16.3.1 Find an expression for the polar moment of inertia for each of the following systems about the specified points. Answer in terms of some or all of the variables given. If a position vector $\vec{\boldsymbol{r}}$ is given then you can assume the coordinates (x, y) are given so that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.

a) A **particle** with mass m at $\vec{\boldsymbol{r}}$ about
   1. Its center of mass
   2. The origin

b) **Two particles** with masses m_1 and m_2 a distance d apart located at $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$, respectively, about
   i) Mass 1
   ii) Mass 2
   i) The center of mass of the system
   iv) The origin

c) A collection of n **particles** m_i at locations $\vec{\boldsymbol{r}}_{i/0}$ about
   i) The origin
   ii) The center of mass G of the system

d) A uniform **line segment** with length ℓ and mass m with center of mass at location $\vec{\boldsymbol{r}}_{G/0}$ about
   i) Its center of mass G
   ii) One end
   iii) The origin

e) A **uniform hoop** with mass m and radius R and center at $\vec{\boldsymbol{r}}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the hoop
   iiii) The origin

f) A **uniform disk** with mass m and radius R and center at $\vec{\boldsymbol{r}}_{G/0}$ about
   i) Its center of mass G
   ii) A point on the perimeter
   iii) The origin

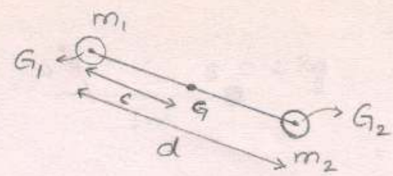
g) A **uniform rectangle** with mass m and sides ℓ and w and with center at $\vec{\boldsymbol{r}}_{G/0}$ about
   i) Its center of mass G
   ii) A corner
   iii) The origin

Answer:
(a) $I_{zz}^{cm} = 0, I_{zz}^{O} = mr^2$.
(b) $I_{zz}^{m1} = m_2 \left|r_{21}^2\right|$, $I_{zz}^{m2} = m_1 \left|r_{12}^2\right|$, $I_{zz}^{cm} = \left(\frac{m_1 m_2}{m_1+m_2}\right) d^2$, and $I_{zz}^{cm} = m1 \left|\vec{\boldsymbol{r}}_1\right|^2 + m2 \left|\vec{\boldsymbol{r}}_2\right|^2$
(c) $I_{zz}^{O} = \sum_{i=1}^{n} m_i \left|\vec{\boldsymbol{r}}_i\right|^2$ and $I_{zz}^{cm} = \sum_{i=1}^{n} m_i \left|\vec{\boldsymbol{r}}_i - \vec{\boldsymbol{r}}_{cm}\right|^2$
(d) $I_{zz}^{G} = \frac{ml^2}{12}$, $I_{zz}^{A} = \frac{ml^2}{3}$, and $I_{zz}^{O} = \frac{ml^2}{12} + m\left|\vec{\boldsymbol{r}}_{G/O}\right|^2$.
(e) $I_{zz}^{G} = mR^2$, $I_{zz}^{P} = 2mR^2$, and $I_{zz}^{O} = I_{zz}^{G} + m\left|\vec{\boldsymbol{r}}_{G/O}\right|^2$.
(f) $I_{zz}^{G} = \frac{mR^2}{2}$, $I_{zz}^{P} = \frac{3}{2}mR^2$, and $I_{zz}^{O} = I_{zz}^{G} + m\left|\vec{\boldsymbol{r}}_{G/O}\right|^2$.
(g) $I_{zz}^{G} = \frac{m}{12}\left(l^2 + w^2\right)$, $I_{zz}^{P} = \frac{m}{3}\left(l^2 + w^2\right)$, and $I_{zz}^{O} = I_{zz}^{G} + m\left|\vec{\boldsymbol{r}}_{G/O}\right|^2$.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.2 Two objects have masses m_1 and m_2 and polar moments of inertia I_1 and I_2 about their respective centers of mass G_1 and G_2 which are a distance d apart. What is the moment of inertia of the system about its center of mass?

Answer:
$I_{zz}^G = I_1 + I_2 + \left(\frac{m_1 m_2}{m_1+m_2}\right) d^2.$



G is the centre of mass of the system

$$m_1 \times c = m_2(d-c)$$

$$c = \left(\frac{m_2}{m_1+m_2}\right) d$$

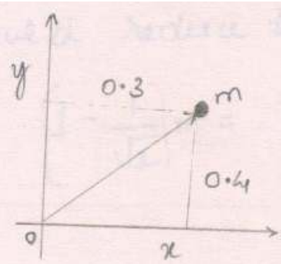
$$I^G = I_1 + m_1 c^2 + I_2 + m_2(d-c)^2$$

$$= I_1 + I_2 + m_1\left(\frac{m_2}{m_1+m_2}\right)^2 d^2 + m_2 \frac{m_1^2}{(m_1+m_2)^2} d^2$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.3 A point mass $m = 0.5\,\mathrm{kg}$ is located at $x = 0.3\,\mathrm{m}$ and $y = 0.4\,\mathrm{m}$ in the xy-plane. Find the moment of inertia of the mass about the z-axis.

Answer:
$I_{zz} = 0.125\,\mathrm{kg}\cdot\mathrm{m}^2$.



$m = 0{\cdot}5\ \mathrm{kg}$

$x = 0{\cdot}3\,m$

$y = 0{\cdot}4\,m$

$$I^{o}_{zz} = m\,(x^2 + y^2)$$

$$= 0{\cdot}5\,(0{\cdot}3^2 + 0{\cdot}4^2)$$

$$= 0{\cdot}125\ \mathrm{Kgm^2}$$

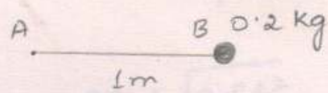
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.4 A small ball of mass 0.2 kg is attached to a 1 m massless rod.

a) What is the value of I_{zz} of the ball about the other end of the rod?

b) How much must you shorten the rod to reduce the moment of inertia of the ball by half?

Answer:
(a) $0.2\,\mathrm{kg}\cdot\mathrm{m}^2$.
(b) 0.29 m.



