg}\cdot\text{m}^2}$$

SAMPLE 16.11 **Moment of inertia of a wheel with a cut-out:** A uniform rigid wheel of radius $r = 1\,\mathrm{ft}$ is made eccentric by cutting out a portion of the wheel. The center-of-mass of the eccentric wheel is at C, a distance $e = \frac{r}{3}$ from the geometric center O. The mass of the wheel (after deducting the cut-out) is 3.2 lbm. The moment of inertia of the wheel about point O, I_{zz}^{O}, is $1.8\ \mathrm{lbm{\cdot}ft^2}$. We are interested in the moment of inertia I_{zz} of the wheel about points A and B on the perimeter.

1. Without any calculations, guess which point, A or B, gives a higher moment of inertia. Why?
2. Calculate I_{zz}^{C}, I_{zz}^{A} and I_{zz}^{B} and compare with the guess in (a).

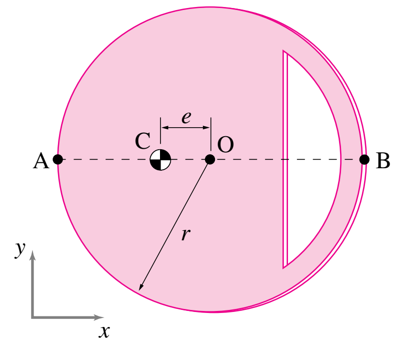


Figure 16.43

Solution

1. The moment of inertia I_{zz}^{B} should be higher. Moment of inertia I_{zz} measures the geometric distribution of mass about the z-axis. But the distance of the mass from the axis counts more than the mass itself ($I_{zz}^{O} = \int_m r_{/O}^2\, dm$). The distance $r_{/O}$ of the mass appears as a quadratic term in I_{zz}^{O}. The total mass is the same whether we take the moment of inertia about point A or about point B. However, the distribution of mass is not the same about the two points. Due to the cut-out being closer to point B there are more "dm's" at greater distances from point B than from point A. So, we guess that

$$\boxed{I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}}$$

2. If we know the moment of inertia I_{zz}^{C} (about the center-of-mass) of the wheel, we can use the parallel axis theorem to find I_{zz}^{A} and I_{zz}^{B}. In the problem, we are given I_{zz}^{O}. But,

$$\begin{aligned}
& I_{zz}^{\mathrm{O}} &=& I_{zz}^{\mathrm{C}} + M r_{O/C}^2 \quad \text{(parallel axis theorem)} \\
\Rightarrow\quad & I_{zz}^{\mathrm{C}} &=& I_{zz}^{\mathrm{O}} - M r_{O/C}^2 \\
& &=& 1.8\,\mathrm{lbm\,ft^2} - 3.2\,\mathrm{lbm} \underbrace{\left(\frac{1\,\mathrm{ft}}{3}\right)^2}_{r_{O/C}=e=\frac{r}{3}} \\
& &=& 1.44\,\mathrm{lbm{\cdot}ft^2} \\
\text{Now,}\quad & I_{zz}^{\mathrm{A}} &=& I_{zz}^{\mathrm{C}} + M r_{A/C}^2 = I_{zz}^{\mathrm{C}} + M\left(\frac{2r}{3}\right)^2 \\
& &=& 1.44\,\mathrm{lbm{\cdot}ft^2} + 3.2\,\mathrm{lbm}\left(\frac{2\,\mathrm{ft}}{3}\right)^2 \\
& &=& 2.86\,\mathrm{lbm{\cdot}ft^2} \\
\text{and}\quad & I_{zz}^{\mathrm{B}} &=& I_{zz}^{\mathrm{C}} + M r_{B/C}^2 = I_{zz}^{\mathrm{C}} + M\left(r + \frac{r}{3}\right)^2 \\
& &=& 1.44\,\mathrm{lbm{\cdot}ft^2} + 3.2\,\mathrm{lbm}\left(1\,\mathrm{ft} + \frac{1\,\mathrm{ft}}{3}\right)^2 \\
& &=& 7.13\,\mathrm{lbm{\cdot}ft^2}
\end{aligned}$$

$$\boxed{I_{zz}^{\mathrm{C}} = 1.44\,\mathrm{lbm{\cdot}ft^2},\ I_{zz}^{\mathrm{A}} = 2.86\,\mathrm{lbm{\cdot}ft^2},\ I_{zz}^{\mathrm{B}} = 7.13\,\mathrm{lbm{\cdot}ft^2}}$$

Clearly, $I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}$, as guessed in (a).

