SAMPLE 16.9 **Moment of inertia of point masses:** A pendulum is made up of two unequal point masses m and $2m$ connected by a massless rigid rod of length $4r$. The pendulum is pivoted at distance r along the rod from the small mass.

1. Find the moment of inertia I_{zz}^{cm} of the pendulum.
2. Find the moment of inertia I_{zz}^{O} of the pendulum.
3. Find the radius of gyration of the pendulum.

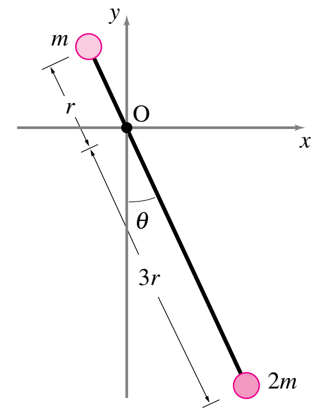

Figure 16.37

Solution

1. First we need to find the center of mass of the system. Let the center of mass C be located at distance r_{cm} from the origin O. An easy way to find the location of C will be to consider the pendulum to lie along the x-axis with the origin at O (see fig. 16.38). Then

$$\begin{aligned} x_{cm} &= \frac{\sum m_i x_i}{\sum m_i} \\ \Rightarrow \quad r_{cm} &= \frac{m(-r) + 2m(3r)}{m + 2m} = \frac{5}{3}r. \end{aligned}$$

So, the moment of inertia about the center of mass is

$$I_{zz}^{\text{cm}} = \sum m_i r_{i/cm}^2 = m\left(r + \frac{5}{3}r\right)^2 + 2m\left(3r - \frac{5}{3}r\right)^2 = \frac{32}{3}mr^2.$$

$$\boxed{I_{zz}^{\text{cm}} = 10.67mr^2}$$

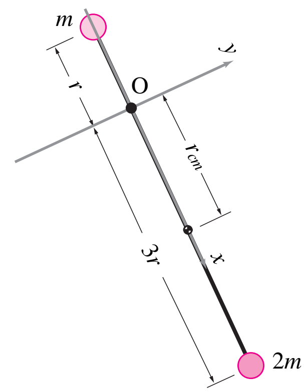

Figure 16.38

2. We can calculate I_{zz}^{O} in two different ways, one directly by summing the contributions of each mass about point O, and the other by using I_{zz}^{cm} and the parallel axis theorem.

$$\begin{aligned} I_{zz}^{\text{O}} &= mr^2 + 2m(3r)^2 = 19mr^2 \\ \text{or} \qquad I_{zz}^{\text{O}} &= I_{zz}^{\text{cm}} + m_{tot} r_{cm/O}^2 \\ &= \frac{32}{3}mr^2 + 3m\left(\frac{5}{3}r\right)^2 = 19mr^2. \end{aligned}$$

$$\boxed{I_{zz}^{O} = 19mr^2}$$

3. The radius of gyration, by definition, is the distance that gives the desired moment of inertia if the entire mass of the system is concentrated there. Thus, if r_{gyr} is the radius of gyration for the moment of inertia about point O, then

$$\begin{aligned} I_{zz}^{\text{O}} &= (3m)\cdot r_{gyr}^2 \\ \Rightarrow \quad 19\cancel{m}r^2 &= 3\cancel{m}r_{gyr}^2 \\ \Rightarrow \quad r_{gyr} &= \sqrt{\frac{19}{3}}r = 2.52r. \end{aligned}$$

Thus the radius of gyration r_{gyr} of the given pendulum is $r_{gyr} = 2.52r$.

