SAMPLE 16.3 A uniform bar AB of length $\ell = 50\,\text{cm}$ rotates counterclockwise about point A with constant angular speed ω. At the instant shown in fig. 16.19 the linear speed v_C of the center-of-mass C is 7.5 cm/ s.

1. What are the angular speed and angular velocity of the bar?
2. What is the linear velocity of point B?
3. By what angles do the angular positions of points C and B change in 2 seconds?

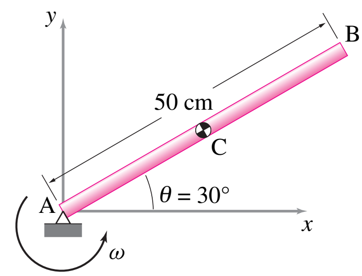

Figure 16.19

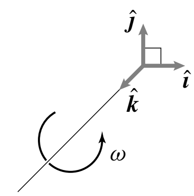

Figure 16.20

Solution Let the angular velocity of the bar be $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}}$ where $\dot{\theta}$ is the angular speed. We first need to find $\dot{\theta}$.

1. The linear speed of point C is given, $v_C = 7.5\,\text{cm/ s}$. Now,

$$\begin{aligned} v_C &= \dot{\theta} r_C \\ \Rightarrow \quad \dot{\theta} &= \frac{v_C}{r_C} = \frac{7.5\,\text{cm/ s}}{25\,\text{cm}} = 0.3\,\text{rad/s}. \end{aligned}$$

Therefore, the angular velocity of the bar is $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}} = 0.3\,\text{rad/s}\hat{\boldsymbol{k}}$.

$$\boxed{\dot{\theta} = 0.3\,\text{rad/s}, \text{ and } \vec{\omega} = 0.3\,\text{rad/s}\hat{\boldsymbol{k}}}$$

2. Point B is at distance ℓ from the pivot point A. Thus it goes around a circle of radius ℓ (see fig. 16.21). Therefore,

$$\begin{aligned} \vec{v}_B &= \vec{\omega} \times \vec{r}_B = \dot{\theta}\hat{\boldsymbol{k}} \times \ell(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) \\ &= \dot{\theta}\ell(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) \\ &= 0.3\,\text{rad/s} \cdot 50\,\text{cm}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right) \\ &= 15\,\text{cm/ s}\left(\frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}\right). \end{aligned}$$

$$\boxed{\vec{v}_B = 15\,\text{cm/ s}\left(\tfrac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \tfrac{1}{2}\hat{\boldsymbol{\imath}}\right)}$$

We can also write $\vec{v}_B = 15\,\text{cm/ s}\,\hat{\boldsymbol{e}}_\theta$ where $\hat{\boldsymbol{e}}_\theta = \frac{\sqrt{3}}{2}\hat{\boldsymbol{\jmath}} - \frac{1}{2}\hat{\boldsymbol{\imath}}$.

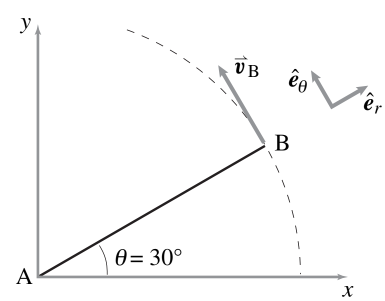

Figure 16.21: From the given geometry, $\hat{\boldsymbol{e}}_r = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$, $\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$, and $\vec{v}_B = |\vec{v}_B|\hat{\boldsymbol{e}}_\theta$.

3. Let θ_1 be the position of point C at some time t_1 and θ_2 be the position at time t_2. We want to find $\Delta\theta = \theta_2 - \theta_1$ for $t_2 - t_1 = 2\,\text{s}$.

$$\begin{aligned} \frac{d\theta}{dt} &= \dot{\theta} = \text{constant} = 0.3\,\text{rad/s}. \\ \Rightarrow \quad d\theta &= (0.3\,\text{rad/s})dt. \\ \Rightarrow \quad \int_{\theta_1}^{\theta_2} d\theta &= \int_{t_1}^{t_2} (0.3\,\text{rad/s})dt. \\ \Rightarrow \quad \theta_2 - \theta_1 &= 0.3\,\text{rad/s}(t_2 - t_1) \\ \text{or} \quad \Delta\theta &= 0.3\frac{\text{rad}}{\cancel{\text{s}}}\cdot 2\cancel{\text{s}} = 0.6\,\text{rad}. \end{aligned}$$