$m = 0{\cdot}2\,\mathrm{kg}$

$\ell = 1\,\mathrm{m}$

a) $I_{zz}^{A} = 0{\cdot}2 \times 1^2$

$= 0{\cdot}2\ \mathrm{kg\text{-}m^2}$

b) $I_{zz}^{A'} = \frac{1}{2} I_{zz}^{A}$

$\frac{1}{2} I_{zz}^{A} = 0{\cdot}2\,\mathrm{kg} \times \ell'^2$

$\frac{1}{2}\mathrm{m}^2 = \ell'^2$

$\ell' = \frac{1}{\sqrt{2}}\ \mathrm{m}$

We should reduce the length by

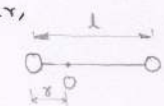
$\left(1 - \frac{1}{\sqrt{2}}\right)\mathrm{m} = 0{\cdot}29\,\mathrm{m}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.5 Two identical point masses are attached to the two ends of a rigid massless bar of length ℓ (one mass at each end). Locate a point along the length of the bar about which the polar moment of inertia of the system is 20% more than that calculated about the mid point of the bar.

Answer:
At 0.72ℓ from either end

Inertia about the midpoint of the bar,



$$I^{cm} = m\left(\frac{l}{2}\right)^2 + m\left(\frac{l}{2}\right)^2 = m\frac{l^2}{2}$$

Inertia about 'O' to be 20% more than I^{cm}, we have

$$I^{o} = mr^2 + m(l-r)^2 = 1.2\,\frac{ml^2}{2}$$

$$\Rightarrow \quad r^2 - lr + 0.2l^2 = 0$$

$$\Rightarrow \quad r = \frac{l \pm \sqrt{l^2 - 0.8l^2}}{2}$$

$$\Rightarrow \quad r = 0.7236\,l \quad \text{(or)} \quad 0.2764\,l \;//$$

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16.3.6 A dumbbell consists of a rigid massless bar of length ℓ and two identical point masses m and m, one at each end of the bar.

a) About which point on the dumbbell is its polar moment of inertia I_{zz} a minimum and what is this minimum value?

b) About which point on the dumbbell is its polar moment of inertia I_{zz} a maximum and what is this maximum value?

Answer:
(a) $(I_{zz})_{min} = m\ell^2/2$, about the midpoint.
(b) $(I_{zz})_{max} = m\ell^2$, about either end

$I_{zz} = mr^2 + m(l-r)^2$

a) Minimum I_{zz}, occurs at $r = ?$

$$\frac{dI_{zz}}{dr} = 0$$

$\Rightarrow \quad 2r + 2(l-r)(-1) = 0$

$\Rightarrow \quad r = \frac{l}{2}$ and $I_{zz}^{min} = m\frac{l^2}{2}$

b) Maximum I_{zz}, occurs at $r = ?$

By inspection. Maximum I_{zz} occurs at

$r = 0$ (or) l and $I_{zz}^{max} = ml^2$

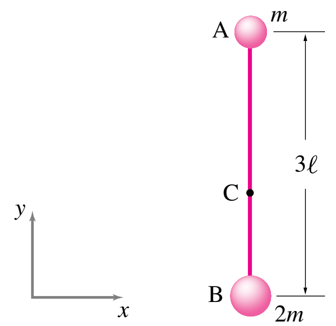
16.3.7 A light rigid rod AB of length 3ℓ has a point mass m at end A and a point mass $2m$ at end B. Point C is the center of mass of the system. First, answer the following questions without any calculations and then do calculations to verify your guesses.

a) About which point A, B, or C, is the polar moment of inertia I_{zz} of the system a minimum?

b) About which point is I_{zz} a maximum?

c) What is the ratio of I_{zz}^A and I_{zz}^B?

d) Is the radius of gyration of the system greater, smaller, or equal to the length of the rod?



Problem 16.7

Answer:
(a) C
(b) A
(c) $I_{zz}^A/I_{zz}^B = 2$
(d) smaller, $r_{gyr} = \sqrt{I_{zz}^C/(3m)} = \sqrt{2}\ell$

a) $I^{min} = I^C = m(2l)^2 + 2m(l)^2 = 6ml^2$

b) $I^{max} = I^A = 2m(3l)^2 = 18ml^2$

c) $\dfrac{I^A}{I^B} = \dfrac{2m(3l)^2}{m(3l)^2} = 2$

d) $Mr^2 = 6ml^2 \Rightarrow$ Radius of gyration, $r = \sqrt{2}\,l$

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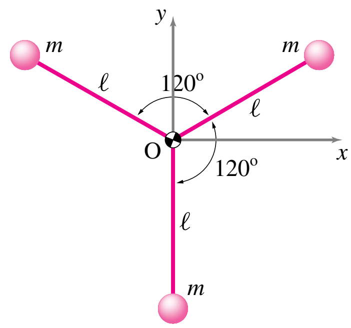
16.3.8 Three identical particles of mass m are connected to three identical massless rods of length ℓ and welded together at point O as shown in the figure.

a) Guess (no calculations) which of the three moment of inertia terms I^O_{xx}, I^O_{yy}, I^O_{zz} is the smallest and which is the biggest.

b) Calculate the three moments of inertia to check your guess.

c) If the orientation of the system is changed, so that one mass is along the x-axis, will your answer to part (a) change?

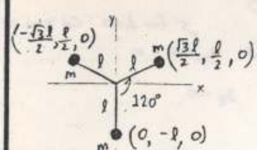
d) Find the radius of gyration of the system for the polar moment of inertia.



Problem 16.8

Answer:
(a) Biggest: I^O_{zz}; smallest: $I^O_{yy} = I^O_{xx}$.
(b) $I^O_{xx} = \frac{3}{2}m\ell^2 = I^O_{yy}$. $I^O_{zz} = 3m\ell^2$.
(c) Answer does not change.
(d) $r_{gyr} = \ell$.

\# 4.96 Perpendicular Axis Theorem



Axis-symmetric body
- massless rods
- identical point masses

(a) Guess that $I_{zz} > I_{xx}$ and I_{yy} because it is seen that masses attain greatest distance from z-axis.
Symmetry may let us guess that $I_{xx} = I_{yy}$.

