16.2 Angular velocity and acceleration of a rigid object in planar circular motion

Angular velocity of a rigid object: $\vec{\omega}$

Thus far we have talked about rotation, but not how it varies in time. Dynamics is about motion, velocities and accelerations, so we need to think about rotation rates and rotational accelerations.

A 2D rigid object's net rotation is measured by the rotation angle θ. Thus, the simplest measure of rotation rate is $\dot{\theta} \equiv \frac{d\theta}{dt}$. Because all marked lines rotate the same amount θ they all have the same rates of change. So $\dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 =$ *etc.* and, as for rotation, the concept of rotation rate of a rigid object transcends the concept of rotation rate of this or that particular line. We give this rotation rate of a rigid object a special name, *angular velocity*, and symbol, ω (omega).

Repeating, for all lines marked on a rigid object,

$$\omega \equiv \dot{\theta}_1 = \dot{\theta}_2 = \dot{\theta}_3 = \cdots = \dot{\theta}. \tag{16.10}$$

Often we think of angular velocity as a vector $\vec{\omega}$. Its direction is the axis of the rotation which, for objects in the xy plane is $\hat{\boldsymbol{k}}$, pointed in the $+z$ direction normal to the xy plane. The scalar part of $\vec{\omega}$ is ω. So, the *angular velocity vector* is

$$\vec{\omega} \equiv \omega\hat{\boldsymbol{k}} \tag{16.11}$$

with ω as defined in eqn. (16.10). Note $\vec{\omega}$ is the angular velocity of the object (and of *every* line on it)[1]. The usual sign convention for ω is the same as that for θ, positive is counter-clockwise (CCW) around the origin. That's the direction your fingers wrap if you point your thumb in the $+z$ direction and grab the z axis.

[1] We could also think of rotation rate as a derivative of the rotation matrix $[R]$. This point of view and its relation to the vector point of view here, is described in Box 16.5 on page 8.

Rate of change of $\hat{\imath}'$, $\hat{\jmath}'$

Our first use of the angular velocity vector $\vec{\omega}$ is to calculate the rate of change of the rotating unit base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$. We can find the rate of change of, say, $\hat{\boldsymbol{\imath}}'$, by taking the time derivative of the first eqn. (16.4), and using the chain rule while recognizing that $\theta = \theta(t)$. We can also make an analogy with polar coordinates (page 725), where we think of $\hat{\boldsymbol{e}}_R$ as like $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{e}}_\theta$ as like $\hat{\boldsymbol{\jmath}}'$. We found there that $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$ and $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. Either from differentiating $\hat{\boldsymbol{\imath}}'$ or from the correlation with polar coordinates, we get

$$
\begin{aligned}
\dot{\hat{\imath}}' &= \ \ \dot\theta\,\hat{\jmath}' &\quad\text{or}\quad& \dot{\hat{\imath}}' = \vec\omega\times\hat{\imath}' \quad\text{and}\\
\dot{\hat{\jmath}}' &= -\dot\theta\,\hat{\imath}' &\quad\text{or}\quad& \dot{\hat{\jmath}}' = \vec\omega\times\hat{\jmath}'
\end{aligned}
\tag{16.12}
$$

because $\hat{\jmath}' = \hat{k}'\times\hat{\imath}'$ and $\hat{\imath}' = -\hat{k}'\times\hat{\jmath}'$. Depending on the tastes of your lecturer, you may find Equations 16.12 are the most used equations from this point onward②.

② Eqn. 16.12 is sometimes considered the definition of $\vec\omega$. In this view, popularized by Tom Kane, $\vec\omega$ is that vector which determines $\dot{\hat{\imath}}'$ and $\dot{\hat{\jmath}}'$ by the formulas $\dot{\hat{\imath}}' = \vec\omega\times\hat{\imath}'$ and $\dot{\hat{\jmath}}' = \vec\omega\times\hat{\jmath}'$. Then one needs to show that such a vector exists and that it is $\vec\omega = \hat{\imath}'\times\dot{\hat{\imath}}'$. Luckily this reasoning leads to the same $\vec\omega$ as our $\vec\omega = \dot\theta\hat{k}$.

Velocity of a point fixed on a rigid object

Let's call some rotating object $\mathcal{B}$ (script capital B) to which is glued a coordinate system $x'y'$ with base vectors $\hat{\imath}'$ and $\hat{\jmath}'$. We now introduce the concept of *derivative in a frame* which we write, for the frame $\mathcal{B}$, as

$$\frac{{}^{\mathcal{B}}d}{dt}$$

which means, in words, the rate of change of something as viewed in the rotating frame $\mathcal{B}$. Now consider a point P at $\vec{r}_{\mathrm{P}}$ that is glued to the object. That is, the x' and y' coordinates of $\vec{r}_{\mathrm{P}}$ do not change in time.

$$\frac{{}^{\mathcal{B}}d\vec{r}_{\mathrm{P}}}{dt} \equiv \overset{\mathcal{B}.}{\vec{r}}_{\mathrm{P}} = \quad \dot{x}'\hat{\imath}' + \dot{y}'\hat{\jmath}' = \vec{\mathbf{0}}.$$

That is, relative to a moving frame, the velocity of a point glued to the frame is zero (no surprise).

We would like to know the velocity of such a point in the fixed frame. We just take the derivative, using the product rule and the differentiation rules we have developed for the rotating base vectors:

16.2 The fixed Newtonian reference frame $\mathcal{F}$

Now we can reconsider the concept of a Newtonian frame, a concept which we had to assume to write the equations of dynamics in the first place. All of mechanics depends, of course, on the laws of mechanics. The laws of mechanics are equations which involve, in part, the positions of things as a function of time. But how position is perceived to change in time depends on your reference frame. And some reference frames are better than others. The best, from our point of view, are reference frames in which Newton's laws are accurate. Such a reference frame is called a *Newtonian frame*. In engineering practice the frames we use as approximations of a Newtonian frame often seem, loosely speaking, somehow still. So we sometimes call such a frame *the fixed frame* and label it with a script capital $\mathcal{F}$. When we talk about velocity and acceleration of mass points, for use in the equations of mechanics, we are always talking about the velocity and acceleration relative to a $\mathcal{F}$ixed, or equivalently, Newtonian frame.

Assume x and y are the coordinates of a vector $\vec{r}_{\mathrm{P}}$ and $\mathcal{F}$ is a fixed frame with fixed axis (with associated constant base vectors $\hat{\imath}$ and $\hat{\jmath}$). When we write $\dot{\vec{r}}_P$ we mean $\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. But we could be more explicit (and notationally ornate) and write the velocity of P in the Newtonian frame as

$$\frac{{}^{\mathcal{F}}d\vec{r}_{\mathrm{P}}}{dt} \equiv \overset{\mathcal{F}.}{\vec{r}}_{\mathrm{P}} \quad \text{by which we mean} \quad \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}.$$

The $\mathcal{F}$ in front of the time derivative (or in front of the dot) means that when we calculate a derivative we hold the base vectors of $\mathcal{F}$ constant. This is no surprise, because for $\mathcal{F}$ the base vectors *are* constant. In general, however, when taking a derivative in a given frame you

- write vectors in terms of base vectors stuck to the frame, and
- only differentiate the components.

But only for the $\mathcal{F}$ixed or $\mathcal{N}$ewtonian frame will accelerations calculated this way be directly applicable to Newton's laws.