The change in angular position of point B is the same as that of point C. In fact, all points on AB undergo the same change in angular position because AB is a rigid body.

$$\boxed{\Delta\theta_C = \Delta\theta_B = 0.6\,\text{rad}}$$

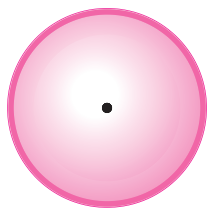

Figure 16.22

SAMPLE 16.4 A flywheel of diameter 2 ft is made of cast iron. To avoid extremely high stresses and cracks it is recommended that the peripheral speed not exceed 6000 to 7000 ft/min. What is the corresponding rpm rating for the wheel?

Solution

$$\begin{aligned} \text{Diameter of the wheel} &= 2\,\text{ft}. \\ \Rightarrow \quad \text{radius of wheel} &= 1\,\text{ft}. \end{aligned}$$

Now,

$$\begin{aligned} v &= \omega r \\ \Rightarrow \quad \omega &= \frac{v}{r} = \frac{6000\,\cancel{f}\text{t/min}}{1\,\cancel{f}\text{t}} \\ &= 6000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\,\text{rad}} \\ &= 955\,\text{rpm}. \end{aligned}$$

Similarly, corresponding to $v = 7000\,\text{ft/min}$

$$\begin{aligned} \omega &= \frac{7000\,\cancel{f}\text{t/min}}{1\,\cancel{f}\text{t}} \\ &= 7000\frac{\text{rad}}{\text{min}} \cdot \frac{1\text{rev}}{2\pi\,\text{rad}} \\ &= 1114\,\text{rpm}. \end{aligned}$$

Thus the rpm rating of the wheel should read 955 – 1114 rpm.

$\omega = 955$ to $1114\,\text{rpm}$.

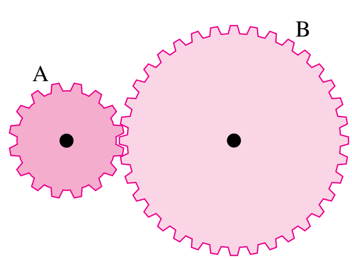

Figure 16.23

SAMPLE 16.5 Two gears A and B have the diameter ratio of 1:2. Gear A drives gear B. If the output at gear B is required to be 150 rpm, what should be the angular speed of the driving gear? Assume no slip at the contact point.

Solution Let C and C′ be the points of contact on gear A and B respectively at some instant t. Since there is no relative slip between C and C′, both points must have the same linear velocity at instant t. If the velocities are the same, then the linear speeds must also be the same. Thus

$$\begin{aligned} v_C &= v_{C'} \\ \Rightarrow \quad \omega_A r_A &= \omega_B r_B \\ \Rightarrow \quad \omega_A &= \omega_B \cdot \frac{r_B}{r_A} \\ &= \omega_B \cdot \frac{2\cancel{r}}{\cancel{r}} = 2\omega_B \\ &= (2)\cdot(150\,\text{rpm}) \\ &= 300\,\text{rpm}. \end{aligned}$$

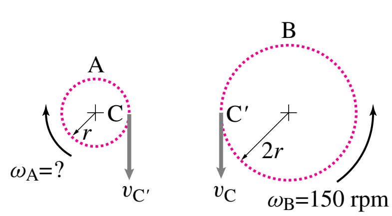

Figure 16.24

$\omega_A = 300\,\text{rpm}$

SAMPLE 16.6 A uniform rigid rod AB of length $\ell = 0.6\,\text{m}$ is connected to two rigid links OA and OB. The assembly rotates at a constant rate about point O in the xy plane. At the instant shown, when rod AB is vertical, the velocities of points A and B are $\vec{\boldsymbol{v}}_A = -4.64\,\text{m/s}\,\hat{\boldsymbol{\jmath}} - 1.87\,\text{m/s}\hat{\boldsymbol{\imath}}$, and $\vec{\boldsymbol{v}}_B = 1.87\,\text{m/s}\hat{\boldsymbol{\imath}} - 4.64\,\text{m/s}\,\hat{\boldsymbol{\jmath}}$. Find the angular velocity of bar AB. What is the length R of the links?

