16.2 Angular velocity and acceleration of a rigid object in planar circular motion

Angular velocity of a rigid object: $\vec{\omega}$

Thus far we have talked about rotation, but not how it varies in time. Dynamics is about motion, velocities and accelerations, so we need to think about rotation rates and rotational accelerations.

A 2D rigid object's net rotation is measured by the rotation angle θ. Thus, the simplest measure of rotation rate is $\dot{\theta} \equiv \frac{d\theta}{dt}$. Because all marked lines rotate the same amount θ they all have the same rates of change. So $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 =$ *etc.* and, as for rotation, the concept of rotation rate of a rigid object transcends the concept of rotation rate of this or that particular line. We give this rotation rate of a rigid object a special name, *angular velocity*, and symbol, ω (omega).

Repeating, for all lines marked on a rigid object,

$$\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \cdots = \dot{\theta}. \tag{16.10}$$

Often we think of angular velocity as a vector $\vec{\boldsymbol{\omega}}$. Its direction is the axis of the rotation which, for objects in the xy plane is $\hat{\boldsymbol{k}}$, pointed in the $+z$ direction normal to the xy plane. The scalar part of $\vec{\boldsymbol{\omega}}$ is ω. So, the *angular velocity vector* is

$$\vec{\boldsymbol{\omega}} \equiv \omega \hat{\boldsymbol{k}} \tag{16.11}$$

with ω as defined in eqn. (16.10). Note $\vec{\boldsymbol{\omega}}$ is the angular velocity of the object (and of *every* line on it)①. The usual sign convention for ω is the same as that for θ, positive is counter-clockwise (CCW) around the origin. That's the direction your fingers wrap if you point your thumb in the $+z$ direction and grab the z axis.

①We could also think of rotation rate as a derivative of the rotation matrix $[R]$. This point of view and its relation to the vector point of view here, is described in Box 16.5 on page 8.

Rate of change of $\hat{\boldsymbol{\imath}}'$, $\hat{\boldsymbol{\jmath}}'$

Our first use of the angular velocity vector $\vec{\boldsymbol{\omega}}$ is to calculate the rate of change of the rotating unit base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$. We can find the rate of change of, say, $\hat{\boldsymbol{\imath}}'$, by taking the time derivative of the first eqn. (16.4), and using the chain rule while recognizing that $\theta = \theta(t)$. We can also make an analogy with polar coordinates (page 725), where we think of $\hat{\boldsymbol{e}}_R$ as like $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{e}}_\theta$ as like $\hat{\boldsymbol{\jmath}}'$. We found there that $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. Either from differentiating $\hat{\boldsymbol{\imath}}'$ or from the correlation with polar coordinates, we get

$$\begin{aligned} \dot{\hat{\imath}}' &= \dot{\theta}\hat{\jmath}' \quad &\text{or} \quad \dot{\hat{\imath}}' &= \vec{\omega}\times\hat{\imath}' \quad \text{and} \\ \dot{\hat{\jmath}}' &= -\dot{\theta}\hat{\imath}' \quad &\text{or} \quad \dot{\hat{\jmath}}' &= \vec{\omega}\times\hat{\jmath}' \end{aligned} \tag{16.12}$$

because $\hat{\jmath}' = \hat{k}' \times \hat{\imath}'$ and $\hat{\imath}' = -\hat{k}' \times \hat{\jmath}'$. Depending on the tastes of your lecturer, you may find Equations 16.12 are the most used equations from this point onward②.

② Eqn. 16.12 is sometimes considered the definition of $\vec{\omega}$. In this view, popularized by Tom Kane, $\vec{\omega}$ is that vector which determines $\dot{\hat{\imath}}'$ and $\dot{\hat{\jmath}}'$ by the formulas $\dot{\hat{\imath}}' = \vec{\omega}\times\hat{\imath}'$ and $\dot{\hat{\jmath}}' = \vec{\omega}\times\hat{\jmath}'$. Then one needs to show that such a vector exists and that it is $\vec{\omega} = \hat{\imath}' \times \dot{\hat{\imath}}'$. Luckily this reasoning leads to the same $\vec{\omega}$ as our $\vec{\omega} = \dot{\theta}\hat{k}$.