We will avoid the ornate notation of labeling frames when it is not needed. For example, if you don't see any script capital letters floating around in front of derivatives, you can assume that we are taking derivatives relative to a fixed Newtonian frame.

$$
\begin{aligned}
\vec{\boldsymbol{r}}_{\text{P}} &= x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}' \\
\Rightarrow \quad \vec{\boldsymbol{v}}_{\text{P}} = \dot{\vec{\boldsymbol{r}}}_{\text{P}} &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) = x'\dot{\hat{\boldsymbol{\imath}}}' + y'\dot{\hat{\boldsymbol{\jmath}}}' = x'(\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\boldsymbol{\omega}} \times (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')
\end{aligned}
$$

where $\dot{\vec{\boldsymbol{r}}}_P$ is the simple way to write $\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\text{P}}}{dt}$. Thus,

$$\vec{\boldsymbol{v}}_{\text{P}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{\text{P}} \tag{16.18}$$

We can rewrite eqn. (16.18) in a minimalist or elaborate notation as

$$
\begin{aligned}
\vec{\boldsymbol{v}} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}} \qquad \text{or} \\
\frac{{}^{\mathcal{F}}d\vec{\boldsymbol{r}}_{\text{P}}}{dt} &= \vec{\boldsymbol{\omega}}_{\mathcal{B}/\mathcal{F}} \times \vec{\boldsymbol{r}}_{\text{P/O}}.
\end{aligned}
$$

Both are correct. In the first case you have to use common sense to know what point you are talking about and that it is on an object rotating with absolute angular velocity $\vec{\boldsymbol{\omega}}$. In the second case everything is laid out perfectly clearly (which is why it looks perfectly confusing). On the left side of the equation it says that we are interested in how point P moves relative to, not just any frame, but the fixed frame $\mathcal{F}$. On the right side we make clear that the rotation rate we are looking at is that of object $\mathcal{B}$ relative to $\mathcal{F}$ and not some other relative rotation. We further make clear that the formula only makes sense if the position of the point P is measured relative to a point which doesn't move, namely 0.

What we have just found largely duplicates what we already learned in section 7.1 for points moving in circles. The slight generalization is that the same angular velocity $\vec{\boldsymbol{\omega}}$ can be used to calculate the velocities of

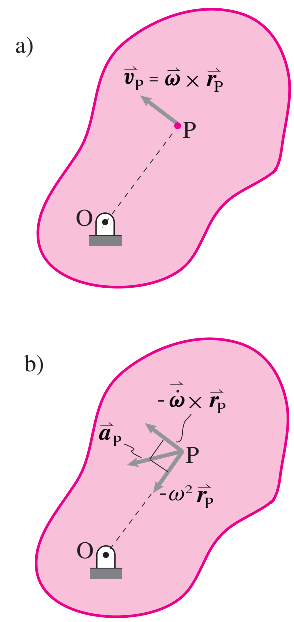


Figure 16.15: Velocity and acceleration of two points on a rigid object rotating about 0.

16.3 Plato's discussion of spinning in circles as motion (or not)

Here is how a smart person, Plato, thought about these things two thousand years ago. Here he writes about an imagined discussion between Socrates and Glaucon about how an object can maintain contradictory attributes simultaneously.

"Socrates: Now let's have a more precise agreement so that we won't have any grounds for dispute as we proceed. If someone were to say of a human being standing still, but moving his hands and head, that the same man at the same time stands still and moves, I don't suppose we'd claim that it should be said like that, but rather that one part of him stands still and another moves. Isn't that so?
Glaucon: Yes it is.
Socrates: Then if the man who says this should become still more charming and make the subtle point that tops as wholes stand still and move at the same time when the peg is fixed in the same place and they spin, or that anything else going around in a circle on the same spot does this too, we wouldn't accept it because it's not with respect to the same part of themselves that such things are at the same time both at rest and in motion. But we'd say that they have in them both a straight and a circumference; and with respect to the straight they stand still since they don't lean in any direction –while with respect to the circumference they move in a circle; and when the straight inclines to the right the left, forward, or backward at the same time that it's spinning, then in no way does it stand still.
Glaucon: And we'd be right."

This chapter is about things that are still with respect to their own parts (they do not distort) but in which the points do move in circles.

multiple points on one rigid object. But the key idea remains: the velocity of a point going in circles is tangent to the circle it is going around and with magnitude proportional both to distance from the center and to the angular rate of rotation (fig. 16.15a).

Acceleration of a point on a rotating rigid object

Let's again consider a point stuck on a rotating object and with position

$$\vec{\boldsymbol{r}}_{\mathrm{P}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'.$$

Relative to the frame $\mathcal{B}$ to which a point is attached, its acceleration is zero (again no surprise). But what is its acceleration in the fixed frame? We find this acceleration by writing the position vector and then differentiating twice, repeatedly using the product rule and eqn. (16.12) we get (see Box 16.4 for the details).

$$\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{\boldsymbol{r}}_{\mathrm{P}}\right) \qquad (16.19)$$

which is hardly intuitive at a glance(3). Recalling that in 2D $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ we can use either the right hand rule or manipulation of unit vectors to rewrite eqn. (16.19) as

$$\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} - \omega^2 \vec{\boldsymbol{r}}_{\mathrm{P}} \qquad (16.20)$$

where $\omega = \dot{\theta}$ and $\dot{\omega} = \ddot{\theta}$(4).

Thus, as we found in section 15.1 for a particle going in circles, the acceleration can be written as the sum of two terms, a tangential acceleration $\dot{\omega}\hat{\boldsymbol{k}} \times \vec{\boldsymbol{r}}_{\mathrm{P}}$ due to increasing tangential speed, and a centrally directed

(3) Although the form eqn. (16.19) is not of much immediate use, if you are going to continue on to the mechanics of mechanisms or three-dimensional mechanics, you should follow the derivation of eqn. (16.19) carefully.

(4) The derivation of eqn. (16.20) can be written more informally and briefly like this (using minimalist notation):

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \dot{\vec{\boldsymbol{v}}} = \frac{d}{dt}(\vec{\omega} \times \vec{\boldsymbol{r}}) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times \dot{\vec{\boldsymbol{r}}} \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}} - \omega^2 \vec{\boldsymbol{r}}.
\end{aligned}$$

16.4 Acceleration of a point on a rotating body, using $\vec{\omega}$

Leaving off the ornate pre-super-script $\mathcal{F}$ for simplicity, we have

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}} = \dot{\vec{\boldsymbol{v}}}_{\mathrm{P}} &= \frac{d}{dt}\left(\frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right)\right) \\
&= \frac{d}{dt}\left(x'(\vec{\omega} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\omega} \times \hat{\boldsymbol{\jmath}}')\right). \qquad (16.13)
\end{aligned}$$

To continue we need to use the product rule of differentiation for the cross product of two time dependent vectors like this:

$$\begin{aligned}
\frac{d}{dt}\left(\vec{\omega} \times \hat{\boldsymbol{\imath}}'\right) &= \dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times \dot{\hat{\boldsymbol{\imath}}}' \qquad (16.14) \\
&= \dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\imath}}'), \qquad (16.15) \\
\frac{d}{dt}\left(\vec{\omega} \times \hat{\boldsymbol{\jmath}}'\right) &= \dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times \dot{\hat{\boldsymbol{\jmath}}}' \qquad (16.16) \\
&= \dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\jmath}}'). \qquad (16.17)
\end{aligned}$$

Substituting back into eqn. (16.13) we get

$$\begin{aligned}
\vec{\boldsymbol{a}}_{\mathrm{P}} &= x'\left(\dot{\vec{\omega}} \times \hat{\boldsymbol{\imath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\imath}}')\right) \\
&\quad + y'\left(\dot{\vec{\omega}} \times \hat{\boldsymbol{\jmath}}' + \vec{\omega} \times (\vec{\omega} \times \hat{\boldsymbol{\jmath}}')\right) \\
&= \dot{\vec{\omega}} \times (x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}') + \vec{\omega} \times \left(\vec{\omega} \times \left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right)\right) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_{\mathrm{P}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{\boldsymbol{r}}_{\mathrm{P}}\right)
\end{aligned}$$

which derives eqn. (16.19).

(centripetal) acceleration $-\omega^2\vec{r}_P$ due to the direction of the velocity continuously changing towards the center (see fig. 16.15b). The generalization we have made in this section is that the same $\vec{\omega}$ can be used to calculate the acceleration for all the different points on one rotating object.