(b) $I_{xx} = \sum_{i=1}^{3} m_i(y_i^2 + \overset{0}{z_i^2}) = \sum_{i=1}^{3} m_i y_i^2$

$= m\left[(\frac{\ell}{2})^2 + (\frac{\ell}{2})^2 + (-\ell)^2\right]$

$\boxed{I_{xx} = \frac{3}{2} m \ell^2}$

$I_{yy} = \sum_{i=1}^{3} m_i (x_i^2 + \overset{0}{z_i^2}) = \sum_{i=1}^{3} m_i x_i^2$

$= m\left[(\frac{\sqrt{3}\ell}{2})^2 + (-\frac{\sqrt{3}\ell}{2})^2 + 0^2\right]$

$\boxed{I_{yy} = \frac{3}{2} m \ell^2}$

$I_{zz} = \sum_{i=1}^{3} m_i(x_i^2 + y_i^2)$

$= \sum_{i=1}^{3} m_i x_i^2 + \sum_{i=1}^{3} m_i x_i^2$

$= I_{yy} + I_{xx}$

$\boxed{!}$ This only works because $z_i = 0$ for all particles.

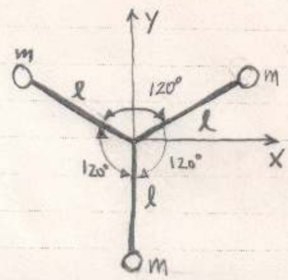
$\boxed{I_{zz} = 3 m \ell^2}$

(c) By symmetry, answer does not change.

(d) $J = I_{zz} = 3m\ell^2 = (3m)R^2$ $\boxed{R = L}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

4.96



a. Guess which is the biggest of the three moment of inertia terms $I^o_{xx}, I^o_{yy}, I^o_{zz}$

$$I^o_{xx} = I^o_{yy} < I^o_{zz}$$

b. Calculate the moments of inertia.
Note: discrete masses, no integral required

$$I^o_{xx} = m\ell^2 + 2m\left[\ell \sin 30°\right]^2 \qquad \sin 30° = \frac{1}{2}$$

$$= m\ell^2 + \frac{m\ell^2}{2} = \boxed{\frac{3m\ell^2}{2} = I^o_{xx}}$$

$$I^o_{yy} = 2m\left[\ell \cos 30°\right]^2 \qquad \cos 30° = \frac{\sqrt{3}}{2}$$

$$= 2m\ell^2\frac{3}{4} = \boxed{\frac{3m\ell^2}{2} = I^o_{yy}}$$

$$\boxed{I^o_{zz} = 3m\ell^2}$$

Check: for planar objects $I^o_{zz} = I^o_{xx} + I^o_{yy}$ ✓

c. If orientation of object were changed so that one mass was along the x-axis, My answer to **a.** would $\boxed{\underline{\text{not}}}$ change.

Can make a general statement (see if you can verify). Any object with $m(>n)$ axes of symmetry in n dimensions will have an infinite number of principal axes.

D. Find radius of gyration for polar moment of inertia. Polar moment of inertia is I^o_{zz}.

$$I^o_{zz} = 3m\ell^2$$

call radius of gyration, r_{gyr}

$$I^o_{zz} = M r_{gyr}^2 \qquad M = 3m$$

Equating the two $\boxed{r_{gyr} = \ell}$

So you could "smear out" all mass symmetrically about O at distance of ℓ from O and have the same moment of inertia.

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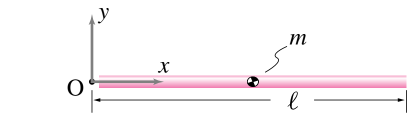
16.3.9 Show that the polar moment of inertia I^O_{zz} of the uniform bar of length ℓ and mass m, shown in the figure, is $\frac{1}{3}m\ell^2$, in two different ways:

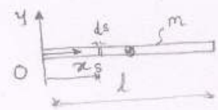
a) by using the basic definition of polar moment of inertia $I^O_{zz} = \int r^2\, dm$, and

b) by computing I^{cm}_{zz} first and then using the parallel axis theorem.



Problem 16.9

a) Polar moment of inertia, $I^o_{zz} = \int r^2 dm$



$$I^o_{zz} = \int_0^l \frac{m}{l} s^2 ds = \frac{m}{l}\cdot\frac{s^3}{3}\Big|_0^l = \frac{1}{3}ml^2$$

b) $$I^o_{zz} = I^{cm} + m\left(\frac{l}{2}\right)^2 = \int_{-l/2}^{l/2} s^2 \frac{m}{l} ds + m\left(\frac{l}{2}\right)^2$$

$$\Rightarrow I^o_{zz} = \frac{m}{l}\,\frac{1}{3}\left(\frac{2l^3}{8}\right) + m\frac{l^2}{4}$$

$$\Rightarrow I^o_{zz} = \frac{1}{12}ml^2 + \frac{1}{4}ml^2 = \frac{1}{3}ml^2$$

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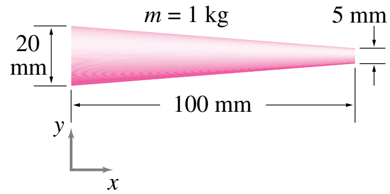
16.3.10 Approximately locate the center of mass of the tapered rod shown in the figure and compute the polar moment of inertia I_{zz}^{cm}. [Hint: use the variable thickness of the rod to define an approximate equivalent variable mass density per unit length.]



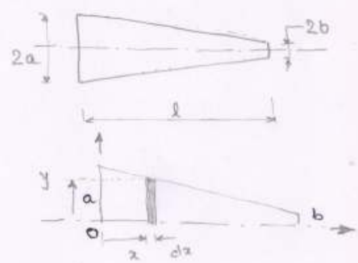
Problem 16.10

Answer:
x_{cm} = 40 mm and I_{zz}^{cm} = 733.3 kg· mm^2.