Velocity of a point fixed on a rigid object

Let's call some rotating object $\mathcal{B}$ (script capital B) to which is glued a coordinate system $x'y'$ with base vectors $\hat{\imath}'$ and $\hat{\jmath}'$. We now introduce the concept of *derivative in a frame* which we write, for the frame $\mathcal{B}$, as

$$\frac{{}^{\mathcal{B}}d}{dt}$$

which means, in words, the rate of change of something as viewed in the rotating frame $\mathcal{B}$. Now consider a point P at $\vec{r}_{\mathrm{P}}$ that is glued to the object. That is, the x' and y' coordinates of $\vec{r}_{\mathrm{P}}$ do not change in time.

$$\frac{{}^{\mathcal{B}}d\vec{r}_{\mathrm{P}}}{dt} \equiv \overset{\mathcal{B}.}{\vec{r}}_{\mathrm{P}} = \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' = \vec{0}.$$

That is, relative to a moving frame, the velocity of a point glued to the frame is zero (no surprise).

We would like to know the velocity of such a point in the fixed frame. We just take the derivative, using the product rule and the differentiation rules we have developed for the rotating base vectors:

16.2 The fixed Newtonian reference frame $\mathcal{F}$

Now we can reconsider the concept of a Newtonian frame, a concept which we had to assume to write the equations of dynamics in the first place. All of mechanics depends, of course, on the laws of mechanics. The laws of mechanics are equations which involve, in part, the positions of things as a function of time. But how position is perceived to change in time depends on your reference frame. And some reference frames are better than others. The best, from our point of view, are reference frames in which Newton's laws are accurate. Such a reference frame is called a *Newtonian frame*. In engineering practice the frames we use as approximations of a Newtonian frame often seem, loosely speaking, somehow still. So we sometimes call such a frame *the fixed frame* and label it with a script capital $\mathcal{F}$. When we talk about velocity and acceleration of mass points, for use in the equations of mechanics, we are always talking about the velocity and acceleration relative to a $\mathcal{F}$ixed, or equivalently, Newtonian frame.

Assume x and y are the coordinates of a vector $\vec{r}_{\mathrm{P}}$ and $\mathcal{F}$ is a fixed frame with fixed axis (with associated constant base vectors $\hat{\imath}$ and $\hat{\jmath}$). When we write $\dot{\vec{r}}_P$ we mean $\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. But we could be more explicit (and notationally ornate) and write the velocity of P in the Newtonian frame as

$$\frac{{}^{\mathcal{F}}d\vec{r}_{\mathrm{P}}}{dt} \equiv \overset{\mathcal{F}.}{\vec{r}}_{\mathrm{P}} \quad \text{by which we mean} \quad \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}.$$

The $\mathcal{F}$ in front of the time derivative (or in front of the dot) means that when we calculate a derivative we hold the base vectors of $\mathcal{F}$ constant. This is no surprise, because for $\mathcal{F}$ the base vectors *are* constant. In general, however, when taking a derivative in a given frame you

- write vectors in terms of base vectors stuck to the frame, and
- only differentiate the components.

But only for the $\mathcal{F}$ixed or $\mathcal{N}$ewtonian frame will accelerations calculated this way be directly applicable to Newton's laws.

We will avoid the ornate notation of labeling frames when it is not needed. For example, if you don't see any script capital letters floating around in front of derivatives, you can assume that we are taking derivatives relative to a fixed Newtonian frame.

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\mathrm{P}} &= x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\mathrm{P}} = \dot{\vec{\boldsymbol{r}}}_{\mathrm{P}} &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) = x'\dot{\hat{\boldsymbol{\imath}}}' + y'\dot{\hat{\boldsymbol{\jmath}}}' = x'(\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\boldsymbol{\omega}} \times (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')
\end{aligned}
$$

where $\dot{\vec{\boldsymbol{r}}}_P$ is the simple way to write $\frac{^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P}}}{dt}$. Thus,

$$\vec{\boldsymbol{v}}_{\mathrm{P}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} \tag{16.18}$$