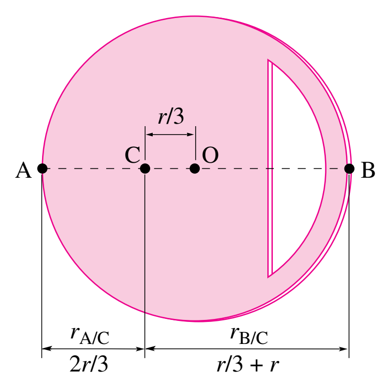


Figure 16.44

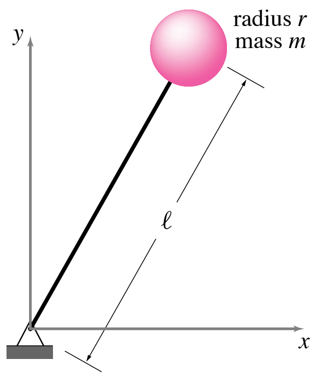


Figure 16.45

SAMPLE 16.12 Moment of inertia: modeling a sphere as a point mass: A uniform solid sphere of mass m and radius r is attached to a massless rigid rod of length ℓ. The sphere swings in the xy plane. Find the error in calculating I_{zz}^{O} as a function of r/ℓ if the sphere is treated as a point mass concentrated at the center-of-mass of the sphere.

Solution The exact moment of inertia of the sphere about point O can be calculated using parallel axis theorem:

$$\begin{aligned} I_{zz}^{O} &= I_{zz}^{cm} + m\ell^2 \\ &= \frac{2}{5}mr^2 + m\ell^2. \quad \text{(See Table IV on inside cover)} \end{aligned}$$

If we treat the sphere as a point mass, the moment of inertia I_{zz}^{O} is

$$\tilde{I}_{zz}^{O} = m\ell^2.$$

Therefore, the relative error in I_{zz}^{O} is

$$\begin{aligned} \text{error} &= \frac{I_{zz}^{O} - \tilde{I}_{zz}^{O}}{I_{zz}^{O}} \\ &= \frac{\frac{2}{5}mr^2 + m\ell^2 - m\ell^2}{\frac{2}{5}mr^2 + m\ell^2} \\ &= \frac{\frac{2}{5}\frac{r^2}{\ell^2}}{\frac{2}{5}\frac{r^2}{\ell^2} + 1} \end{aligned}$$

From the above expression we see that for $r \ll \ell$ the error is very small. From the graph of error in Fig. 16.46 we see that even for $r = \ell/5$, the error in I_{zz}^{O} due to approximating the sphere as a point mass is less than 2%.

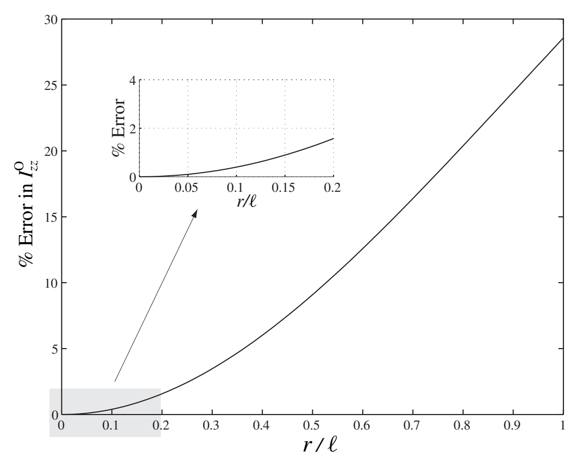


Figure 16.46: Relative error in I_{zz}^{O} of the sphere as a function of r/ℓ.