Find x_{cm} and I^{cm}

x_{cm}

$$\bar{m} A x_{cm} = \int_0^l x\, dm = \bar{m}\int_0^l 2xy\, dx$$



$$\Rightarrow \quad A x_{cm} = 2\int_0^l x\left(a-(a-b)\frac{x}{l}\right) dx$$

$$\Rightarrow \quad x_{cm} = \frac{2}{(a+b)l}\left\{a\frac{l^2}{2} - (a-b)\frac{l^2}{3}\right\}$$

$$\Rightarrow \quad x_{cm} = \frac{(a+2b)}{(a+b)}\frac{l}{3} = 40\text{ mm}$$

$y = a-(a-b)\frac{x}{l}$

Let $\bar{m} = \frac{M}{A}$

$\Rightarrow dm = \bar{m}\, 2y\, dx$

& $A = (a+b)l$

& $2a = 20$mm, $2b = 5$mm

$M = 1$Kg, $l = 100$mm

I^{cm}

Using parallel axis theorem.

$$I^{cm} = I^o - M x_{cm}^2$$

$$I^o = \int_0^l x^2\, dm = \bar{m}\int_0^l 2x^2 y\, dx$$

$$\Rightarrow \quad I^o = \frac{2M}{A}\int_0^l x^2\left(a-(a-b)\frac{x}{l}\right)dx = \frac{1}{6}Ml^2\left[\frac{a+3b}{a+b}\right] = 2333.3\text{ Kg mm}^2$$

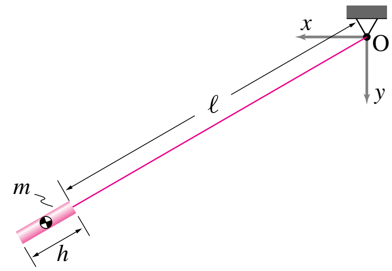
$$\Rightarrow \quad I^{cm} = \frac{1}{6}Ml^2\left[\frac{a+3b}{a+b}\right] - M\left[\frac{a+2b}{a+b}\cdot\frac{l}{3}\right]^2 = 733.3\text{ Kg mm}^2$$

16.3.11 A short rod of mass m and length h hangs from an inextensible and massless rod of length ℓ.

a) Find the moment of inertia I_{zz}^{O} of the rod.

b) Find the moment of inertia of the rod I_{zz}^{O} by considering it as a point mass located at its center of mass.

c) Find the percent error in I_{zz}^{O} in treating the bar as a point mass by comparing the expressions in parts (a) and (b). Plot the percent error versus h/ℓ. For what values of h/ℓ is the percentage error less than 5%?



Problem 16.11

Answer:
(a) $I_{zz}^{O} = \frac{mh^2}{12} + m\left(\ell + \frac{h}{2}\right)^2$.
(b) $I_{zz}^{O} = m\left(\ell + \frac{h}{2}\right)^2$.
(c) Percent error in I_{zz}^{O} is $100\left(1 + 12(\frac{\ell}{h} + 0.5)^2\right)^{-1}$. The percentage error less than 5% if $h/\ell < 1.32$.

15.3.11

a) Using parallel axis theorem,

$$\underline{I}^{o} = \underline{I}^{cm} + mr^2$$

$$\Rightarrow \quad \underline{I}^{o} = \frac{mh^2}{12} + m\left(l + \frac{h}{2}\right)^2$$

b) By considering the short rod as a point mass at its center of mass,

$$\underline{I}^{o} = m\left(l + \frac{h}{2}\right)^2$$

c) Percent error in $\underline{I}^{o}$ in treating the rod as a point mass,

$$\% e = \frac{\frac{mh^2}{12}}{\frac{mh^2}{12} + m\left(l + \frac{h}{2}\right)^2} \cdot 100$$

or

$$e = \frac{(h/l)^2}{(h/l)^2 + 12(1 + 0.5\, h/l)^2}$$

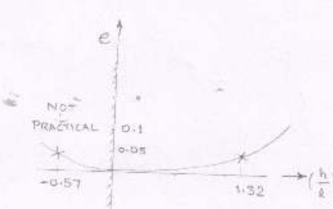
for the error to be less than 5%, we have

$$\frac{1}{4 + 12\left(\frac{1}{h/l} + \frac{1}{(h/l)^2}\right)} < 0.05$$

$$\Rightarrow \quad 4\left(\frac{h}{l}\right)^2 - 3\left(\frac{h}{l}\right) - 3 < 0$$

$$\Rightarrow \quad \left(\frac{h}{l} - 1.32\right)\left(\frac{h}{l} + 0.57\right) < 0$$

$$\Rightarrow \quad \frac{h}{l} < 1.32 \quad //$$



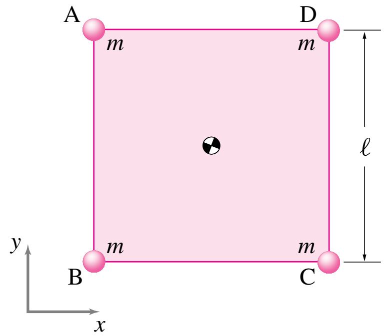
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.12 Parallel axis theorem? A massless square plate $ABCD$ has four identical point masses located at its corners.

a) Find the polar moment of inertia I_{zz}^{cm}.

b) Find a point P on the plate about which the system's moment of inertia I_{zz} is maximum.

c) Find the radius of gyration of the system about the rectangle center.

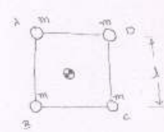


Problem 16.12

Answer:
(a) $I_{zz}^{cm} = 2m\ell^2$
(b) $P \equiv A,\ B,\ C$, or D, often, but not always, diagonally opposite the corner with the largest mass
(c) $r_{gyr} = \ell/\sqrt{2}$

Mass less square plate with 4 identical point masses at its corners.

a) $I^{cm} = \sum_i m_i r_i^2 = 4 \cdot m \left[\left(\frac{l}{2}\right)^2 + \left(\frac{l}{2}\right)^2 \right] = 2ml^2$



b) Systems' moment of inertia is maximum about either A, B, C or D.