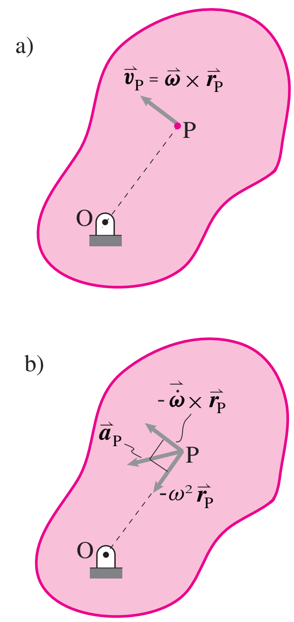

Figure 16.15: Velocity and acceleration of two points on a rigid object rotating about 0.

We can rewrite eqn. (16.18) in a minimalist or elaborate notation as

$$
\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \qquad \text{or} \\
\frac{^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\mathrm{P}}}{dt} &= \vec{\boldsymbol{\omega}}_{_{\mathcal{B}/\mathcal{F}}} \times \vec{\boldsymbol{r}}_{\mathrm{P/O}}.
\end{aligned}
$$

Both are correct. In the first case you have to use common sense to know what point you are talking about and that it is on an object rotating with absolute angular velocity $\vec{\boldsymbol{\omega}}$. In the second case everything is laid out perfectly clearly (which is why it looks perfectly confusing). On the left side of the equation it says that we are interested in how point P moves relative to, not just any frame, but the fixed frame $\mathcal{F}$. On the right side we make clear that the rotation rate we are looking at is that of object $\mathcal{B}$ relative to $\mathcal{F}$ and not some other relative rotation. We further make clear that the formula only makes sense if the position of the point P is measured relative to a point which doesn't move, namely 0.

What we have just found largely duplicates what we already learned in section 7.1 for points moving in circles. The slight generalization is that the same angular velocity $\vec{\boldsymbol{\omega}}$ can be used to calculate the velocities of

16.3 Plato's discussion of spinning in circles as motion (or not)

Here is how a smart person, Plato, thought about these things two thousand years ago. Here he writes about an imagined discussion between Socrates and Glaucon about how an object can maintain contradictory attributes simultaneously.

"Socrates: Now let's have a more precise agreement so that we won't have any grounds for dispute as we proceed. If someone were to say of a human being standing still, but moving his hands and head, that the same man at the same time stands still and moves, I don't suppose we'd claim that it should be said like that, but rather that one part of him stands still and another moves. Isn't that so?
Glaucon: Yes it is.
Socrates: Then if the man who says this should become still more charming and make the subtle point that tops as wholes stand still and move at the same time when the peg is fixed in the same place and they spin, or that anything else going around in a circle on the same spot does this too, we wouldn't accept it because it's not with respect to the same part of themselves that such things are at the same time both at rest and in motion. But we'd say that they have in them both a straight and a circumference; and with respect to the straight they stand still since they don't lean in any direction –while with respect to the circumference they move in a circle; and when the straight inclines to the right the left, forward, or backward at the same time that it's spinning, then in no way does it stand still.
Glaucon: And we'd be right."

This chapter is about things that are still with respect to their own parts (they do not distort) but in which the points do move in circles.

multiple points on one rigid object. But the key idea remains: the velocity of a point going in circles is tangent to the circle it is going around and with magnitude proportional both to distance from the center and to the angular rate of rotation (fig. 16.15a).

Acceleration of a point on a rotating rigid object

Let's again consider a point stuck on a rotating object and with position

$$\vec{\boldsymbol{r}}_{\mathrm{P}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'.$$

Relative to the frame $\mathcal{B}$ to which a point is attached, its acceleration is zero (again no surprise). But what is its acceleration in the fixed frame? We find this acceleration by writing the position vector and then differentiating twice, repeatedly using the product rule and eqn. (16.12) we get (see Box 16.4 for the details).

$$\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\vec{\boldsymbol{\omega}}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} + \vec{\boldsymbol{\omega}} \times (\vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}}) \tag{16.19}$$

which is hardly intuitive at a glance③. Recalling that in 2D $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$ we can use either the right hand rule or manipulation of unit vectors to rewrite eqn. (16.19) as

$$\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} - \omega^2\vec{\boldsymbol{r}}_{\mathrm{P}} \tag{16.20}$$

where $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$④.