16.3 Polar moment of inertia

Preparatory Problems

16.3.1 Find an expression for the polar moment of inertia for each of the following systems about the specified points. Answer in terms of some or all of the variables given. If a position vector $\vec{r}$ is given then you can assume the coordinates (x, y) are given so that $\vec{r} = x\hat{\imath} + y\hat{\jmath}$.

a) A **particle** with mass m at $\vec{r}$ about
   1. Its center of mass
   2. The origin

   *

b) **Two particles** with masses m_1 and m_2 a distance d apart located at $\vec{r}_1$ and $\vec{r}_2$, respectively, about
   i) Mass 1
   ii) Mass 2
   i) The center of mass of the system
   iv) The origin

   *

c) A collection of n **particles** m_i at locations $\vec{r}_{i/0}$ about
   i) The origin
   ii) The center of mass G of the system

   *

d) A uniform **line segment** with length ℓ and mass m with center of mass at location $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) One end
   iii) The origin

   *

e) A **uniform hoop** with mass m and radius R and center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the hoop
   iiii) The origin

   *

f) A **uniform disk** with mass m and radius R and center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the perimeter
   iii) The origin

   *

g) A **uniform rectangle** with mass m and sides ℓ and w and with center at $\vec{r}_{G/0}$ about
   i) Its center of mass G
   ii) A corner
   iii) The origin

   *

16.3.2 Two objects have masses m_1 and m_2 and polar moments of inertia I_1 and I_2 about their respective centers of mass G_1 and G_2 which are a distance d apart. What is the moment of inertia of the system about its center of mass? *

16.3.3 A point mass $m = 0.5$ kg is located at $x = 0.3$ m and $y = 0.4$ m in the xy-plane. Find the moment of inertia of the mass about the z-axis. *

16.3.4 A small ball of mass 0.2 kg is attached to a 1 m massless rod.

a) What is the value of I_{zz} of the ball about the other end of the rod? *

b) How much must you shorten the rod to reduce the moment of inertia of the ball by half? *

More-Involved Problems

16.3.5 Two identical point masses are attached to the two ends of a rigid massless bar of length ℓ (one mass at each end). Locate a point along the length of the bar about which the polar moment of inertia of the system is 20% more than that calculated about the mid point of the bar. *

16.3.6 A dumbbell consists of a rigid massless bar of length ℓ and two identical point masses m and m, one at each end of the bar.

a) About which point on the dumbbell is its polar moment of inertia I_{zz} a minimum and what is this minimum value? *

b) About which point on the dumbbell is its polar moment of inertia I_{zz} a maximum and what is this maximum value? *

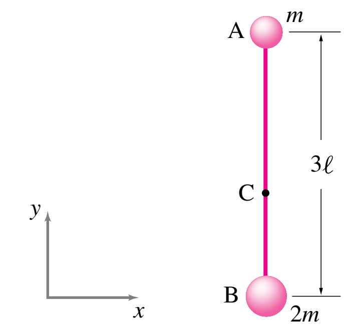
16.3.7 A light rigid rod AB of length 3ℓ has a point mass m at end A and a point mass $2m$ at end B. Point C is the center of mass of the system. First, answer the following questions without any calculations and then do calculations to verify your guesses.

a) About which point A, B, or C, is the polar moment of inertia I_{zz} of the system a minimum? *

b) About which point is I_{zz} a maximum? *

c) What is the ratio of I_{zz}^A and I_{zz}^B? *

d) Is the radius of gyration of the system greater, smaller, or equal to the length of the rod? *



Problem 16.3.7

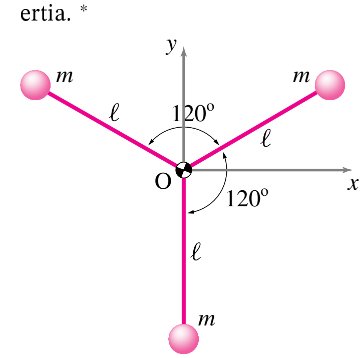
16.3.8 Three identical particles of mass m are connected to three identical massless rods of length ℓ and welded together at point O as shown in the figure.