$I^{max} = ml^2 + ml^2 + m(2l^2) = 4ml^2$

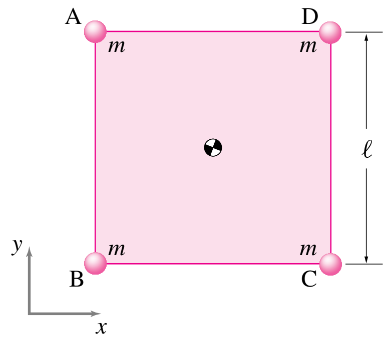
c) Radius of gyration about the center, r.

$(4m) r^2 = 2ml^2$

$\Rightarrow \quad r = \frac{l}{\sqrt{2}}$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.13 For the massless square plate with four point masses on the corners, the polar moment of inertia $I_{zz}^{cm} = 0.6\,\text{kg}\cdot\text{m}^2$. Find I_{xx}^{cm} of the system.



Problem 16.13

Answer:
$I_{xx}^{O} = I_{yy}^{O} = 0.3\,\text{kg}\cdot\text{m}^2$.

Given $I_{zz}^{cm} = 0.6$ Kg m^2

Using perpendicular axis theorem,

$$I_{zz}^{cm} = I_{xx}^{cm} + I_{yy}^{cm}$$

As the masses are symmetrically placed, $I_{xx}^{cm} = I_{yy}^{cm}$

$$\therefore \quad I_{zz}^{cm} = 2 I_{xx}^{cm}$$

$$\Rightarrow \quad I_{xx}^{cm} = \frac{1}{2} I_{zz}^{cm} = 0.3 \text{ Kg m}^2$$

(or) $$I_{xx}^{cm} = 2m\left(\frac{l}{2}\right)^2 + 2m\left(\frac{l}{2}\right)^2 = ml^2$$

As $$I_{zz}^{cm} = 2ml^2 = 0.6 \text{ Kg m}^2$$

$$I_{xx}^{cm} = ml^2 = 0.3 \text{ Kg m}^2$$

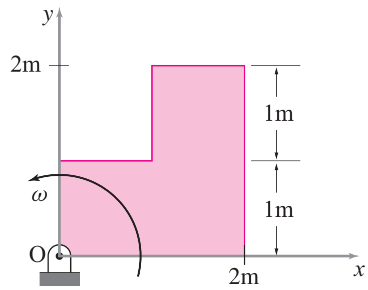
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

16.3.14 A uniform square plate (2 m on edge) has a corner cut out. The total mass of the remaining plate is 3 kg.

a) Where is the center of mass of the plate at the instant shown?

b) What is the moment of inertia of the plate about point O?

c) What is the moment of inertia of the plate about its center of mass?



Problem 16.14

Answer:
(a) $x_{cm} = 1.67\,\text{m}, y_{cm} = 0.83\,\text{m}$.
(b) $I_{zz}^{O} = 8\,\text{kg}\cdot\text{m}^2$.
(c) $I_{zz}^{cm} = 1.83\,\text{kg}\cdot\text{m}^2$.

15.3.14

a) Center of mass at the instant shown.

$m_{tot}\cdot R_x = m_1\cdot 0.5\,m + m_2\cdot 1.5\,m + m_3\cdot 1.5\,m$

$\Rightarrow \quad R_x = \frac{3.5}{3}\,m = 1.1667\,m$

III'ly $\quad m_{tot}\cdot R_y = m_1\cdot 0.5\,m + m_2\cdot 0.5\,m + m_3\cdot 1.5\,m$

$\Rightarrow \quad R_y = \frac{2.5\ \text{kg}\cdot m}{3\ \text{kg}} = 0.8333\,m.$

b) Find I^O.

$I^O = I^O_{a\times b\ \text{plate}} - I^O_{\frac{a}{2}\times\frac{b}{2}\ \text{cutout}}$

$= \frac{1}{3}M(a^2+b^2) - \left\{\left[\frac{1}{12}m\left(\frac{b}{2}\right)^2 + m\left(\frac{3b}{4}\right)^2\right] + \frac{1}{3}m\left(\frac{a}{2}\right)^2\right\}$

Substituting, $M = 4\,\text{kg}$, $a = 2\,m$, $b = 2\,m$, $m = 1\,\text{kg}$

$I^O = \frac{1}{3}4[8] - \left[\frac{1}{3} + \frac{1}{12} + \frac{9}{4}\right] = 4\left(\frac{8}{3}\right) - \frac{8}{3} = 8$ // in kg m²

c) $I^{cm} = ?$ Using parallel axis theorem

$I^O = I^{cm} + m_{tot}R^2$

$\Rightarrow I^{cm} = I^O - m_{tot}\left[R_x^2 + R_y^2\right]$

$= 8 - 3\left[1.167^2 + 0.833^2\right]$ kg m².

$I^{cm} = 1.834$ kg m² //

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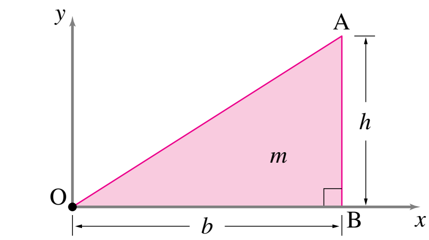
16.3.15 A uniform thin triangular plate of mass m, height h , and base b lies in the xy-plane.

a) Set up the integral to find the polar moment of inertia I_{zz}^O of the plate.

b) Show that $I_{zz}^O = \dfrac{m}{6}(h^2 + 3b^2)$ by evaluating the integral in part (a).

c) Locate the center of mass of the plate and calculate I_{zz}^{cm}.