Thus, as we found in section 15.1 for a particle going in circles, the acceleration can be written as the sum of two terms, a tangential acceleration $\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{P}}$ due to increasing tangential speed, and a centrally directed

③ Although the form eqn. (16.19) is not of much immediate use, if you are going to continue on to the mechanics of mechanisms or three-dimensional mechanics, you should follow the derivation of eqn. (16.19) carefully.

④The derivation of eqn. (16.20) can be written more informally and briefly like this (using minimalist notation):

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \dot{\vec{\boldsymbol{v}}} = \frac{d}{dt}(\vec{\omega} \times \vec{\boldsymbol{r}}) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times \dot{\vec{\boldsymbol{r}}} \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} - \omega^2\vec{\boldsymbol{r}}.
\end{aligned}$$

16.4 Acceleration of a point on a rotating body, using $\vec{\omega}$

Leaving off the ornate pre-super-script $\mathcal{F}$ for simplicity, we have

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\vec{\boldsymbol{v}}}_{\mathrm{P}} &= \frac{d}{dt}\left(\frac{d}{dt}(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')\right) \\
&= \frac{d}{dt}\left(x'(\vec{\omega} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\omega} \times \hat{\boldsymbol{\jmath}}')\right). \qquad (16.13)
\end{aligned}$$

To continue we need to use the product rule of differentiation for the cross product of two time dependent vectors like this:

$$\frac{d}{dt}(\vec{\omega} \times \hat{\boldsymbol{\imath}}') = \dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times \dot{\hat{\boldsymbol{\imath}}}' \tag{16.14}$$

$$= \dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\imath}}'), \tag{16.15}$$

$$\frac{d}{dt}(\vec{\omega} \times \hat{\boldsymbol{\jmath}}') = \dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times \dot{\hat{\boldsymbol{\jmath}}}' \tag{16.16}$$

$$= \dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\jmath}}'). \tag{16.17}$$

Substituting back into eqn. (16.13) we get

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}} &= x'\left(\dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\imath}}')\right) \\
&\quad + y'\left(\dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\jmath}}')\right) \\
&= \dot{\vec{\omega}} \times (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') + \vec{\omega} \times \left(\vec{\omega} \times (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')\right) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_{\mathrm{P}})
\end{aligned}$$

which derives eqn. (16.19).

(centripetal) acceleration $-\omega^2\vec{r}_P$ due to the direction of the velocity continuously changing towards the center (see fig. 16.15b). The generalization we have made in this section is that the same $\vec{\omega}$ can be used to calculate the acceleration for all the different points on one rotating object.

Relative motion of points on a rigid object

As you well know by now, the position of point B relative to point A is $\vec{r}_{B/A} \equiv \vec{r}_B - \vec{r}_A$. Similarly the relative velocity and acceleration of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad \text{and} \quad \vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A \tag{16.21}$$

So, the relative velocity (as calculated relative to a fixed frame) of two points glued to one spinning rigid object $\mathcal{B}$ is given by

$$\begin{aligned} \vec{v}_{B/A} &\equiv \vec{v}_B - \vec{v}_A \\ &= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} \\ &= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) \\ &= \vec{\omega} \times \vec{r}_{B/A}, \end{aligned}$$

where point O is the point in the Newtonian frame on the fixed axis of rotation and $\vec{\omega} = \vec{\omega}_{\mathcal{B}}$ is the angular velocity of $\mathcal{B}$. Repeating,

$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \tag{16.22}$$

Because points A and B are fixed on $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{C}$ are all $\vec{\mathbf{0}}$. But, point A has an absolute velocity that is different from that of point B. So they have a relative velocity as seen in the fixed frame. And it is what you would get if B was just going in circles around A. Similarly, the relative acceleration of two points glued to one rigid object spinning at constant rate is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \tag{16.23}$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. These kinematics results, 16.22 and 16.23, are useful for calculating angular momentum relative to the center-of-mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

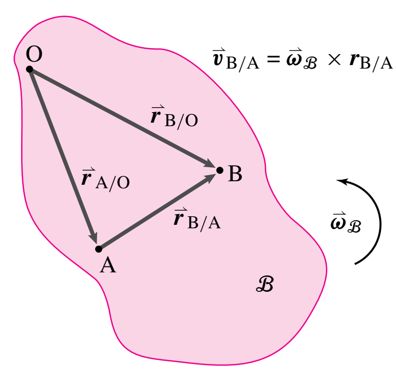

Figure 16.16: The acceleration of B relative to A if they are both on the same rotating rigid object.