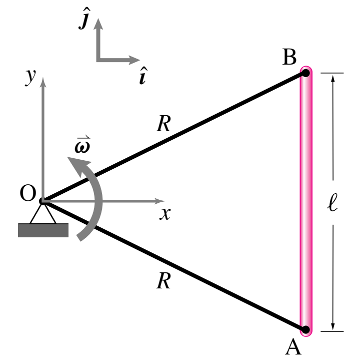

Figure 16.25

Solution Let the angular velocity of the rod AB be $\vec{\boldsymbol{\omega}} = \omega\hat{\boldsymbol{k}}$. ① Since we are given the velocities of two points on the rod we can use the relative velocity formula to find $\vec{\boldsymbol{\omega}}$:

$$
\begin{aligned}
\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} &= \vec{\boldsymbol{v}}_B - \vec{\boldsymbol{v}}_A \\
\text{or} \quad \underbrace{\omega\hat{\boldsymbol{k}}}_{\vec{\boldsymbol{\omega}}} \times \underbrace{\ell\,\hat{\boldsymbol{\jmath}}}_{\vec{\boldsymbol{r}}_{B/A}} &= (1.87\hat{\boldsymbol{\imath}} - 4.64\hat{\boldsymbol{\jmath}})\,\text{m/s} - (-4.64\hat{\boldsymbol{\jmath}} - 1.87\hat{\boldsymbol{\imath}})\,\text{m/s} \\
\text{or} \quad \omega\ell(-\hat{\boldsymbol{\imath}}) &= (1.87\hat{\boldsymbol{\imath}} + 1.87\hat{\boldsymbol{\imath}})\,\text{m/s} - (4.\cancel{6}4\hat{\boldsymbol{\jmath}} - 4.\cancel{6}4\hat{\boldsymbol{\jmath}})\,\text{m/s} \\
&= 3.74\hat{\boldsymbol{\imath}}\,\text{m/s} \\
\Rightarrow \qquad \omega &= -\frac{3.74\,\text{m/s}}{\ell} \\
&= -\frac{3.74}{0.6}\,\text{rad/s} \\
&= -6.233\,\text{rad/s} \qquad (16.27)
\end{aligned}
$$

Thus,

$$
\vec{\boldsymbol{\omega}} = -6.233\,\text{rad/s}\hat{\boldsymbol{k}}. \qquad (16.28)
$$

$$
\boxed{\vec{\boldsymbol{\omega}} = -6.23\,\text{rad/s}\hat{\boldsymbol{k}}}
$$

① We know that the rod rotates about the z-axis but we do not know the sense of the rotation *i.e.*, $+\hat{\boldsymbol{k}}$ or $-\hat{\boldsymbol{k}}$. Here we have assumed that $\vec{\boldsymbol{\omega}}$ is in the positive $\hat{\boldsymbol{k}}$ direction, although just by sketching $\vec{\boldsymbol{v}}_A$ we can easily see that $\vec{\boldsymbol{\omega}}$ must be in the $-\hat{\boldsymbol{k}}$ direction.

Let θ be the angle between link OA and the horizontal axis. Now,

$$
\begin{aligned}
\vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_A &= \omega\hat{\boldsymbol{k}} \times \underbrace{R(\cos\theta\hat{\boldsymbol{\imath}} - \sin\theta\,\hat{\boldsymbol{\jmath}})}_{\vec{\boldsymbol{r}}_A} \\
\text{or} \quad (-4.64\hat{\boldsymbol{\jmath}} - 1.87\hat{\boldsymbol{\imath}})\,\text{m/s} &= \omega R(\cos\theta\,\hat{\boldsymbol{\jmath}} + \sin\theta\hat{\boldsymbol{\imath}})
\end{aligned}
$$

Dotting both sides of the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get

$$
\begin{aligned}
-1.87\,\text{m/s} &= \omega R\sin\theta \qquad (16.29) \\
-4.64\,\text{m/s} &= \omega R\cos\theta \qquad (16.30)
\end{aligned}
$$