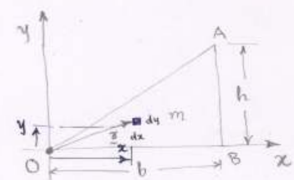
Problem 16.15

Answer:

(a) $I_{zz}^O = \dfrac{2m}{bh}\displaystyle\int_0^b \int_0^{hx/b} (x^2+y^2)\,dy\,dx.$

(c) $x_{cm} = \frac{2b}{3}, y_{cm} = \frac{h}{3}.$

a) Moment of inertia about 'O'



$$I_{zz}^O = \int r^2\,dm$$

$$I_{zz}^O = \int (x^2+y^2)\,\bar{m}\,dx\,dy$$

Let $\bar{m} = \dfrac{m}{A}$

$dm = \bar{m}\,dx\,dy$

$$\Rightarrow I_{zz}^O = \frac{2m}{bh}\int_0^b \int_0^{\frac{hx}{b}} (x^2+y^2)\,dy\,dx.$$

$\dfrac{b}{h} = \dfrac{x}{y} \Rightarrow y = \dfrac{hx}{b}$

b) $$I_{zz}^O = \frac{2m}{bh}\int_0^b \left\{\frac{h}{b}x^3 + \frac{h}{b}\left[\frac{h^2}{b^2}\cdot\frac{x^3}{3}\right]\right\} dx$$

$$\Rightarrow I_{zz}^O = \frac{2m}{b^2}\left\{ \frac{x^4}{4}\Big|_0^b + \frac{h^2}{b^2}\cdot\frac{1}{3}\,\frac{x^4}{4}\Big|_0^b \right\}$$

$$\Rightarrow I_{zz}^O = \frac{m}{6}(3b^2+h^2)$$

c) Center of mass. $(x_{cm}, y_{cm}) = ?$

$$m\,x_{cm} = \int x\,dm$$

$$\Rightarrow x_{cm} = \frac{1}{m}\int x\,\bar{m}\,\frac{hx}{b}\,dx = \frac{2}{3}\,\frac{x^3}{b^2}\Big|_0^b = \frac{2b}{3}$$

Similarly, $$y_{cm} = \frac{1}{m}\int y\,\bar{m}\,(b-x)\,dy = \frac{2}{h}\int\left(y-\frac{y^2}{h}\right)dy = \frac{h}{3}$$

Using parallel axis theorem,

$$I_{zz}^O = I^{cm} + m\,r_{O/cm}^2$$

$$\Rightarrow I^{cm} = I_{zz}^O - m\,r_{O/cm}^2$$

$$= \frac{m}{6}(3b^2+h^2) - m\left[\frac{4b^2}{9}+\frac{h^2}{9}\right]$$

$$= \frac{m}{\ }(b^2+h^2)$$

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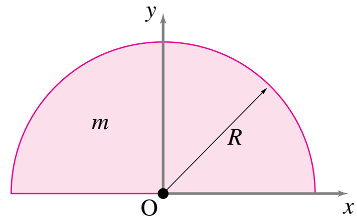
16.3.16 A uniform thin plate of mass m is cast in the shape of a semi-circular disk of radius R as shown in the figure.

a) Find the location of the center of mass of the plate

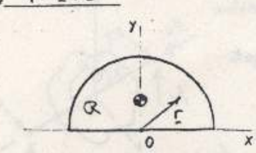
b) Find the polar moment of inertia of the plate, I_{zz}^{cm}. [Hint: It may be easier to set up and evaluate the integral for I_{zz}^{O} and then use the parallel axis theorem to calculate I_{zz}^{cm}.]



Problem 16.16

Answer:
(a) $x_{cm} = 0, y_{cm} = \frac{4R}{3\pi}$.
(b) $I_{zz}^{cm} = \frac{(9\pi^2-31)mR^2}{18\pi^2}$.

4.103



$$A_R = \int_0^\pi \int_0^R r\,dr\,d\theta = \frac{\pi R^2}{2}$$

$$\underline{r} = r(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})$$

(a) Center of Mass

$$m\underline{r}_{cm} = \int_R \underline{r}\,dm \qquad dm = \frac{m}{A_R} r\,dr\,d\theta$$

$$= \frac{2m}{\pi R^2}\int_0^\pi \int_0^R r(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\, r\,dr\,d\theta$$

$$= \frac{2m}{\pi R^2}\left(\frac{R^3}{3}\right)\int_0^\pi (\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath})\,d\theta$$

$$= \frac{2mR}{3\pi}\left([\sin\theta]_0^\pi\,\hat{\imath} + [-\cos\theta]_0^\pi\,\hat{\jmath}\right)$$

$$\boxed{\underline{r}_{cm} = \frac{4R}{3\pi}\hat{\jmath}}$$

(b) $I_{zz}^{cm} = I_{zz}^{o} - m|\underline{r}_{cm/o}|^2$

$$I_{zz}^{o} = \int_R |\underline{r}|^2\,dm = \int_R \underline{r}\cdot\underline{r}\,dm$$

$$= \frac{2m}{\pi R^2}\int_0^\pi \int_0^R r^2(\cos^2\theta + \sin^2\theta)\, r\,dr\,d\theta$$

$$= \frac{2m}{\pi R^2}\int_0^\pi \int_0^R r^3\,dr\,d\theta$$

$$= \frac{2m}{\pi R^2}(\pi)\frac{R^4}{4} = \frac{1}{2}mR^2.$$

Qu: Do you see why this makes sense?

$$I_{zz}^{cm} = \frac{mR^2}{2} - m\left(\frac{4R}{3\pi}\right)^2$$

$$\boxed{I_{zz}^{cm} = mR^2\left(\frac{1}{2} - \frac{16}{9\pi^2}\right)}$$

$$\approx .32\,mR^2$$

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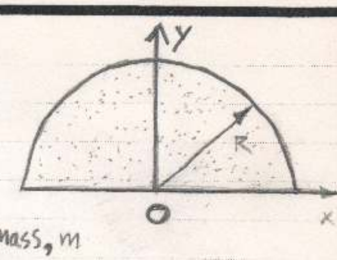
4.103



$\boxed{a.}$ Find CM of plate.