$$\boxed{r_{gyr} = 2.52r}$$

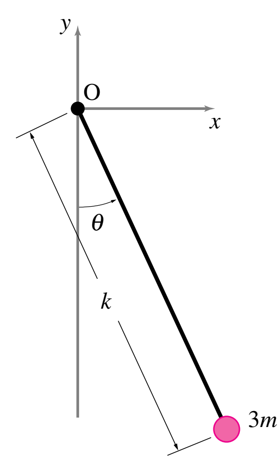

Figure 16.39

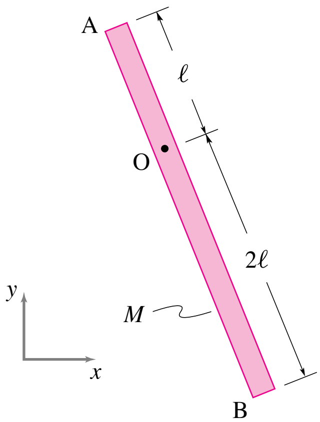

Figure 16.40

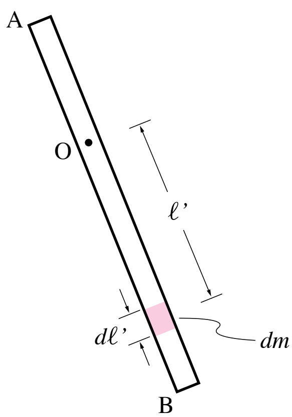

Figure 16.41

SAMPLE 16.10 **Moment of inertia of a rod:** A uniform rigid rod AB of mass $M = 2\,\mathrm{kg}$ and length $3\ell = 1.5\,\mathrm{m}$ swings about the z-axis passing through the pivot point O.

1. Find the moment of inertia I^O_{zz} of the bar using the fundamental definition $I^O_{zz} = \int_m r^2_{/O}\, dm$.
2. Find I^O_{zz} using the parallel axis theorem given that $I^{\mathrm{cm}}_{zz} = \frac{1}{12}m\ell^2$ where m = total mass, and ℓ = total length of the rod. (You can find I^{cm}_{zz} for many commonly encountered objects in the table on the inside backcover of the text).

Solution

1. Since we need to carry out the integral, $I^O_{zz} = \int_m r^2_{/O}\, dm$, to find I^O_{zz}, let us consider an infinitesimal length segment $d\ell'$ of the bar at distance ℓ' from the pivot point O. (see Figure 16.41). Let the mass of the infinitesimal segment be dm.
Now the mass of the segment may be written as

$$\begin{aligned} dm &= \text{(mass per unit length of the bar)} \cdot \text{(length of the segment)} \\ &= \frac{M}{3\ell}\, d\ell' \qquad \left(\text{Note: } \frac{\text{mass}}{\text{unit length}} = \frac{\text{total mass}}{\text{total length}}\right). \end{aligned}$$

We also note that the distance of the segment from point O, $r_{/O} = \ell$. Substituting the values found above for $r_{/O}$ and dm in the formula we get

$$\begin{aligned} I^O_{zz} &= \int_{-\ell}^{2\ell} \underbrace{(\ell')^2}_{r^2_{/O}} \underbrace{\frac{M}{3\ell}d\ell'}_{dm} \\ &= \frac{M}{3\ell}\int_{-\ell}^{2\ell} (\ell')^2 d\ell' = \frac{M}{3\ell}\left[\frac{\ell'^3}{3}\right]_{-\ell}^{2\ell} \\ &= \frac{M}{3\ell}\left[\frac{8\ell^3}{3} - \left(-\frac{\ell^3}{3}\right)\right] = M\ell^2 \\ &= 2\,\mathrm{kg}\cdot(0.5\,\mathrm{m})^2 = 0.5\,\mathrm{kg}\cdot\mathrm{m}^2. \end{aligned}$$

$$I^O_{zz} = 0.5\,\mathrm{kg}\cdot\mathrm{m}^2$$

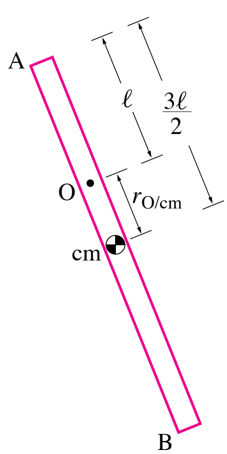

Figure 16.42

2. The parallel axis theorem states that

$$I^O_{zz} = I^{\mathrm{cm}}_{zz} + M r^2_{O/\mathrm{cm}}.$$

Since the rod is uniform, its center-of-mass is at its geometric center, *i.e.*, at distance $\frac{3\ell}{2}$ from either end. From the Fig 16.42 we can see that