The fundamental $\vec{\omega}$ equation

Equation 16.22 actually applies to any vector $\vec{Q}$ that has the property that it is fixed in the frame which is rotating at $\vec{\omega}$ (fig. 16.17). That is, all we needed in the derivation was that $\vec{Q}$'s components $(Q_{x'}, Q_{y'})$ in the $\hat{\imath}'$-$\hat{\jmath}'$ reference frame be constant. Examples of such $\vec{Q}$ would include more than just the relative position of two points fixed on the object, $\vec{r}_{B/A}$. Both $\hat{\imath}'$ and $\hat{\jmath}'$ are also are fixed in the rotating frame. Also imagine a bug moving at a constant speed on a straight line marked on a body. That bug's velocity relative to the rotating frame would also be constant as seen in the frame. For any $\vec{Q}$ whose representation as an arrow moves like a line segment drawn on a rigid object rotating at $\vec{\omega}$:

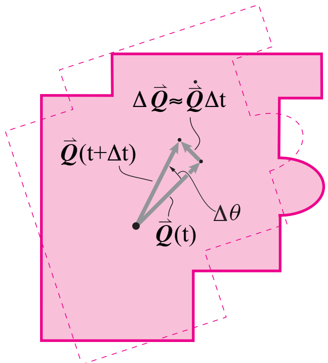

Figure 16.17: A vector $\vec{Q}$ is fixed to a rotating object. So its rate of change is the velocity of its tip. The magnitude is $\dot{\theta}|\vec{Q}|$ and the direction is orthogonal to $\vec{Q}$.

$$\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}. \qquad (16.24)$$

This is the most fundamental equation of rigid-object kinematics. Know it well. For future reference, equation (16.24) is also valid in three dimensions. Equation 16.24 is the generalization of the three equations

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}, \qquad \dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}' \qquad \text{and} \qquad \dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'.$$

Calculating relative velocity directly, using rotating frames

A coordinate system $x'y'$ attached to a rotating rigid object $\mathcal{C}$, defines a reference frame $\mathcal{C}$ (fig. 16.6 on page 760). Recall, the base vectors in this frame change in time just like any other vector fixed in the rotating frame (eqn. (16.24))

$$\frac{d}{dt}\hat{\imath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\imath}' \quad \text{and} \quad \frac{d}{dt}\hat{\jmath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}'.$$

Once we know how the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ change with time, as per the above equations, we can find velocity and acceleration by differentiation of position. For example, consider two points A and B fixed on a rotating object. The position of B relative to A is

$$\vec{r}_{B/A} = x'\hat{\imath}' + y'\hat{\jmath}'.$$

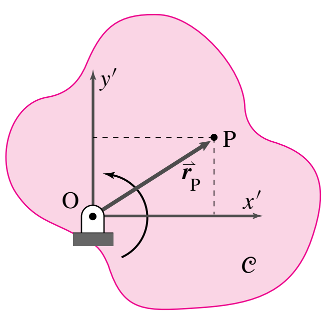

Figure 16.18: Coordinates in a rotating frame.

where we are using x' and y' as a shorthand notation for $x'_{B/A}$ and $y'_{B/A}$ (see fig. 16.18). The coordinates x' and y' are constant so

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

Now we just differentiate the relative position $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ with respect to time,

$$\begin{aligned}
\frac{d}{dt}(\vec{\boldsymbol{r}}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\boldsymbol{\imath}}' + x'\frac{d}{dt}\hat{\boldsymbol{\imath}}' + \underbrace{\dot{y}'}_{0}\hat{\boldsymbol{\jmath}}' + y'\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \underbrace{(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')}_{\vec{\boldsymbol{r}}_{B/A}} \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}.
\end{aligned}$$

We could similarly calculate $\vec{\boldsymbol{a}}_{\mathrm{B/A}}$ by taking another derivative to get, after a calculation much like that above,

$$\vec{\boldsymbol{a}}_{\mathrm{B/A}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \left(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}\right) + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}.$$

For the above calculation the points A and B were fixed on a rotating part. Most machines have many parts, at least some of which move in more complex ways than just circular motion. We will be able to understand such machines by considering points and parts that are moving relative to a part that is itself rotating. Such will be considered in Chapter 19.