By symmetry, CM is located on y axis. $x_{cm/o} = 0$

$$y_{cm/o} = \frac{1}{m}\int_{\substack{\text{all}\\ \text{mass}}} y_{/o}\, dm = \frac{1}{m}\int_0^{\pi}\int_0^{R} r\sin\theta\ \rho\ r\,dr\,d\theta$$

ρ is density

$$= \frac{\rho}{m}\ -\cos\theta\Big|_0^{\pi}\int_0^R r^2\,dr = \frac{\rho}{m}\ -(-1)-(-1)\ \frac{r^3}{3}\Big|_0^R$$

$$= \frac{\rho}{m}\ \frac{2R^3}{3} \quad \text{but } \rho = \frac{2M}{\pi R^2}$$

$$= \frac{4R}{3\pi}$$

$$\boxed{\underline{r}_{cm/o} = \frac{4R}{3\pi}\hat{\jmath}}$$

can also integrate $[\rho\, r\,dr\,d\theta]$

$$\underline{r}_{cm/o} = \frac{1}{m}\int_0^{\pi}\int_0^{R}(r\cos\theta\,\hat{\imath} + r\sin\theta\,\hat{\jmath})$$

$\boxed{b.}$ $I_{zz}^{cm} = ?$ Use hint and calculate I_{zz}^{o} followed by application of parallel axis theorem and the result from $\boxed{a.}$ $\rho = \frac{2M}{\pi R^2}$ an areal density

$$I_{zz}^{o} = \int_{\substack{\text{all}\\ \text{mass}}} (x^2+y^2)\,dm = \int_0^{\pi}\int_0^{R} r^2\ \rho\ r\,dr\,d\theta$$

$$= \pi\rho\ \frac{r^4}{4}\Big|_0^R = \pi\ \frac{2M}{\pi R^2}\ \frac{R^4}{4} = \frac{MR^2}{2}$$

Parallel axis theorem states $I_{zz}^{o} = I_{zz}^{cm} + md^2$
where $d = \left|\underline{r}_{cm/o}\right| = \frac{4R}{3}$

$$I_{zz}^{cm} = \frac{mR^2}{2} - m\left(\frac{4R}{3\pi}\right)^2 = \frac{mR^2}{2} - \frac{16mR^2}{9\pi^2}$$

$$\boxed{I_{zz}^{cm} = \frac{(9\pi^2-32)\,mR^2}{18\,\pi^2}}$$

Can it really be this bad?

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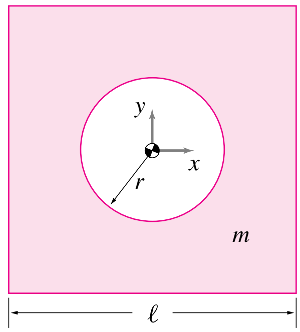
16.3.17 A uniform square plate of side $\ell = 250\,\text{mm}$ has a circular cut-out of radius $r = 50\,\text{mm}$. The mass of the plate is $m = \frac{1}{2}\,\text{kg}$.

a) Find the polar moment of inertia of the plate about its center.

b) Plot I_{zz}^{cm} versus r/ℓ.

c) Find the limiting values of I_{zz}^{cm} for $r = 0$ and $r = \ell$.



Problem 16.17

Answer:
(a) $I_{zz}^{cm} = \frac{m\ell^2}{6}\left(\frac{1-3\pi(r/\ell)^4)}{1-\pi(r/\ell)^2}\right)$.
(c) $\lim_{r\to 0} I_{zz}^{cm} = \dfrac{m\ell^2}{6}$ and
$\lim_{r\to \ell} I_{zz}^{cm} = \dfrac{m\ell^2}{6}\left(\dfrac{3\pi - 1}{\pi - 1}\right)$.

15.3.17

a) Polar moment of inertia about center:

$$I_{zz}^{cm} = I_{plate} - I_{cutout}$$

$$= \frac{M\ell^2}{6} - \frac{m_c r^2}{2}$$

$$= \frac{m\ell^4}{6(\ell^2-\pi r^2)} - \frac{m\pi r^4}{2(\ell^2-\pi r^2)}$$

$$= \frac{m}{\ell^2-\pi r^2}\left[\frac{\ell^4 - 3\pi r^4}{6}\right]$$

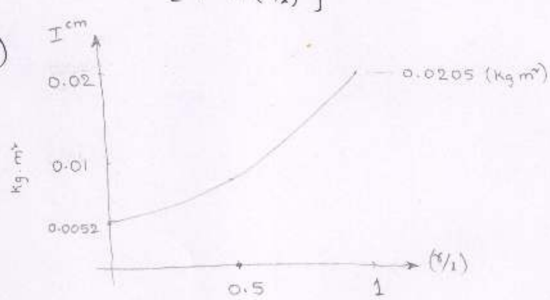
$$= \frac{m\ell^2}{6}\left[\frac{1-3\pi(r/\ell)^4}{1-\pi(r/\ell)^2}\right]$$

Let M = mass of plate

m_c = mass of disk

$$\frac{M}{\ell^2} = \frac{m}{\ell^2-\pi r^2} = \frac{m_c}{\pi r^2}$$

b)



c) $\lim_{r\to 0} I_{zz}^{cm} = \frac{1}{6} m\ell^2$ = Moment of inertia of a square plate

$$\lim_{r\to \ell} I_{zz}^{cm} = \frac{m\ell^2}{6}\left[\frac{3\pi - 1}{\pi - 1}\right] = 0.0205\ \text{Kg m}^2$$

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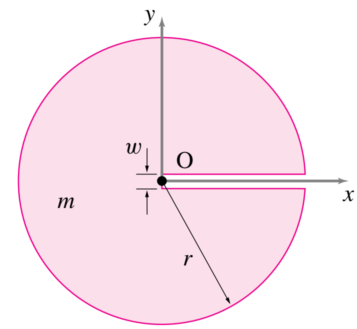
16.3.18 A uniform thin circular disk of radius $r = 100\,\text{mm}$ and mass $m = 2\,\text{kg}$ has a rectangular slot of width $w = 10\,\text{mm}$ cut into it as shown in the figure. You can approximate the right edge of the cutout as a straight line (thus your calculation would be exact for a system that has a very thin sliver of material closing the gap at the right).

a) Find the polar moment of inertia I^O_{zz} of the disk.

b) Locate the center of mass of the disk and calculate I^{cm}_{zz}.