a) Guess (no calculations) which of the three moment of inertia terms I_{xx}^O, I_{yy}^O, I_{zz}^O is the smallest and which is the biggest. *

b) Calculate the three moments of inertia to check your guess. *

c) If the orientation of the system is changed, so that one mass is along the x-axis, will your answer to part (a) change? *

d) Find the radius of gyration of the system for the polar moment of inertia. *

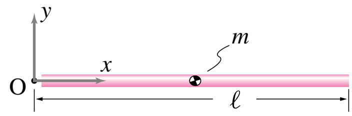


Problem 16.3.8

16.3.9 Show that the polar moment of inertia I_{zz}^{O} of the uniform bar of length ℓ and mass m, shown in the figure, is $\frac{1}{3}m\ell^2$, in two different ways:

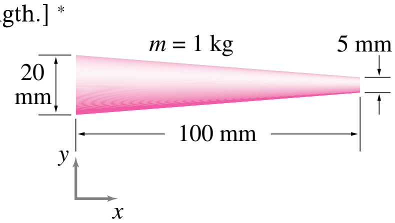
a) by using the basic definition of polar moment of inertia $I_{zz}^{O} = \int r^2\, dm$, and

b) by computing I_{zz}^{cm} first and then using the parallel axis theorem.



Problem 16.3.9

16.3.10 Approximately locate the center of mass of the tapered rod shown in the figure and compute the polar moment of inertia I_{zz}^{cm}. [Hint: use the variable thickness of the rod to define an approximate equivalent variable mass density per unit length.] *



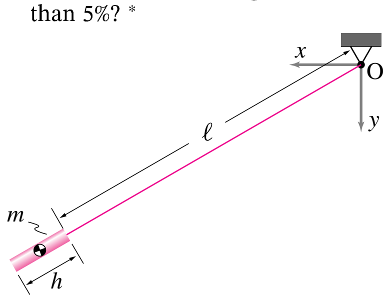
Problem 16.3.10

16.3.11 A short rod of mass m and length h hangs from an inextensible and massless rod of length ℓ.

a) Find the moment of inertia I_{zz}^{O} of the rod. *

b) Find the moment of inertia of the rod I_{zz}^{O} by considering it as a point mass located at its center of mass. *

c) Find the percent error in I_{zz}^{O} in treating the bar as a point mass by comparing the expressions in parts (a) and (b). Plot the percent error versus h/ℓ. For what values of h/ℓ is the percentage error less than 5%? *



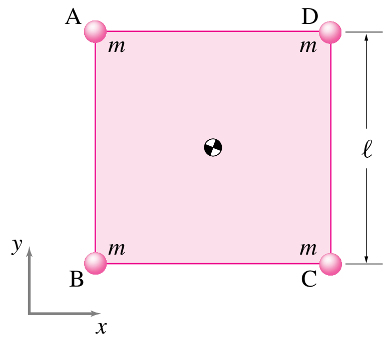
Problem 16.3.11

16.3.12 **Parallel axis theorem?** A massless square plate $ABCD$ has four identical point masses located at its corners.

a) Find the polar moment of inertia I_{zz}^{cm}. *

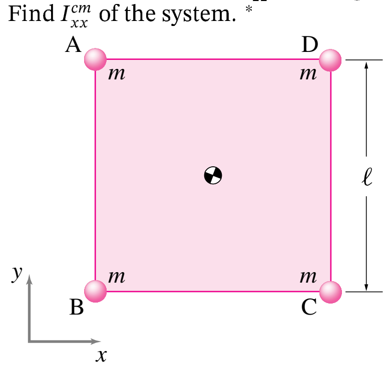
b) Find a point P on the plate about which the system's moment of inertia I_{zz} is maximum. *

c) Find the radius of gyration of the system about the rectangle center. *



Problem 16.3.12

16.3.13 For the massless square plate with four point masses on the corners, the polar moment of inertia $I_{zz}^{cm} = 0.6\,\text{kg}\cdot\text{m}^2$. Find I_{xx}^{cm} of the system. *