$$r_{O/\mathrm{cm}} = AG - AO = \frac{3\ell}{2} - \ell = \frac{\ell}{2}$$

$$\begin{aligned} \text{Therefore,}\quad I^O_{zz} &= \underbrace{\frac{1}{12}M(3\ell)^2}_{I^{\mathrm{cm}}_{zz}} + M\left(\frac{\ell}{2}\right)^2 \\ &= \frac{9}{12}M\ell^2 + M\frac{\ell^2}{4} = M\ell^2 \\ &= 0.5\,\mathrm{kg}\cdot\mathrm{m}^2 \quad \text{(same as in (a), of course)} \end{aligned}$$

$$I^O_{zz} = 0.5\,\mathrm{kg}\cdot\mathrm{m}^2$$

SAMPLE 16.11 Moment of inertia of a wheel with a cut-out: A uniform rigid wheel of radius $r = 1\,\mathrm{ft}$ is made eccentric by cutting out a portion of the wheel. The center-of-mass of the eccentric wheel is at C, a distance $e = \frac{r}{3}$ from the geometric center O. The mass of the wheel (after deducting the cut-out) is 3.2 lbm. The moment of inertia of the wheel about point O, I_{zz}^{O}, is $1.8\ \mathrm{lbm{\cdot}ft^2}$. We are interested in the moment of inertia I_{zz} of the wheel about points A and B on the perimeter.

1. Without any calculations, guess which point, A or B, gives a higher moment of inertia. Why?
2. Calculate I_{zz}^{C}, I_{zz}^{A} and I_{zz}^{B} and compare with the guess in (a).

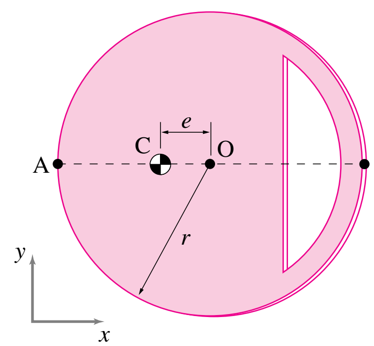

Figure 16.43

Solution

1. The moment of inertia I_{zz}^{B} should be higher. Moment of inertia I_{zz} measures the geometric distribution of mass about the z-axis. But the distance of the mass from the axis counts more than the mass itself ($I_{zz}^{\mathrm{O}} = \int_m r_{/O}^2\,dm$). The distance $r_{/O}$ of the mass appears as a quadratic term in I_{zz}^{O}. The total mass is the same whether we take the moment of inertia about point A or about point B. However, the distribution of mass is not the same about the two points. Due to the cut-out being closer to point B there are more "dm's" at greater distances from point B than from point A. So, we guess that

$$\boxed{I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}}$$

2. If we know the moment of inertia I_{zz}^{C} (about the center-of-mass) of the wheel, we can use the parallel axis theorem to find I_{zz}^{A} and I_{zz}^{B}. In the problem, we are given I_{zz}^{O}. But,

$$\begin{array}{rrcl}
 & I_{zz}^{\mathrm{O}} & = & I_{zz}^{\mathrm{C}} + Mr_{O/C}^2 \quad \text{(parallel axis theorem)} \\
\Rightarrow & I_{zz}^{\mathrm{C}} & = & I_{zz}^{\mathrm{O}} - Mr_{O/C}^2 \\
 & & = & 1.8\,\mathrm{lbm\,ft^2} - 3.2\,\mathrm{lbm}\underbrace{\left(\frac{1\,\mathrm{ft}}{3}\right)^2}_{r_{O/C}=e=\frac{r}{3}} \\
 & & = & 1.44\,\mathrm{lbm{\cdot}ft^2} \\
\text{Now,} & I_{zz}^{\mathrm{A}} & = & I_{zz}^{\mathrm{C}} + Mr_{A/C}^2 = I_{zz}^{\mathrm{C}} + M\left(\frac{2r}{3}\right)^2 \\
 & & = & 1.44\,\mathrm{lbm{\cdot}ft^2} + 3.2\,\mathrm{lbm}\left(\frac{2\,\mathrm{ft}}{3}\right)^2 \\
 & & = & 2.86\,\mathrm{lbm{\cdot}ft^2} \\
\text{and} & I_{zz}^{\mathrm{B}} & = & I_{zz}^{\mathrm{C}} + Mr_{B/C}^2 = I_{zz}^{\mathrm{C}} + M\left(r + \frac{r}{3}\right)^2 \\
 & & = & 1.44\,\mathrm{lbm{\cdot}ft^2} + 3.2\,\mathrm{lbm}\left(1\,\mathrm{ft} + \frac{1\,\mathrm{ft}}{3}\right)^2 \\
 & & = & 7.13\,\mathrm{lbm{\cdot}ft^2}
\end{array}$$

$$\boxed{I_{zz}^{\mathrm{C}} = 1.44\,\mathrm{lbm{\cdot}ft^2},\ I_{zz}^{\mathrm{A}} = 2.86\,\mathrm{lbm{\cdot}ft^2},\ I_{zz}^{\mathrm{B}} = 7.13\,\mathrm{lbm{\cdot}ft^2}}$$