Squaring and adding Eqns (16.29) and (16.30) together we get

$$
\begin{aligned}
\omega^2 R^2 &= (-4.64\,\text{m/s})^2 + (-1.187\,\text{m/s})^2 \\
&= 25.026\,\text{m}^2/\,\text{s}^2 \\
\Rightarrow \qquad R^2 &= \frac{25.026\,\text{m}^2/\,\text{s}^2}{(-6.23\,\text{rad/s})^2} \\
&= 0.645\,\text{m}^2 \\
\Rightarrow \qquad R &= 0.8\,\text{m}
\end{aligned}
$$

$$
\boxed{R = 0.8\,\text{m}}
$$

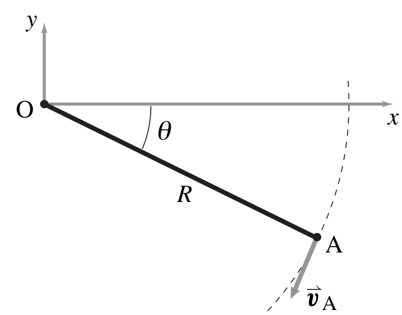

Figure 16.26: The given velocity vector $\vec{\boldsymbol{v}}_A$. Since $\vec{\boldsymbol{v}}_\text{A} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_\text{A}$ where $\vec{\boldsymbol{r}}_\text{A} = R\cos\theta\hat{\boldsymbol{\imath}} + R\sin\theta\,\hat{\boldsymbol{\jmath}}$, we can find R from the given $\vec{\boldsymbol{v}}_\text{A}$.

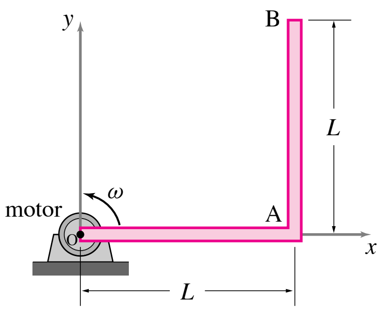

Figure 16.27: An 'L' shaped bar rotates at speed ω about point O.

SAMPLE 16.7 Verify the relative velocity formula: The motor at O in Fig. 16.27 rotates the 'L' shaped bar OAB in counterclockwise direction at an angular speed which increases at $\dot{\omega} = 2.5\,\mathrm{rad/s^2}$. At the instant shown, the angular speed $\omega = 4.5\,\mathrm{rad/s}$. Each arm of the bar is of length $\mathrm{L} = 2\,\mathrm{ft}$.

1. Find the velocity of point A.
2. Find the relative velocity $\vec{\boldsymbol{v}}_{B/A}$ $(= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A})$ and use the result to find the absolute velocity of point B $(\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A})$.
3. Find the velocity of point B directly. Check the answer obtained in part (b) against the new answer.

Solution

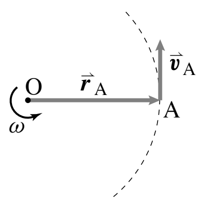

Figure 16.28: $\vec{\boldsymbol{v}}_A = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_A$ is tangential to the circular path of point A.

1. As the bar rotates, every point on the bar goes in circles centered at point O. Therefore, we can easily find the velocity of any point on the bar using circular motion formula $\vec{\boldsymbol{v}} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}$. Thus,

$$\begin{aligned} \vec{\boldsymbol{v}}_A &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_A = \omega \hat{\boldsymbol{k}} \times L\hat{\boldsymbol{\imath}} = \omega L \hat{\boldsymbol{\jmath}} \\ &= 4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\,\hat{\boldsymbol{\jmath}} = 9\,\mathrm{ft/s}\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

The velocity vector $\vec{\boldsymbol{v}}_A$ is shown in Fig. 16.28.

$$\boxed{\vec{\boldsymbol{v}}_A = 9\,\mathrm{ft/s}\,\hat{\boldsymbol{\jmath}}}$$

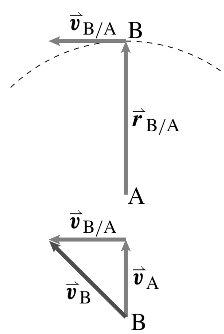

Figure 16.29: $\vec{\boldsymbol{v}}_{B/A} = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A}$ and $\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A}$.