Relative motion of points on a rigid object

As you well know by now, the position of point B relative to point A is $\vec{r}_{B/A} \equiv \vec{r}_B - \vec{r}_A$. Similarly the relative velocity and acceleration of two points A and B is defined to be

$$\vec{v}_{B/A} \equiv \vec{v}_B - \vec{v}_A \quad \text{and} \quad \vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A \tag{16.21}$$

So, the relative velocity (as calculated relative to a fixed frame) of two points glued to one spinning rigid object $\mathcal{B}$ is given by

$$\begin{aligned}
\vec{v}_{B/A} &\equiv \vec{v}_B - \vec{v}_A \\
&= \vec{\omega} \times \vec{r}_{B/O} - \vec{\omega} \times \vec{r}_{A/O} \\
&= \vec{\omega} \times (\vec{r}_{B/O} - \vec{r}_{A/O}) \\
&= \vec{\omega} \times \vec{r}_{B/A},
\end{aligned}$$

where point O is the point in the Newtonian frame on the fixed axis of rotation and $\vec{\omega} = \vec{\omega}_{\mathcal{B}}$ is the angular velocity of $\mathcal{B}$. Repeating,

$$\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}. \tag{16.22}$$

Because points A and B are fixed on $\mathcal{B}$ their velocities and hence their relative velocity as observed in a reference frame fixed to $\mathcal{C}$ are all $\vec{\mathbf{0}}$. But, point A has an absolute velocity that is different from that of point B. So they have a relative velocity as seen in the fixed frame. And it is what you would get if B was just going in circles around A. Similarly, the relative acceleration of two points glued to one rigid object spinning at constant rate is

$$\vec{a}_{B/A} \equiv \vec{a}_B - \vec{a}_A = \dot{\vec{\omega}} \times \vec{r}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{r}_{B/A}). \tag{16.23}$$

Again, the relative acceleration is due to the difference in the points' positions relative to the point O fixed on the axis. These kinematics results, 16.22 and 16.23, are useful for calculating angular momentum relative to the center-of-mass. They are also sometimes useful for the understanding of the motions of machines with moving connected parts.

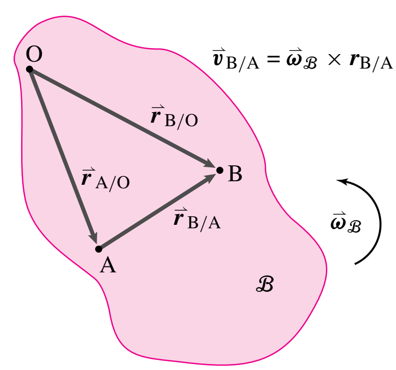


Figure 16.16: The acceleration of B relative to A if they are both on the same rotating rigid object.

The fundamental $\vec{\omega}$ equation

Equation 16.22 actually applies to any vector $\vec{Q}$ that has the property that it is fixed in the frame which is rotating at $\vec{\omega}$ (fig. 16.17). That is, all we needed in the derivation was that $\vec{Q}$'s components $(Q_{x'}, Q_{y'})$ in the $\hat{\imath}'$-$\hat{\jmath}'$ reference frame be constant. Examples of such $\vec{Q}$ would include more than just the relative position of two points fixed on the object, $\vec{r}_{B/A}$. Both $\hat{\imath}'$ and $\hat{\jmath}'$ are also are fixed in the rotating frame. Also imagine a bug moving at a constant speed on a straight line marked on a body. That bug's velocity relative to the rotating frame would also be constant as seen in the frame. For any $\vec{Q}$ whose representation as an arrow moves like a line segment drawn on a rigid object rotating at $\vec{\omega}$:

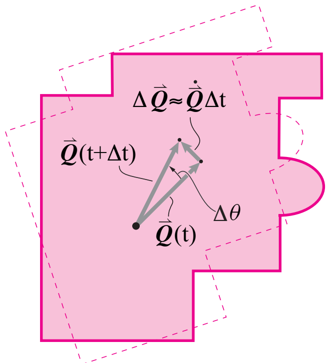


Figure 16.17: A vector $\vec{Q}$ is fixed to a rotating object. So its rate of change is the velocity of its tip. The magnitude is $\dot{\theta}|\vec{Q}|$ and the direction is orthogonal to $\vec{Q}$.

$$\dot{\vec{Q}} = \vec{\omega} \times \vec{Q}. \qquad (16.24)$$

This is the most fundamental equation of rigid-object kinematics. Know it well. For future reference, equation (16.24) is also valid in three dimensions. Equation 16.24 is the generalization of the three equations

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}, \qquad \dot{\hat{\imath}}' = \vec{\omega} \times \hat{\imath}' \qquad \text{and} \qquad \dot{\hat{\jmath}}' = \vec{\omega} \times \hat{\jmath}'.$$

Calculating relative velocity directly, using rotating frames

A coordinate system $x'y'$ attached to a rotating rigid object $\mathcal{C}$, defines a reference frame $\mathcal{C}$ (fig. 16.6 on page 760). Recall, the base vectors in this frame change in time just like any other vector fixed in the rotating frame (eqn. (16.24))

$$\frac{d}{dt}\hat{\imath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\imath}' \quad \text{and} \quad \frac{d}{dt}\hat{\jmath}' = \vec{\omega}_{\mathcal{C}} \times \hat{\jmath}'.$$

Once we know how the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ change with time, as per the above equations, we can find velocity and acceleration by differentiation of position. For example, consider two points A and B fixed on a rotating object. The position of B relative to A is

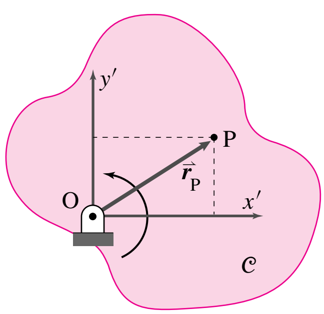


Figure 16.18: Coordinates in a rotating frame.

$$\vec{r}_{B/A} = x'\hat{\imath}' + y'\hat{\jmath}'.$$

where we are using x' and y' as a shorthand notation for $x'_{B/A}$ and $y'_{B/A}$ (see fig. 16.18). The coordinates x' and y' are constant so

$$\dot{x}' = 0 \text{ and } \dot{y}' = 0.$$

Now we just differentiate the relative position $\vec{\boldsymbol{r}}_{\mathrm{B/A}}$ with respect to time,

$$
\begin{aligned}
\frac{d}{dt}(\vec{\boldsymbol{r}}_{\mathrm{B/A}}) &= \frac{d}{dt}\left(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}'\right) \\
&= \underbrace{\dot{x}'}_{0}\hat{\boldsymbol{\imath}}' + x'\frac{d}{dt}\hat{\boldsymbol{\imath}}' + \underbrace{\dot{y}'}_{0}\hat{\boldsymbol{\jmath}}' + y'\frac{d}{dt}\hat{\boldsymbol{\jmath}}' \\
&= x'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\imath}}') + y'(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \hat{\boldsymbol{\jmath}}') \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \underbrace{(x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}')}_{\vec{\boldsymbol{r}}_{B/A}} \\
&= \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{\mathrm{B/A}}.
\end{aligned}
$$

We could similarly calculate $\vec{\boldsymbol{a}}_{\mathrm{B/A}}$ by taking another derivative to get, after a calculation much like that above,

$$\vec{\boldsymbol{a}}_{\mathrm{B/A}} = \vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \left(\vec{\boldsymbol{\omega}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}\right) + \dot{\vec{\boldsymbol{\omega}}}_{\mathcal{C}} \times \vec{\boldsymbol{r}}_{B/A}.$$

For the above calculation the points A and B were fixed on a rotating part. Most machines have many parts, at least some of which move in more complex ways than just circular motion. We will be able to understand such machines by considering points and parts that are moving relative to a part that is itself rotating. Such will be considered in Chapter 19.

16.5 Angular velocity $\vec{\omega}$ and the rotation matrix $[R]$

The rotation matrix $[R]$ and the angular velocity vector $\vec{\omega} = \omega\hat{\boldsymbol{k}}$ are related. Because angular velocity is a rotation rate, in some sense it must be the derivative of the rotation. But the situation is a bit subtle because $\vec{\omega}$ is a vector and $[R]$ is a matrix.