16.5 Angular velocity $\vec{\omega}$ and the rotation matrix $[R]$

The rotation matrix $[R]$ and the angular velocity vector $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ are related. Because angular velocity is a rotation rate, in some sense it must be the derivative of the rotation. But the situation is a bit subtle because $\vec{\omega}$ is a vector and $[R]$ is a matrix.

On the one hand we have the rotation equation:

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{[\vec{r}]_{xy}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\left[\vec{r}^{\,\text{ref}}\right]_{xy}} .$$

This gives the coordinates of $\vec{r}$ after rotation in terms of its coordinates before the rotation. Because the coordinates $\left[\vec{r}^{\,\text{ref}}\right]_{xy}$ before rotation are fixed we can take the time derivative of both sides to get

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \dot{\theta}\underbrace{\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}}_{[\dot{R}]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\left[\vec{r}^{\,\text{ref}}\right]_{xy}} .$$

On the other hand we have the vector equation

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}.$$

where $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. We can relate this vector equation to the matrix component equation above it by expressing the cross product with a matrix multiplication as described in box 1.6 on 84. Applying that result to this 2D case where

$$[\vec{\omega}]_{xyz} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \text{and} \quad [\vec{r}]_{xyz} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

we get

$$\underbrace{\begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \underbrace{\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{\omega})]} \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}_{\left[\vec{r}^{\,\text{ref}}\right]_{xy}} .$$

Just keeping the 2D terms we have

$$\begin{aligned} [\vec{\omega} \times \vec{r}]_{xy} &= [\mathcal{S}(\vec{\omega})] \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

But $[R]$ tells us the coordinates in terms of the reference coordinates. So

$$[\vec{\omega} \times \vec{r}]_{xy} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \underbrace{[R]\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\begin{bmatrix} x \\ y \end{bmatrix}}.$$

Comparing the equations above we see that

$$[\dot{R}] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} [R]. \tag{16.25}$$

Using the notation from box 1.6 we can rewrite this as

$$[\dot{R}] = [\mathcal{S}(\vec{\omega})]\,[R]. \tag{16.26}$$

Equation 16.26 is the way that angular velocity $\vec{\omega}$ and the rate of change $[\dot{R}]$ of the rotation matrix are related.

We can also solve for $[\mathcal{S}(\vec{\omega})]$ as

$$[\mathcal{S}(\vec{\omega})] = [\dot{R}][R]^{-1}$$

from which we can find $\pm\omega$ in the off diagonal elements if we are given $[R]$ and $[\dot{R}]$.

We can understand eqn. (16.26) better, perhaps, by thinking of the rotation matrix $[R]$ as having two columns which are, respectively, the fixed-coordinate-system components of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$:

$$[R] = \left[[\hat{\boldsymbol{\imath}}']_{xy} \mid [\hat{\boldsymbol{\jmath}}']_{xy} \right]$$

Thus the columns of $[\dot{R}]$ are the rates of change of these two unit vectors:

$$[\dot{R}] = \left[[\dot{\hat{\boldsymbol{\imath}}}']_{xy} \mid [\dot{\hat{\boldsymbol{\jmath}}}']_{xy} \right] = \left[[\vec{\omega} \times \hat{\boldsymbol{\imath}}']_{xy} \mid [\vec{\omega} \times \hat{\boldsymbol{\jmath}}']_{xy} \right].$$

With a total abuse of notation (in general one does not allow cross products of a vectors with matrices) we can write the above equation more memorably as

$$\text{“}\,[\dot{R}] = \vec{\omega} \times [R].\,\text{”}$$

Thus eqn. (16.26) is just a version of the fundamental kinematic equation 16.24 on page 6.