2. Point B and A are on the same rigid body. Therefore, with respect to point A, point B goes in circles about A. Hence the relative velocity of B with respect to A is

$$\begin{aligned} \vec{\boldsymbol{v}}_{B/A} &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_{B/A} \\ &= \omega \hat{\boldsymbol{k}} \times L\hat{\boldsymbol{\jmath}} = -\omega L \hat{\boldsymbol{\imath}} \\ &= -4.5\,\mathrm{rad/s} \cdot 2\,\mathrm{ft}\,\hat{\boldsymbol{\imath}} = -9\,\mathrm{ft/s}\,\hat{\boldsymbol{\imath}}. \\ \text{and} \quad \vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{v}}_A + \vec{\boldsymbol{v}}_{B/A} \\ &= 9\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}). \end{aligned}$$

These velocities are shown in Fig. 16.29.

$$\boxed{\vec{\boldsymbol{v}}_{B/A} = -9\,\mathrm{ft/s}\,\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{v}}_B = 9\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

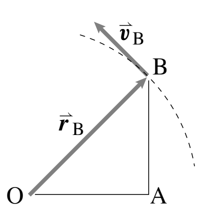

Figure 16.30: $\vec{\boldsymbol{v}}_B = \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_B$.

3. Since point B goes in circles of radius OB about point O, we can find its velocity directly using circular motion formula:

$$\begin{aligned} \vec{\boldsymbol{v}}_B &= \vec{\boldsymbol{\omega}} \times \vec{\boldsymbol{r}}_B \\ &= \omega \hat{\boldsymbol{k}} \times (L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) = \omega L(\hat{\boldsymbol{\jmath}} - \hat{\boldsymbol{\imath}}) \\ &= 9\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}). \end{aligned}$$

The velocity vector is shown in Fig. 16.30. Of course this velocity is the same velocity as obtained in part (b) above.

$$\boxed{\vec{\boldsymbol{v}}_B = 9\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

Note: Nothing in this sample uses $\dot{\omega}$!

SAMPLE 16.8 **Verify the relative acceleration formula:** Consider the 'L' shaped bar of Sample 16.7 again. At the instant shown, the bar is rotating at 4 rad/s and is slowing down at the rate of 2 $\mathrm{rad/s^2}$.

(i) Find the acceleration of point A.

(ii) Find the relative acceleration $\vec{\boldsymbol{a}}_{B/A}$ of point B with respect to point A and use the result to find the absolute acceleration of point B ($\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A}$).

(iii) Find the acceleration of point B directly and verify the result obtained in (ii).

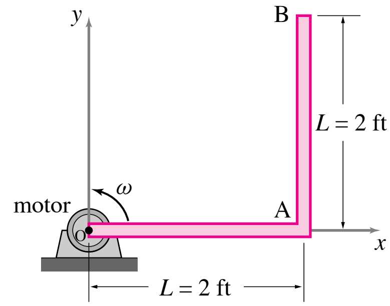

Figure 16.31: The 'L' shaped bar rotates counterclockwise while slowing down.

Solution We are given:

$$\vec{\omega} = \omega\hat{\boldsymbol{k}} = 4\,\mathrm{rad/s}\hat{\boldsymbol{k}}, \quad \text{and} \quad \dot{\vec{\omega}} = -\dot{\omega}\hat{\boldsymbol{k}} = -2\,\mathrm{rad/s^2}\hat{\boldsymbol{k}}.$$

(i) Point A is going in circles of radius L. Hence,

$$\begin{aligned}
\vec{\boldsymbol{a}}_A &= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_A + \vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}}_A) = \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_A - \omega^2\vec{\boldsymbol{r}}_A \\
&= -\dot{\omega}\hat{\boldsymbol{k}}\times L\hat{\boldsymbol{\imath}} - \omega^2 L\hat{\boldsymbol{\imath}} = -\dot{\omega}L\hat{\boldsymbol{\jmath}} - \omega^2 L\hat{\boldsymbol{\imath}} \\
&= -2\,\mathrm{rad/s}\cdot 2\,\mathrm{ft}\hat{\boldsymbol{\jmath}} - (4\,\mathrm{rad/s})^2\cdot 2\,\mathrm{ft}\hat{\boldsymbol{\imath}} \\
&= -(4\hat{\boldsymbol{\jmath}} + 32\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -(4\hat{\boldsymbol{\jmath}} + 32\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}}$$