On the one hand we have the rotation equation:

$$\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{[\vec{r}]_{xy}} = \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\left[\vec{r}^{\,\mathrm{ref}}\right]_{xy}} .$$

This gives the coordinates of $\vec{r}$ after rotation in terms of its coordinates before the rotation. Because the coordinates $\left[\vec{r}^{\,\mathrm{ref}}\right]_{xy}$ before rotation are fixed we can take the time derivative of both sides to get

$$\underbrace{\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \dot{\theta} \underbrace{\begin{bmatrix} -\sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}}_{[\dot{R}]} \underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{\left[\vec{r}^{\,\mathrm{ref}}\right]_{xy}} .$$

On the other hand we have the vector equation

$$\dot{\vec{r}} = \vec{\omega} \times \vec{r}.$$

where $\vec{\omega} = \omega\hat{\boldsymbol{k}}$. We can relate this vector equation to the matrix component equation above it by expressing the cross product with a matrix multiplication as described in box 1.6 on 84. Applying that result to this 2D case where

$$[\vec{\omega}]_{xyz} = \begin{bmatrix} 0 \\ 0 \\ \omega \end{bmatrix} \quad \text{and} \quad [\vec{r}]_{xyz} = \begin{bmatrix} x \\ y \\ 0 \end{bmatrix}$$

we get

$$\underbrace{\begin{bmatrix} -\omega y \\ \omega x \\ 0 \end{bmatrix}}_{[\dot{\vec{r}}]_{xy}} = \underbrace{\begin{bmatrix} 0 & -\omega & 0 \\ \omega & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{[\mathcal{S}(\vec{\omega})]} \underbrace{\begin{bmatrix} x \\ y \\ 0 \end{bmatrix}}_{\left[\vec{r}^{\,\mathrm{ref}}\right]_{xy}} .$$

Just keeping the 2D terms we have

$$\begin{aligned} [\vec{\omega} \times \vec{r}]_{xy} &= [\mathcal{S}(\vec{\omega})] \begin{bmatrix} x \\ y \end{bmatrix} \\ &= \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \end{aligned}$$

But $[R]$ tells us the coordinates in terms of the reference coordinates. So

$$[\vec{\omega} \times \vec{r}]_{xy} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \underbrace{[R] \begin{bmatrix} x' \\ y' \end{bmatrix}}_{\begin{bmatrix} x \\ y \end{bmatrix}} .$$

Comparing the equations above we see that

$$[\dot{R}] = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} [R]. \tag{16.25}$$

Using the notation from box 1.6 we can rewrite this as

$$[\dot{R}] = [\mathcal{S}(\vec{\omega})]\,[R]. \tag{16.26}$$

Equation 16.26 is the way that angular velocity $\vec{\omega}$ and the rate of change $[\dot{R}]$ of the rotation matrix are related.

We can also solve for $[\mathcal{S}(\vec{\omega})]$ as

$$[\mathcal{S}(\vec{\omega})] = [\dot{R}][R]^{-1}$$

from which we can find $\pm\omega$ in the off diagonal elements if we are given $[R]$ and $[\dot{R}]$.

We can understand eqn. (16.26) better, perhaps, by thinking of the rotation matrix $[R]$ as having two columns which are, respectively, the fixed-coordinate-system components of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$:

$$[R] = \left[[\hat{\boldsymbol{\imath}}']_{xy} \mid [\hat{\boldsymbol{\jmath}}']_{xy} \right]$$

Thus the columns of $[\dot{R}]$ are the rates of change of these two unit vectors:

$$[\dot{R}] = \left[[\dot{\hat{\boldsymbol{\imath}}}']_{xy} \mid [\dot{\hat{\boldsymbol{\jmath}}}']_{xy} \right] = \left[[\vec{\omega} \times \hat{\boldsymbol{\imath}}']_{xy} \mid [\vec{\omega} \times \hat{\boldsymbol{\jmath}}']_{xy} \right].$$

With a total abuse of notation (in general one does not allow cross products of a vectors with matrices) we can write the above equation more memorably as

$$\text{“}\,[\dot{R}] = \vec{\omega} \times [R].\,\text{”}$$

Thus eqn. (16.26) is just a version of the fundamental kinematic equation 16.24 on page 6.

SAMPLE 16.3 A uniform bar AB of length $\ell = 50\,\mathrm{cm}$ rotates counterclockwise about point A with constant angular speed ω. At the instant shown in fig. 16.19 the linear speed v_C of the center-of-mass C is 7.5 cm/ s.

1. What are the angular speed and angular velocity of the bar?
2. What is the linear velocity of point B?
3. By what angles do the angular positions of points C and B change in 2 seconds?

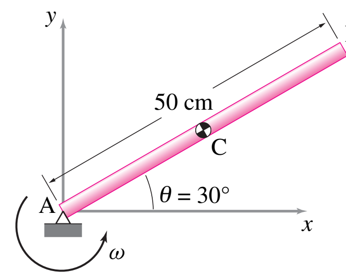


Figure 16.19

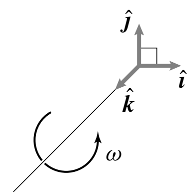


Figure 16.20

Solution Let the angular velocity of the bar be $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}}$ where $\dot{\theta}$ is the angular speed. We first need to find $\dot{\theta}$.

1. The linear speed of point C is given, $v_C = 7.5\,\mathrm{cm/\,s}$. Now,

$$\begin{aligned} v_C &= \dot{\theta}\, r_C \\ \Rightarrow \quad \dot{\theta} &= \frac{v_C}{r_C} = \frac{7.5\,\mathrm{cm/\,s}}{25\,\mathrm{cm}} = 0.3\,\mathrm{rad/s}. \end{aligned}$$

Therefore, the angular velocity of the bar is $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}} = 0.3\,\mathrm{rad/s}\hat{\boldsymbol{k}}$.

$$\boxed{\dot{\theta} = 0.3\,\mathrm{rad/s}, \text{ and } \vec{\omega} = 0.3\,\mathrm{rad/s}\hat{\boldsymbol{k}}}$$

2. Point B is at distance ℓ from the pivot point A. Thus it goes around a circle of radius ℓ (see fig. 16.21). Therefore,

$$\begin{aligned} \vec{\boldsymbol{v}}_B &= \vec{\omega} \times \vec{\boldsymbol{r}}_B = \dot{\theta}\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\ &= \dot{\theta}\ell(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\ &= 0.3\,\mathrm{rad/s} \cdot 50\,\mathrm{cm}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right) \\ &= 15\,\mathrm{cm/\,s}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right). \end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_B = 15\,\mathrm{cm/\,s}\left(\tfrac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \tfrac{1}{2}\hat{\boldsymbol{\imath}}\right)}$$

We can also write $\vec{\boldsymbol{v}}_B = 15\,\mathrm{cm/\,s}\,\hat{\boldsymbol{e}}_\theta$ where $\hat{\boldsymbol{e}}_\theta = \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}$.

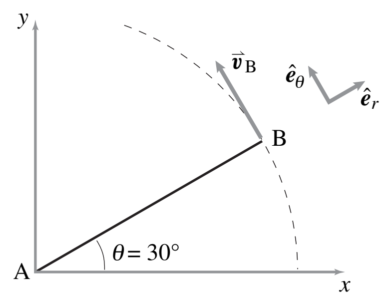


Figure 16.21: From the given geometry, $\hat{\boldsymbol{e}}_r = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$, and $\vec{\boldsymbol{v}}_B = |\vec{\boldsymbol{v}}_B|\hat{\boldsymbol{e}}_\theta$.

3. Let θ_1 be the position of point C at some time t_1 and θ_2 be the position at time t_2. We want to find $\Delta\theta = \theta_2 - \theta_1$ for $t_2 - t_1 = 2\,\mathrm{s}$.

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} = \text{constant} = 0.3\,\mathrm{rad/s}. \\ \Rightarrow \quad d\theta &= (0.3\,\mathrm{rad/s})dt. \\ \Rightarrow \quad \int_{\theta_1}^{\theta_2} d\theta &= \int_{t_1}^{t_2} (0.3\,\mathrm{rad/s})dt. \\ \Rightarrow \quad \theta_2 - \theta_1 &= 0.3\,\mathrm{rad/s}(t_2 - t_1) \\ \text{or} \quad \Delta\theta &= 0.3\frac{\mathrm{rad}}{\cancel{\mathrm{s}}}\cdot 2\cancel{\mathrm{s}} = 0.6\,\mathrm{rad}. \end{aligned}$$

The change in angular position of point B is the same as that of point C. In fact, all points on AB undergo the same change in angular position because AB is a rigid body.

$$\boxed{\Delta\theta_C = \Delta\theta_B = 0.6\,\mathrm{rad}}$$

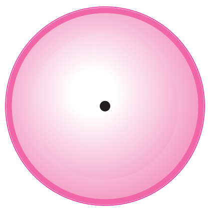


Figure 16.22

SAMPLE 16.4 A flywheel of diameter 2 ft is made of cast iron. To avoid extremely high stresses and cracks it is recommended that the peripheral speed not exceed 6000 to 7000 ft/min. What is the corresponding rpm rating for the wheel?