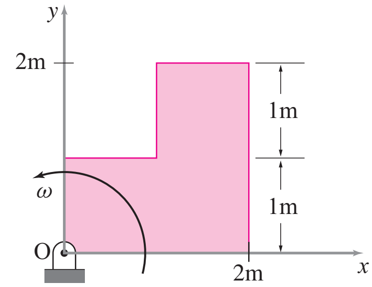
Problem 16.3.13

16.3.14 A uniform square plate (2 m on edge) has a corner cut out. The total mass of the remaining plate is 3 kg.

a) Where is the center of mass of the plate at the instant shown? *

b) What is the moment of inertia of the plate about point O? *

c) What is the moment of inertia of the plate about its center of mass? *



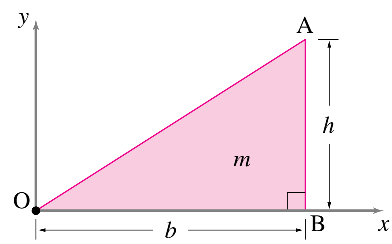
Problem 16.3.14

16.3.15 A uniform thin triangular plate of mass m, height h, and base b lies in the xy-plane.

a) Set up the integral to find the polar moment of inertia I_{zz}^{O} of the plate. *

b) Show that $I_{zz}^{O} = \dfrac{m}{6}(h^2 + 3b^2)$ by evaluating the integral in part (a).

c) Locate the center of mass of the plate and calculate I_{zz}^{cm}. *

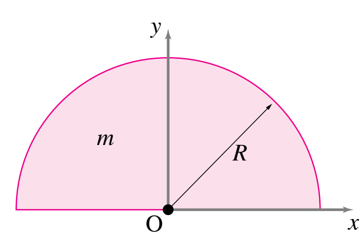


Problem 16.3.15

16.3.16 A uniform thin plate of mass m is cast in the shape of a semi-circular disk of radius R as shown in the figure.

a) Find the location of the center of mass of the plate *

b) Find the polar moment of inertia of the plate, I_{zz}^{cm}. [Hint: It may be easier to set up and evaluate the integral for I_{zz}^{O} and then use the parallel axis theorem to calculate I_{zz}^{cm}.] *



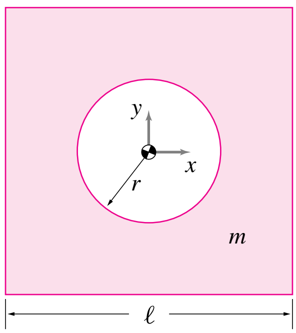
Problem 16.3.16

16.3.17 A uniform square plate of side $\ell = 250\,\text{mm}$ has a circular cut-out of radius $r = 50\,\text{mm}$. The mass of the plate is $m = \frac{1}{2}\,\text{kg}$.

a) Find the polar moment of inertia of the plate about its center. *

b) Plot I_{zz}^{cm} versus r/ℓ.

c) Find the limiting values of I_{zz}^{cm} for $r = 0$ and $r = \ell$. *



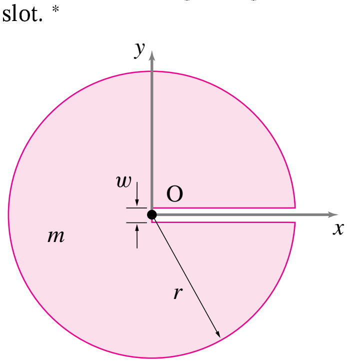
Problem 16.3.17

16.3.18 A uniform thin circular disk of radius $r = 100\,\text{mm}$ and mass $m = 2\,\text{kg}$ has a rectangular slot of width $w = 10\,\text{mm}$ cut into it as shown in the figure. You can approximate the right edge of the cutout as a straight line (thus your calculation would be exact for a system that has a very thin sliver of material closing the gap at the right).

a) Find the polar moment of inertia I_{zz}^{O} of the disk. *

b) Locate the center of mass of the disk and calculate I_{zz}^{cm}. *

c) estimate the error in your calculation of the mass, center of mass location and moment of inertia due to not subtracting out the sliver of material at the right edge of the slot. *



Problem 16.3.18

16.3.19 Calculate the polar moment of inertia of a uniform square plate with mass m and side ℓ various ways. Choose numerical values for m and ℓ if you want to address this as a numerical calculation rather than a theoretical one.

a) As a single uniform plate

b) As a composite of 4 plates, each with sides $\ell/2$

c) As a composite of 100 plates, each with sides $\ell/10$ (These first three methods should give exactly the same answer).

d) For the composite of 100 plates now neglect the terms which concern the moments of inertia of each small square about its center of mass. What is the error in making this neglect? *