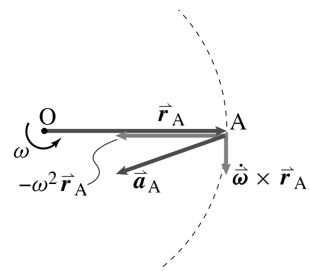

Figure 16.32

(ii) The relative acceleration of point B with respect to point A is found by considering the motion of B with respect to A. Since both the points are on the same rigid body, point B executes circular motion with respect to point A. Therefore,

$$\begin{aligned}
\vec{\boldsymbol{a}}_{B/A} &= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_{B/A} + \vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}}_{B/A}) = \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_{B/A} - \omega^2 L\hat{\boldsymbol{\jmath}} \\
&= -\dot{\omega}\hat{\boldsymbol{k}}\times L\hat{\boldsymbol{\jmath}} - \omega^2 L\hat{\boldsymbol{\jmath}} \\
&= \dot{\omega}L\hat{\boldsymbol{\imath}} - \omega^2 L\hat{\boldsymbol{\jmath}} = 2\,\mathrm{rad/s^2}\cdot 2\,\mathrm{ft}\hat{\boldsymbol{\imath}} - (4\,\mathrm{rad/s})^2\cdot 2\,\mathrm{ft}\hat{\boldsymbol{\jmath}} \\
&= (4\hat{\boldsymbol{\imath}} - 32\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2},
\end{aligned}$$

and

$$\vec{\boldsymbol{a}}_B = \vec{\boldsymbol{a}}_A + \vec{\boldsymbol{a}}_{B/A} = (-28\hat{\boldsymbol{\imath}} - 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}.$$

$$\boxed{\vec{\boldsymbol{a}}_B = -(28\hat{\boldsymbol{\imath}} + 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}}$$

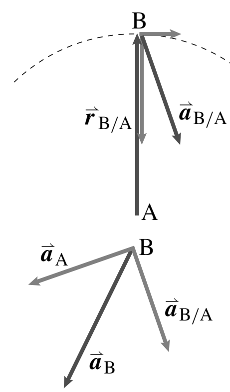

Figure 16.33

(iii) Since point B is going in circles of radius OB about point O, we can find the acceleration of B as follows.

$$\begin{aligned}
\vec{\boldsymbol{a}}_B &= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_B + \vec{\omega}\times(\vec{\omega}\times\vec{\boldsymbol{r}}_B) \\
&= \dot{\vec{\omega}}\times\vec{\boldsymbol{r}}_B - \omega^2\vec{\boldsymbol{r}}_B \\
&= -\dot{\omega}\hat{\boldsymbol{k}}\times(L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) - \omega^2(L\hat{\boldsymbol{\imath}} + L\hat{\boldsymbol{\jmath}}) \\
&= (-\dot{\omega}L - \omega^2 L)\hat{\boldsymbol{\jmath}} + (\dot{\omega}L - \omega^2 L)\hat{\boldsymbol{\imath}} \\
&= (-4 - 32)\,\mathrm{ft/s^2}\hat{\boldsymbol{\jmath}} + (4 - 32)\,\mathrm{ft/s^2}\hat{\boldsymbol{\imath}} \\
&= (-36\hat{\boldsymbol{\jmath}} - 28\hat{\boldsymbol{\imath}})\,\mathrm{ft/s^2}.
\end{aligned}$$

This acceleration is, naturally again, the same acceleration as found in (ii) above.

$$\boxed{\vec{\boldsymbol{a}}_B = -(28\hat{\boldsymbol{\imath}} + 36\hat{\boldsymbol{\jmath}})\,\mathrm{ft/s^2}}$$

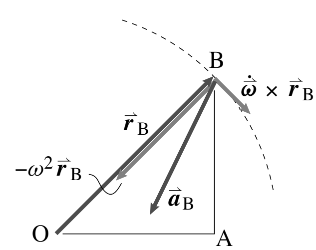

Figure 16.34