Solution

$$\begin{aligned} \text{Diameter of the wheel} &= 2\,\text{ft.} \\ \Rightarrow \quad \text{radius of wheel} &= 1\,\text{ft.} \end{aligned}$$

Now,

$$\begin{aligned} v &= \omega r \\ \Rightarrow \quad \omega &= \frac{v}{r} = \frac{6000\,\cancel{f}t/\text{min}}{1\,\cancel{f}t} \\ &= 6000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\,\text{rad}} \\ &= 955\,\text{rpm.} \end{aligned}$$

Similarly, corresponding to $v = 7000\,\text{ft/min}$

$$\begin{aligned} \omega &= \frac{7000\,\cancel{f}t/\text{min}}{1\,\cancel{f}t} \\ &= 7000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\,\text{rad}} \\ &= 1114\,\text{rpm.} \end{aligned}$$

Thus the rpm rating of the wheel should read 955 – 1114 rpm.

$\omega = 955$ to $1114\,\text{rpm}$.

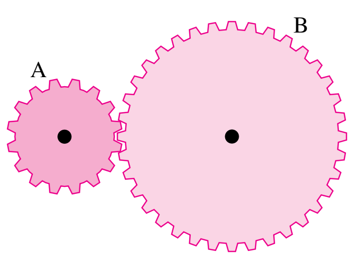


Figure 16.23

SAMPLE 16.5 Two gears A and B have the diameter ratio of 1:2. Gear A drives gear B. If the output at gear B is required to be 150 rpm, what should be the angular speed of the driving gear? Assume no slip at the contact point.

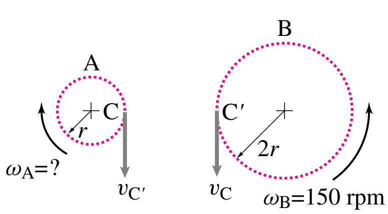


Figure 16.24

Solution Let C and C′ be the points of contact on gear A and B respectively at some instant t. Since there is no relative slip between C and C′, both points must have the same linear velocity at instant t. If the velocities are the same, then the linear speeds must also be the same. Thus

$$\begin{aligned} v_C &= v_{C'} \\ \Rightarrow \quad \omega_A r_A &= \omega_B r_B \\ \Rightarrow \quad \omega_A &= \omega_B \cdot \frac{r_B}{r_A} \\ &= \omega_B \cdot \frac{2\cancel{r}}{\cancel{r}} = 2\omega_B \\ &= (2)\cdot(150\,\text{rpm}) \\ &= 300\,\text{rpm.} \end{aligned}$$

$\omega_A = 300\,\text{rpm}$

SAMPLE 16.6 A uniform rigid rod AB of length $\ell = 0.6\,\text{m}$ is connected to two rigid links OA and OB. The assembly rotates at a constant rate about point O in the xy plane. At the instant shown, when rod AB is vertical, the velocities of points A and B are $\vec{v}_A = -4.64\,\text{m/s}\,\hat{\jmath} - 1.87\,\text{m/s}\,\hat{\imath}$, and $\vec{v}_B = 1.87\,\text{m/s}\,\hat{\imath} - 4.64\,\text{m/s}\,\hat{\jmath}$. Find the angular velocity of bar AB. What is the length R of the links?

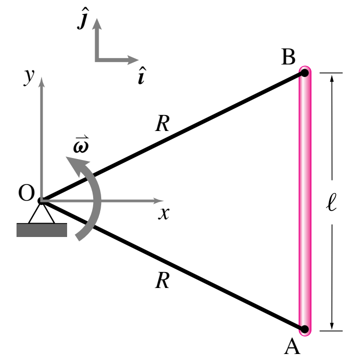


Figure 16.25

Solution Let the angular velocity of the rod AB be $\vec{\omega} = \omega\hat{k}$. ① Since we are given the velocities of two points on the rod we can use the relative velocity formula to find $\vec{\omega}$:

$$
\begin{aligned}
\vec{v}_{B/A} = \vec{\omega}\times\vec{r}_{B/A} &= \vec{v}_B - \vec{v}_A \\
\text{or} \quad \underbrace{\omega\hat{k}}_{\vec{\omega}} \times \underbrace{\ell\,\hat{\jmath}}_{\vec{r}_{B/A}} &= (1.87\hat{\imath} - 4.64\hat{\jmath})\,\text{m/s} - (-4.64\hat{\jmath} - 1.87\hat{\imath})\,\text{m/s} \\
\text{or} \quad \omega\ell(-\hat{\imath}) &= (1.87\hat{\imath} + 1.87\hat{\imath})\,\text{m/s} - (4.\cancel{6}4\hat{\jmath} - 4.\cancel{6}4\hat{\jmath})\,\text{m/s} \\
&= 3.74\hat{\imath}\,\text{m/s} \\
\Rightarrow \qquad \omega &= -\frac{3.74\,\text{m/s}}{\ell} \\
&= -\frac{3.74}{0.6}\,\text{rad/s} \\
&= -6.233\,\text{rad/s} \qquad (16.27)
\end{aligned}
$$

Thus,

$$\vec{\omega} = -6.233\,\text{rad/s}\hat{k}. \qquad (16.28)$$

$$\boxed{\vec{\omega} = -6.23\,\text{rad/s}\hat{k}}$$

① We know that the rod rotates about the z-axis but we do not know the sense of the rotation *i.e.*, $+\hat{k}$ or $-\hat{k}$. Here we have assumed that $\vec{\omega}$ is in the positive $\hat{k}$ direction, although just by sketching $\vec{v}_A$ we can easily see that $\vec{\omega}$ must be in the $-\hat{k}$ direction.

Let θ be the angle between link OA and the horizontal axis. Now,

$$
\begin{aligned}
\vec{v}_A = \vec{\omega}\times\vec{r}_A &= \omega\hat{k} \times \underbrace{R(\cos\theta\hat{\imath} - \sin\theta\,\hat{\jmath})}_{\vec{r}_A} \\
\text{or} \quad (-4.64\hat{\jmath} - 1.87\hat{\imath})\,\text{m/s} &= \omega R(\cos\theta\,\hat{\jmath} + \sin\theta\hat{\imath})
\end{aligned}
$$

Dotting both sides of the equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$
\begin{aligned}
-1.87\,\text{m/s} &= \omega R\sin\theta \qquad (16.29) \\
-4.64\,\text{m/s} &= \omega R\cos\theta \qquad (16.30)
\end{aligned}
$$

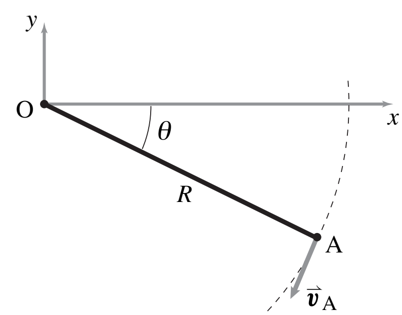


Figure 16.26: The given velocity vector $\vec{v}_A$. Since $\vec{v}_A = \vec{\omega}\times\vec{r}_A$ where $\vec{r}_A = R\cos\theta\hat{\imath} + R\sin\theta\,\hat{\jmath}$, we can find R from the given $\vec{v}_A$.

Squaring and adding Eqns (16.29) and (16.30) together we get

$$
\begin{aligned}
\omega^2 R^2 &= (-4.64\,\text{m/s})^2 + (-1.187\,\text{m/s})^2 \\
&= 25.026\,\text{m}^2/\text{s}^2 \\
\Rightarrow \qquad R^2 &= \frac{25.026\,\text{m}^2/\text{s}^2}{(-6.23\,\text{rad/s})^2} \\
&= 0.645\,\text{m}^2 \\
\Rightarrow \qquad R &= 0.8\,\text{m}
\end{aligned}
$$

$$\boxed{R = 0.8\,\text{m}}$$

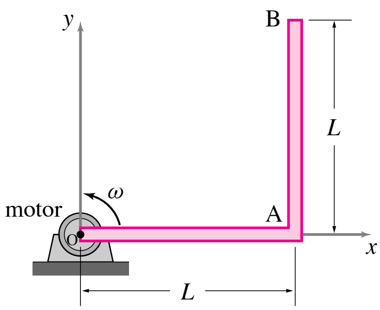


Figure 16.27: An ‘L’ shaped bar rotates at speed ω about point O.