Clearly, $I_{zz}^{\mathrm{B}} > I_{zz}^{\mathrm{A}}$, as guessed in (a).

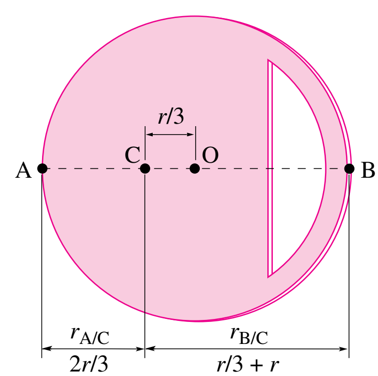

Figure 16.44

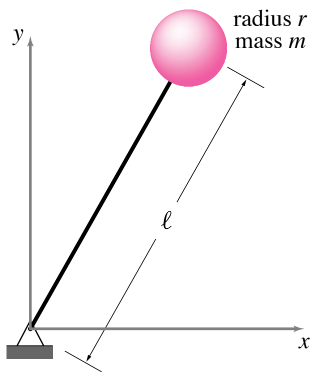

Figure 16.45

SAMPLE 16.12 Moment of inertia: modeling a sphere as a point mass: A uniform solid sphere of mass m and radius r is attached to a massless rigid rod of length ℓ. The sphere swings in the xy plane. Find the error in calculating I_{zz}^{O} as a function of r/ℓ if the sphere is treated as a point mass concentrated at the center-of-mass of the sphere.

Solution The exact moment of inertia of the sphere about point O can be calculated using parallel axis theorem:

$$
\begin{aligned}
I_{zz}^{O} &= I_{zz}^{cm} + m\ell^2 \\
&= \frac{2}{5}mr^2 + m\ell^2. \quad \text{(See Table IV on inside cover)}
\end{aligned}
$$

If we treat the sphere as a point mass, the moment of inertia I_{zz}^{O} is

$$\tilde{I}_{zz}^{O} = m\ell^2.$$

Therefore, the relative error in I_{zz}^{O} is

$$
\begin{aligned}
\text{error} &= \frac{I_{zz}^{O} - \tilde{I}_{zz}^{O}}{I_{zz}^{O}} \\
&= \frac{\frac{2}{5}mr^2 + m\ell^2 - m\ell^2}{\frac{2}{5}mr^2 + m\ell^2} \\
&= \frac{\frac{2}{5}\frac{r^2}{\ell^2}}{\frac{2}{5}\frac{r^2}{\ell^2} + 1}
\end{aligned}
$$

From the above expression we see that for $r \ll \ell$ the error is very small. From the graph of error in Fig. 16.46 we see that even for $r = \ell/5$, the error in I_{zz}^{O} due to approximating the sphere as a point mass is less than 2%.

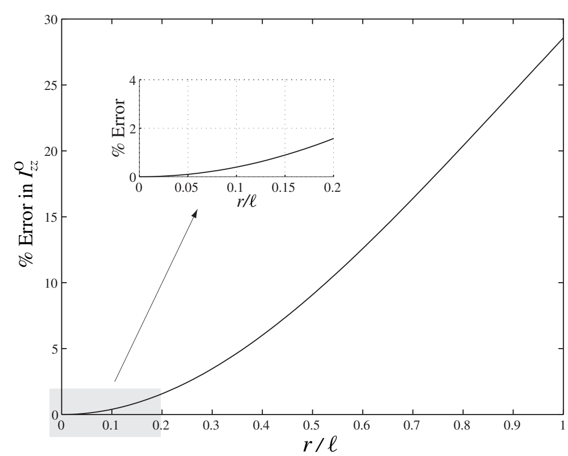

Figure 16.46: Relative error in I_{zz}^{O} of the sphere as a function of r/ℓ.