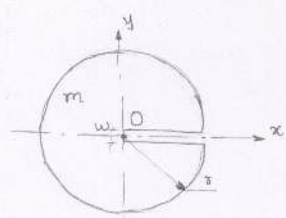
c) estimate the error in your calculation of the mass, center of mass location and moment of inertia due to not subtracting out the sliver of material at the right edge of the slot.



Problem 16.18

Answer:
(a) $I^O_{zz} = 0.01010904\,\text{kg}\cdot\text{m}^2$.
(b) $x_{cm} = -1.64\,\text{mm}$, $y_{cm} = 0\,\text{mm}$, and $I^{cm}_{zz} = 0.01010364\,\text{kg}\cdot\text{m}^2$.
(c) Not subtracting out the sliver of material at the right edge of the slot leads to an error of 0.053%.

a) Polar moment of inertia about 'O'



$$I^O_{zz} = \frac{Mr^2}{2} - \left[\frac{m_c}{12}(r^2+w^2) + m_c\left(\frac{r}{2}\right)^2\right]$$

$$= \frac{Mr^2}{2} - \frac{m_c}{12}(4r^2+w^2)$$

$$\Rightarrow I^O_{zz} = \frac{m\pi r^4}{2(\pi r^2 - rw)} - \frac{m\,rw}{12(\pi r^2 - rw)}(4r^2+w^2)$$

$$= \frac{m}{\pi r^2 - rw}\left[\frac{\pi r^4}{2} - \frac{rw}{12}(4r^2+w^2)\right]$$

Let $\bar{m}$ = mass / unit area

$$\frac{M}{\pi r^2} = \frac{m}{\pi r^2 - rw} = \frac{m_c}{rw}$$

where,
M = mass of the disk
m_c = mass of cut portion

Given data: $r = 0.1\,\text{m}$; $m = 2\,\text{Kg}$; $W = 0.01\,\text{m}$

$\Rightarrow I^O_{zz} = 0.0101\ \text{kg m}^2$ (or) $0.01010904\ \text{kg m}^2$

b) Center of mass: (x_{cm}, y_{cm})

As the object is symmetric about x-axis, $y_{cm} = 0\,\text{m}$ //

$x_{cm} = ?$ $\quad m\,x_{cm} = M(0) - m_c\left(\frac{r}{2}\right)$

$$\Rightarrow x_{cm} = -\frac{rw}{\pi r^2 - rw}\cdot\frac{r}{2} = \frac{-rw}{(2\pi r - 2w)} = -0.00164\ \text{m} \ //$$

c) $$I^{cm}_{zz} = I^O - m x_{cm}^2 = \frac{m}{(\pi r^2 - rw)}\left[\frac{\pi r^4}{2} - \frac{rw}{12}(4r^2+w^2) - \frac{r^4 w^2}{4(\pi r^2 - rw)}\right]$$

$= 0.0101\ \text{kgm}^2$ more precisely $0.01010364\ \text{kg m}^2$

16.3.19 Calculate the polar moment of inertia of a uniform square plate with mass m and side ℓ various ways. Choose numerical values for m and ℓ if you want to address this as a numerical calculation rather than a theoretical one.

a) As a single uniform plate
b) As a composite of 4 plates, each with sides $\ell/2$
c) As a composite of 100 plates, each with sides $\ell/10$ (These first three methods should give exactly the same answer).
d) For the composite of 100 plates now neglect the terms which concern the moments of inertia of each small square about its center of mass. What is the error in making this neglect?

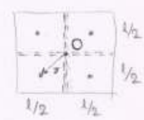
Answer:
(d) Exactly 1% .

a) Single uniform plate with mass 'm', side 'l',

$$I^{cm} = \frac{m}{12}(l^2+l^2) = \frac{1}{6}ml^2 = I^o_{zz}$$

b) Composite of 4 plates, each with sides $\frac{l}{2}$

Mass of each square : $\frac{m}{4}$



Distance to the center of mass of each square from 'O' : $\sqrt{(\frac{l}{4})^2+(\frac{l}{4})^2} = \sqrt{2}\,\frac{l}{4}$

$I^{cm} = \frac{1}{6}(\frac{m}{4})(\frac{l}{2})^2$ for each square

$$\Rightarrow I^o = 4I^{cm} + 4(\frac{m}{4})(\sqrt{2}\,\frac{l}{4})^2 = \frac{ml^2}{24} + m\frac{l^2}{8} = \frac{1}{6}ml^2$$

c) Plate formed by 100 squares.

These 100 squares can be categorised into 2 sets, with one set containing 4 squares having same distance from 'O' and other set with 8 squares having same distance from 'O'. Consider lower left quarter of the plate with 5×5 elements. Let r_{ij} be the distance from O to i-j element, where i = 1 to 5 & j = 1 to 5.

$$\Rightarrow r_{ii} = \sqrt{2}\left\{\frac{l(11-2i)}{20}\right\} \quad i = 1 \text{ to } 5$$

(or) and $r_{ij}^2 = \left[(11-2i)\frac{l}{20}\right]^2 + \left[(11-2j)\frac{l}{20}\right]^2$

Moment of Inertia, $I^o = I^{cm} + mr^2$ (or) $I^o = \sum_k^{100} I_k^{cm} + \sum_k^{100} m_k r_k^2$

$$\Rightarrow I^o = 100\cdot\frac{m}{100}\cdot\frac{(l/10)^2}{6} + 4\sum_i^5 \frac{m}{100} r_{ii}^2 + 8\sum_i^5\sum_j^5 \frac{m}{100} r_{ij}^2 = ml^2\left\{\frac{1}{6\cdot 10^2} + \frac{4\cdot 3\cdot 3}{20^2} + \frac{8\cdot 6\cdot 6}{20^2}\right\}$$

$$\Rightarrow I^o = \frac{ml^2}{10^2}\left\{\frac{1}{6} + \frac{66}{4}\right\} = \frac{1}{6}ml^2$$

d) Error if I^{cm} is ignored, $e = \dfrac{\frac{1}{6}ml^2 - \frac{66}{20^2}ml^2}{\frac{1}{6}ml^2} = 0.01$

$\Rightarrow$ % e = 1 % exactly.

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