SAMPLE 16.7 Verify the relative velocity formula: The motor at O in Fig. 16.27 rotates the ‘L’ shaped bar OAB in counterclockwise direction at an angular speed which increases at $\dot{\omega} = 2.5\,\mathrm{rad/s^2}$. At the instant shown, the angular speed $\omega = 4.5\,\mathrm{rad/s}$. Each arm of the bar is of length $L = 2\,\mathrm{ft}$.

1. Find the velocity of point A.
2. Find the relative velocity $\vec{v}_{B/A}$ $(= \vec{\omega} \times \vec{r}_{B/A})$ and use the result to find the absolute velocity of point B ($\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$).
3. Find the velocity of point B directly. Check the answer obtained in part (b) against the new answer.

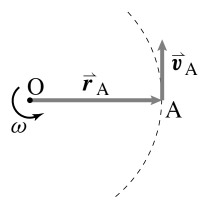


Figure 16.28: $\vec{v}_A = \vec{\omega} \times \vec{r}_A$ is tangential to the circular path of point A.

Solution

1. As the bar rotates, every point on the bar goes in circles centered at point O. Therefore, we can easily find the velocity of any point on the bar using circular motion formula $\vec{v} = \vec{\omega} \times \vec{r}$. Thus,

$$\begin{aligned} \vec{v}_A &= \vec{\omega} \times \vec{r}_A = \omega\hat{k} \times L\hat{\imath} = \omega L\hat{\jmath} \\ &= 4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\,\hat{\jmath} = 9\,\mathrm{ft/s}\,\hat{\jmath}. \end{aligned}$$

The velocity vector $\vec{v}_A$ is shown in Fig. 16.28.

$$\boxed{\vec{v}_A = 9\,\mathrm{ft/s}\,\hat{\jmath}}$$

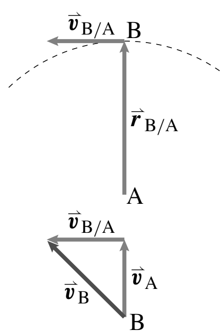


Figure 16.29: $\vec{v}_{B/A} = \vec{\omega} \times \vec{r}_{B/A}$ and $\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$.

2. Point B and A are on the same rigid body. Therefore, with respect to point A, point B goes in circles about A. Hence the relative velocity of B with respect to A is

$$\begin{aligned} \vec{v}_{B/A} &= \vec{\omega} \times \vec{r}_{B/A} \\ &= \omega\hat{k} \times L\hat{\jmath} = -\omega L\hat{\imath} \\ &= -4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\,\hat{\imath} = -9\,\mathrm{ft/s}\,\hat{\imath}. \\ \text{and} \quad \vec{v}_B &= \vec{v}_A + \vec{v}_{B/A} \\ &= 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

These velocities are shown in Fig. 16.29.

$$\boxed{\vec{v}_{B/A} = -9\,\mathrm{ft/s}\,\hat{\imath},\ \vec{v}_B = 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

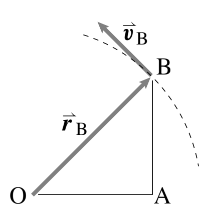


Figure 16.30: $\vec{v}_B = \vec{\omega} \times \vec{r}_B$.

3. Since point B goes in circles of radius OB about point O, we can find its velocity directly using circular motion formula:

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B \\ &= \omega\hat{k} \times (L\hat{\imath} + L\hat{\jmath}) = \omega L(\hat{\jmath} - \hat{\imath}) \\ &= 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath}). \end{aligned}$$

The velocity vector is shown in Fig. 16.30. Of course this velocity is the same velocity as obtained in part (b) above.

$$\boxed{\vec{v}_B = 9\,\mathrm{ft/s}(-\hat{\imath} + \hat{\jmath})}$$

Note: Nothing in this sample uses $\dot{\omega}$!

SAMPLE 16.8 **Verify the relative acceleration formula:** Consider the ‘L’ shaped bar of Sample 16.7 again. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 rad/s^2.

(i) Find the acceleration of point A.

(ii) Find the relative acceleration $\vec{\boldsymbol{a}}_{B/A}$ of point B with respect to point A and use the result to find the absolute acceleration of point B ($\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A}$).

(iii) Find the acceleration of point B directly and verify the result obtained in (ii).

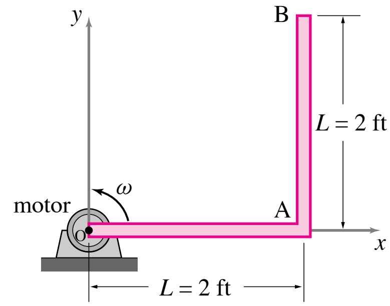


Figure 16.31: The ‘L’ shaped bar rotates counterclockwise while slowing down.

Solution We are given:

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} = 4\,\mathrm{rad/s}\hat{\boldsymbol{k}}, \quad \text{and} \quad \dot{\vec{\omega}} = -\dot{\omega}\hat{\boldsymbol{k}} = -2\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}.$$

(i) Point A is going in circles of radius L. Hence,

$$\begin{aligned}
\vec{\boldsymbol{a}}_A &= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_A + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_A) = \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_A - \omega^2 \vec{\boldsymbol{r}}_A \\
&= -\dot{\omega}\hat{\boldsymbol{k}} \times L\hat{\boldsymbol{\imath}} - \omega^2 L\hat{\boldsymbol{\imath}} = -\dot{\omega}L\hat{\boldsymbol{\jmath}} - \omega^2 L\hat{\boldsymbol{\imath}} \\
&= -2\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\hat{\boldsymbol{\jmath}} - (4\,\mathrm{rad/s})^2 \cdot 2\,\mathrm{ft}\hat{\boldsymbol{\imath}} \\
&= -(4\hat{\boldsymbol{\jmath}} + 32\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -(4\hat{\boldsymbol{\jmath}} + 32\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}}$$

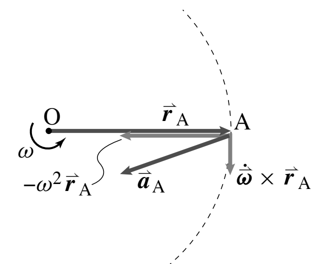


Figure 16.32

(ii) The relative acceleration of point B with respect to point A is found by considering the motion of B with respect to A. Since both the points are on the same rigid body, point B executes circular motion with respect to point A. Therefore,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_{B/A} + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_{B/A}) = \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_{B/A} - \omega^2 L\hat{\boldsymbol{\jmath}} \\
&= -\dot{\omega}\hat{\boldsymbol{k}} \times L\hat{\boldsymbol{\jmath}} - \omega^2 L\hat{\boldsymbol{\jmath}} \\
&= \dot{\omega}L\hat{\boldsymbol{\imath}} - \omega^2 L\hat{\boldsymbol{\jmath}} = 2\,\mathrm{rad/s^2} \cdot 2\,\mathrm{ft}\hat{\boldsymbol{\imath}} - (4\,\mathrm{rad/s})^2 \cdot 2\,\mathrm{ft}\hat{\boldsymbol{\jmath}} \\
&= (4\hat{\boldsymbol{\imath}} - 32\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2},
\end{aligned}$$

and

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} = (-28\hat{\boldsymbol{\imath}} - 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}.$$

$$\boxed{\vec{\boldsymbol{a}}_B = -(28\hat{\boldsymbol{\imath}} + 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}}$$

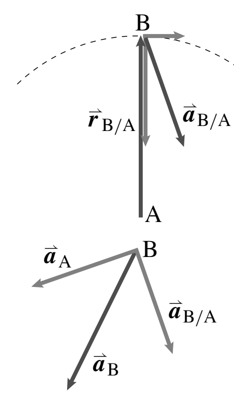


Figure 16.33

(iii) Since point B is going in circles of radius OB about point O, we can find the acceleration of B as follows.

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_B + \vec{\omega} \times (\vec{\omega} \times \vec{\boldsymbol{r}}_B) \\
&= \dot{\vec{\omega}} \times \vec{\boldsymbol{r}}_B - \omega^2 \vec{\boldsymbol{r}}_B \\
&= -\dot{\omega}\hat{\boldsymbol{k}} \times (L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) - \omega^2 (L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) \\
&= (-\dot{\omega}L - \omega^2 L)\hat{\boldsymbol{\jmath}} + (\dot{\omega}L - \omega^2 L)\hat{\boldsymbol{\imath}} \\
&= (-4 - 32)\,\mathrm{ft/s^2}\hat{\boldsymbol{\jmath}} + (4 - 32)\,\mathrm{ft/s^2}\hat{\boldsymbol{\imath}} \\
&= (-36\hat{\boldsymbol{\jmath}} - 28\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}.
\end{aligned}$$

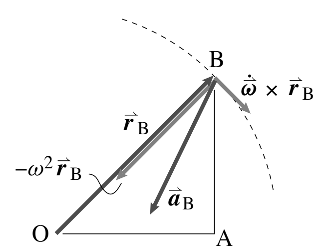


Figure 16.34

This acceleration is, naturally again, the same acceleration as found in (ii) above.

$$\boxed{\vec{\boldsymbol{a}}_B = -(28\hat{\boldsymbol{\imath}} + 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}}$$

16.2 Angular velocity

Preparatory Problems

16.2.1 Give the definition of each term below in words and, but for the first, with an equation.

a) angular velocity *

b) angular acceleration *

16.2.2 Assume that in the reference configuration, when $\theta = 0$, that the $x'y'$ axes attached to an object are aligned with the xy axes. Consider a point P attached to the moving frame (object) that has coordinates (x', y'). Find each of the quantities below in terms of some or all of $x', y', \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

a) $\vec{\boldsymbol{r}}_{\mathrm{P}}$ *

b) $\vec{\boldsymbol{v}}_{\mathrm{P}}$ *

c) $\vec{\boldsymbol{a}}_{\mathrm{P}}$ *

16.2.3 Assume that in the reference configuration, when $\theta = 0$, the $x'y'$ axes are aligned with the xy axes. Consider two points $\mathrm{P_1}$ and $\mathrm{P_2}$ attached to the moving frame. $\mathrm{P_1}$ and $\mathrm{P_2}$ have coordinates (x'_1, y'_1) (x'_2, y'_2). Find each of the quantities below in terms of some or all of $x'_1, y'_1, x'_2, y'_2, \hat{\boldsymbol{\imath}}, \hat{\boldsymbol{\jmath}}, \theta, \dot{\theta}$ and $\ddot{\theta}$.

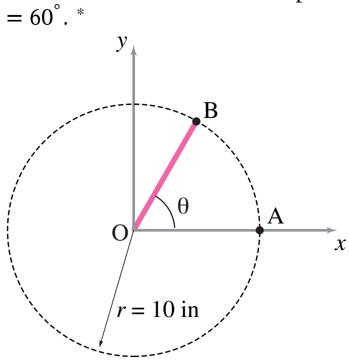
a) $\vec{\boldsymbol{r}}_{\mathrm{P_1/P_2}}$ *

b) $\vec{\boldsymbol{v}}_{\mathrm{P_1/P_2}}$ *

c) $\vec{\boldsymbol{a}}_{\mathrm{P_1/P_2}}$ *

16.2.4 Find $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$, if $\vec{\boldsymbol{\omega}} = 1.5\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and $\vec{\boldsymbol{r}} = 2\,\mathrm{m}\hat{\boldsymbol{\imath}} - 3\,\mathrm{m}\hat{\boldsymbol{\jmath}}$. *

16.2.5 A rod OB rotates with its end O fixed as shown in the figure with angular velocity $\vec{\boldsymbol{\omega}} = 5\,\mathrm{rad/s}\hat{\boldsymbol{k}}$ and angular acceleration $\vec{\boldsymbol{\alpha}} = 2\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}$ at the instant of interest. Find, draw, and label the tangential and normal acceleration of point B at $\theta = 60^\circ$. *



Problem 16.2.5

16.2.6 A disc rotates at 15 rpm. How many seconds does it take to rotate by 180 degrees? What is the angular speed of the disc in rad/s? *

16.2.7 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What is the angular displacement $\theta(t)$ of the disc if it starts at $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \omega$? What are the velocity and acceleration of a point P at position $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$? *

16.2.8 Find the angular velocities of the second, minute, and hour hands of a clock. *

16.2.9 A disc $\mathcal{C}$ spins at a constant rate of two revolutions per second counter-clockwise about its geometric center, G, which is fixed. A point P is marked on the disk at a radius of one meter. At the instant of interest, point P is on the x-axis of an xy-coordinate system centered at point G.

a) Draw a neat diagram showing the disk, the particle, and the coordinate axes.

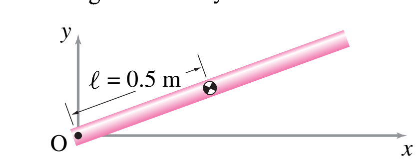
b) What is the angular velocity of the disk, $\vec{\boldsymbol{\omega}}_{\mathcal{C}}$? *

c) What is the angular acceleration of the disk, $\dot{\vec{\boldsymbol{\omega}}}_{\mathrm{C}}$? *

d) What is the velocity $\vec{\boldsymbol{v}}_P$ of point P? *

e) What is the acceleration $\vec{\boldsymbol{a}}_P$ of point P? *

16.2.10 A uniform rigid rod rotates at constant speed in the xy-plane about a peg at point O. The center of mass of the rod may not exceed a specified acceleration $a_{max} = 0.5\,\mathrm{m/s^2}$. Find the maximum angular velocity of the rod. *

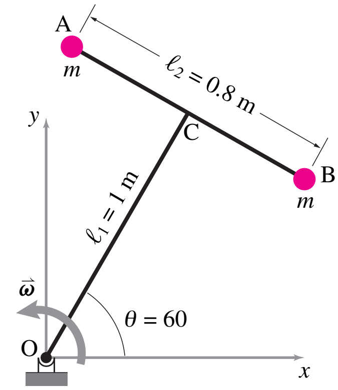


Problem 16.2.10

16.2.11 A dumbbell AB is welded to a rigid arm OC such that OC is perpendicular to AB. Arm OC rotates about O at a constant angular velocity $\vec{\boldsymbol{\omega}} = 10\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. At the instant when $\theta = 60^\circ$,

a) Find the relative velocity of B with respect to A. *

b) Find the acceleration of point B relative to point A. *



Problem 16.2.11

More-Involved Problems

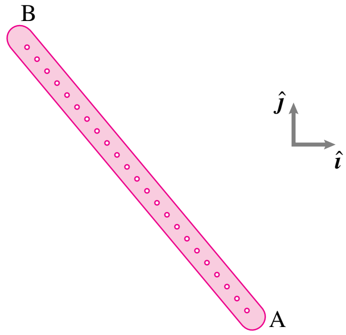
16.2.12 Two discs $\mathcal{A}$ and $\mathcal{B}$ rotate at constant speeds about their centers. Disc $\mathcal{A}$ rotates at 100 rpm and disc $\mathcal{B}$ rotates at 10 rad/s. Which is rotating faster? *

16.2.13 A motor turns a uniform disc of radius R counter-clockwise about its mass center at a constant rate ω. The disc lies in the xy-plane and its angular displacement θ is measured (positive counter-clockwise) from the x-axis. What are the velocity and acceleration of a point P at position $\vec{\boldsymbol{r}}_P = c\hat{\boldsymbol{\imath}} + d\hat{\boldsymbol{\jmath}}$ relative to the velocity and acceleration of a point Q at position $\vec{\boldsymbol{r}}_Q = 0.5(-d\hat{\boldsymbol{\imath}} + c\hat{\boldsymbol{\jmath}})$ on the disk? $(c^2 + d^2 < R^2.)$ *

16.2.14 A 0.4 m long rod AB has many holes along its length such that it can be pegged at any of the various locations. It rotates counter-clockwise at a constant angular speed about a peg whose location is not known. At some instant t, the velocity of end B is $\vec{\boldsymbol{v}}_B = -3\,\mathrm{m/s}\hat{\boldsymbol{\jmath}}$. After $\frac{\pi}{20}$ s, the velocity of end B is $\vec{\boldsymbol{v}}_B = -3\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$. If the rod has not completed one revolution during this period,

a) find the angular velocity of the rod, and *

b) find the location of the peg along the length of the rod. *



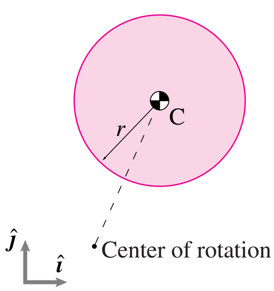
Problem 16.2.14

16.2.15 A circular disc of radius $r = 250\,\mathrm{mm}$ rotates in the xy-plane about a point which is at a distance $d = 2r$ away from the center of the disk. At the instant of interest, the linear speed of the center C is $0.60\,\mathrm{m/s}$ and the magnitude of its centripetal acceleration is $0.72\,\mathrm{m/s^2}$.

a) Find the rotational speed of the disk. *

b) Is the given information enough to locate the center of rotation of the disk? *

c) If the acceleration of the center has no component in the $\hat{\jmath}$ direction at the instant of interest, can you locate the center of rotation? If yes, is the point you locate unique? If not, what other information is required to make the point unique? *

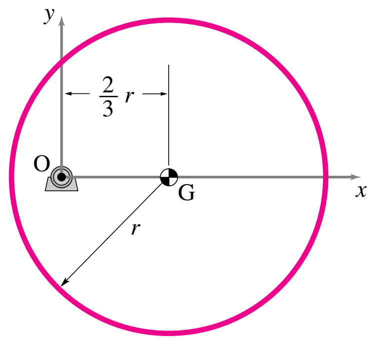


Problem 16.2.15

16.2.16 A uniform disc of radius $r = 200\,\mathrm{mm}$ is mounted eccentrically on a motor shaft at point O. The motor rotates the disc at a constant angular speed. At the instant shown, the velocity of the center of mass is $\vec{v}_G = -1.5\,\mathrm{m/s}\,\hat{\jmath}$.

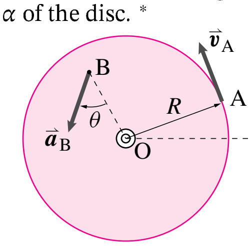
a) Find the angular velocity of the disc. *

b) Find the point with the highest linear speed on the disc. What is its velocity? *



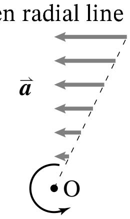
Problem 16.2.16

16.2.17 The circular disc of radius $R = 100\,\mathrm{mm}$ rotates about its center O. At a given instant, point A on the disk has a velocity $v_A = 0.8\,\mathrm{m/s}$ in the direction shown. At the same instant, the tangent of the angle θ made by the total acceleration vector of any point B with its radial line to O is 0.6. Compute the angular acceleration α of the disc. *



Problem 16.2.17

16.2.18 Show that, for non-constant rate circular motion, the acceleration of all points in a given radial line are parallel.



Problem 16.2.18

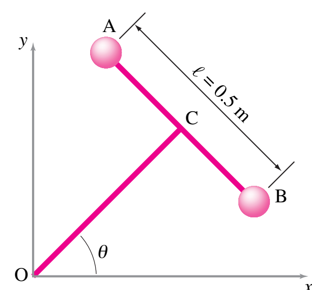
16.2.19 A motor turns a uniform disc of radius R about its mass center at a variable angular rate ω with rate of change $\dot{\omega}$, counter-clockwise. The disc lies in the xy-plane and its angular displacement θ is measured from the x-axis, positive counter-clockwise. What are the velocity and acceleration of a point P at position $\vec{r}_P = c\hat{\imath} + d\hat{\jmath}$ relative to the velocity and acceleration of a point Q at position $\vec{r}_Q = 0.5(-d\hat{\imath} + c\hat{\jmath})$ on the disk? ($c^2 + d^2 < R^2$.) *

16.2.20 The dumbbell AB shown in the figure rotates counterclockwise about point O with angular acceleration $3\,\mathrm{rad/s^2}$. Bar AB is perpendicular to bar OC. At the instant of interest, $\theta = 45^\circ$ and the angular speed is $2\ \mathrm{rad/s}$.

a) Find the velocity of point B relative to point A. Will this relative velocity be different if the dumbbell were rotating at a *constant* rate of $2\ \mathrm{rad/s}$? *

b) Without calculations, draw a vector approximately representing the acceleration of B relative to A.

c) Find the acceleration of point B relative to A. What can you say about the direction of this vector as the motion progresses in time? *



Problem 16.2.20

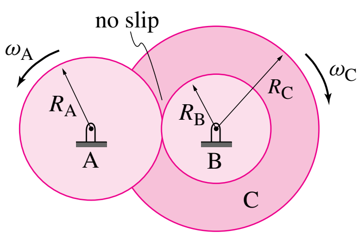
16.2.21 Bit-stream kinematics of a CD. A Compact Disk (CD) has bits of data etched on concentric circular tracks. The data from a track is read by a beam of light from a head that is positioned under the track. The angular speed of the disk remains constant as long as the head is positioned over a particular track. As the head moves to the next track, the angular speed of the disk changes, so that the *linear speed* at any track is always the same. The data stream comes out at a constant rate 4.32×10^6 bits/second. When the head is positioned on the outermost track, for which $r = 56\,\mathrm{mm}$, the disk rotates at 200 rpm.

a) What is the number of bits of data on the outermost track? *

b) find the angular speed of the disk when the head is on the innermost track ($r = 22$ mm), and *

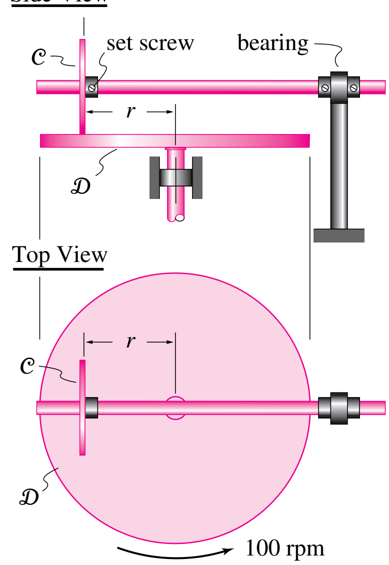
c) find the number of bits on the innermost track. *

16.2.22 2-D constant rate gear train. The angular velocity of the input shaft (driven by a motor not shown) is a constant, $\omega_{\text{input}} = \omega_A$. What is the angular velocity $\omega_{\text{output}} = \omega_C$ of the output shaft and the speed of a point on the outer edge of disc C, in terms of R_A, R_B, R_C, and ω_A? *



Problem 16.2.22: Gear B is welded to C and engages with A.

16.2.23 A horizontal disk $\mathcal{D}$ of diameter $d = 500$ mm is driven at a constant speed of 100 rpm. A small disk $\mathcal{C}$ can be positioned anywhere between $r = 10$ mm and $r = 240$ mm on disk $\mathcal{D}$ by sliding it along the overhead shaft and then fixing it at the desired position with a set screw (see the figure). Disk $\mathcal{C}$ rolls without slip on disk $\mathcal{D}$. The overhead shaft rotates with disk $\mathcal{C}$ and, therefore, its rotational speed can be varied by varying the position of disk $\mathcal{C}$. This gear system is called *brush gearing*. Find the maximum and minimum rotational speeds of the overhead shaft. *



Problem 16.2.23

16.2.24 Two points A and B are on the same machine part that is hinged at an as yet unknown location C. Assume you are given that points at positions $\vec{r}_A$ and $\vec{r}_B$ are supposed to move in given directions, indicated by unit vectors $\hat{\lambda}_A$ and $\hat{\lambda}_B$. For each of the problem parts below, illustrate your results with two numerical examples (in consistent units): i) $\vec{r}_A = 1\hat{\imath}$, $\vec{r}_B = 1\hat{\jmath}$, $\hat{\lambda}_A = 1\hat{\jmath}$, and $\hat{\lambda}_B = -1\hat{\imath}$ (thus $\vec{r}_C = \vec{0}$), and ii) a more complex example of your choosing.

a) Describe in detail what equations must be satisfied by the point $\vec{r}_C$.

b) Write a computer program that takes as input the 4 pairs of numbers $[\vec{r}_A]$, $[\vec{r}_B]$, $[\hat{\lambda}_A]$ and $[\hat{\lambda}_B]$ and gives as output the pair of numbers $[\vec{r}_C]$.

c) Find a formula of the form $\vec{r}_C = \ldots$ that explicitly gives the position vector for point C in terms of the 4 given vectors. *