16.1 Rotation of a rigid object in planar circular motion

When two parts are glued together or attached by welding, gluing, several tight screws, bolts, rivets or the like we call the connection a 'rigid attachment'. And, for the purposes of mechanics analysis, the two connected parts make up one bigger object. But most machines have various parts that are connected to each other, but not welded to each other. The most common such non-rigid attachment in engineering is a hinge. In 2D,

> a *hinge* attachment between two objects keeps two points, one from each object, on top of each other while freely allowing relative rotation of the two objects about the hinge point.

In 3D a hinge keeps two lines, one on each body, coincident and allows relative rotation about that line. The common line, or in 2D the line orthogonal to the plane through the points, is called the hinge, the hinge axis, or the axis of relative rotation.

One example of a hinge is a car axle which allows rotation of a wheel relative to the car suspension. The hinge axis is the wheel axle[1]. Physically, hinges are made various ways, sometimes by poking a cylindrical pin through the two objects and sometimes with ball bearings (see box 5.7 on page 273). So hinges are also called *pin connections* or *bearings* (fig. 16.3). In this chapter we limit our attention to a simple use of a hinge: one rigid part is hinged to a second fixed part that doesn't move at all. Such a non-moving part can be thought of as connected to, or an extension of, the ground or 'fixed frame'. In the simple case of interest in this chapter, one point on the moving part does not move and the rest of the part rotates about that point.

For definiteness and simplicity let's call the hinge location 0 and the hinge axis through 0 the z axis. One function of the hinge is to make the part's only possible motion to be rotation about O. Thus to understand the dynamics of a hinged part we need to understand the position, velocity and acceleration of points on a rigid object which rotates. This whole section is about the kinematics (the geometry of motion) for this rotation. We will measure the amount of rotation by the angle θ, and the rate of change of θ by the *angular velocity* $\vec{\omega}$ ('omega'), and the rate of change of this angular velocity $\vec{\omega}$ by the *angular acceleration* $\vec{\alpha}$ ('alpha').

As simple as this topic seems at first glance, you should pay close attention to the meanings and uses of these quantities. The rest of the book completely depends on the material in this section.

Rotation of an object counterclockwise by θ

We start by imagining the object in a distinguished configuration which we call the *reference configuration*, *reference state* or *reference position*. For

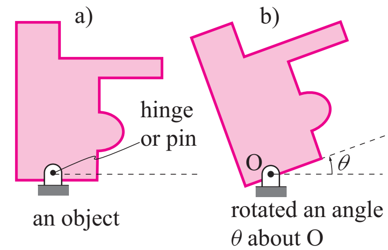


Figure 16.3: a) An object, b) rotated counterclockwise an angle θ about 0.

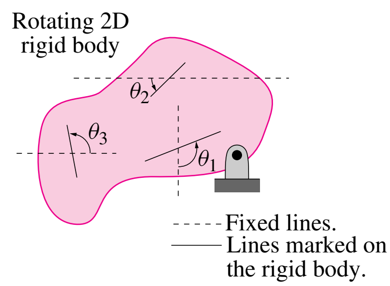


Figure 16.4: **Rotation of lines on a rotating rigid object.** Some real or imagined lines marked on the rigid object are shown. They make the angles θ_1, θ_2, θ_3, ... with respect to various fixed lines which do not rotate. As the object rotates, each of these angles increases by the same amount.

[1] The word axis is obviously related to the word axle. More generally the word axis means 'line'. For example the x and y axes are generally not axles about which anything rotates, they are just lines.

example we could take the left figure in fig. 16.3 as the reference configuration. If possible it's usually best to pick the reference state to be one in which a prominent feature of the object is aligned with the x or y axes. The reference state may or may not be the start of the motion of interest. Even if not, we measure an object's rotation by the change, relative to the reference state, in the counterclockwise angle θ of a reference line marked in the object relative to a fixed line outside. Which reference line? Fortunately,

> *All real or imagined lines marked on a rotating rigid object rotate by the same angle, the* rotation angle*, θ. (See box 16.1).*

In three dimensions things are more complicated. General rotation of a rigid object is then represented not with a single angle θ, but rather with 3 angles, or with a unit vector and an angle, or a 3×3 matrix. So we wait to discuss which of the 2D ideas here generalize to 3D and which do not.

16.1 Rotation is uniquely defined for a rigid object (2D)

Most people will find it self-evident that, starting with a rigid object at a reference orientation, all lines marked on the object rotate by the same angle θ. Here, for the doubting, we demonstrate this fact.

A rigid object is defined this way:

> For every pair of material points A and B on a rigid object the distance |AB| between them does not change as the object moves.

In particular, when a rigid object rotates all distances between all pairs of points are preserved. Thus, by the "side-side-side" similar triangle theorem of elementary geometry, all relative angles between marked line segments are preserved by the rotation. For example, for a triangle ABC the angle at B is constant as the object rotates. Now consider any pair of line segments on the object.

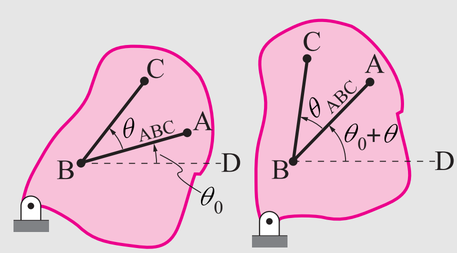
[By 'on' the object we don't mean a projected image drawn with a light pen that can move around on the object relative to the atoms. Rather, by 'on the object' we mean something defined by a particular set of atoms that make up the object.]

If the segments do not cross we can extend them to a point of intersection B. Such a pair of intersecting lines is shown here before and after rotation.Initially BA makes an angle θ_0 with a horizontal reference line. BC then makes an angle of $\theta_{ABC} + \theta_0$. After rotation we measure the angle to the line BD. BA now makes an angle of $\theta_0 + \theta$. By the addition of angles in the rotated configuration line BC now makes an angle of $\theta_{ABC} + \theta_0 + \theta$ which makes an increase by θ of the angle made by BC with the horizontal reference line. So both BA and BC rotate by the same angle θ.

We could use one of these two lines and compare it with an arbitrary third line through B and show that the third line also has equal rotation, and then a fourth, and so on. So all lines on the object through a point B rotate by the same angle θ. The demonstration for a pair of parallel lines, one of them through B, is easy, they stay parallel so always make a common angle with any reference line.

Because any line on the object either goes through B or is parallel to a line through B, all lines marked on a rigid object rotate by the same angle θ.

> The rotation of a rigid object in 2D is thus unambiguously defined as *the* angle through which all lines on the object rotate.



Rotated coordinates and base vectors $\hat{\imath}'$ and $\hat{\jmath}'$

We pick two orthogonal lines on the rotating object and give them distinguished status as *object-fixed* (or body-fixed) rotating coordinate axes x' and y'. Think of these axes as $x'y'$ coordinate axes on a piece of graph paper that is glued to the object (see fig. 16.5a&b). It's easiest if we start by assuming that the $x'y'$ axes have the same origin 0 as the xy axes and are parallel with the fixed xy axes when the object is in the reference configuration (when $\theta = 0$).

These rotating coordinate axes, x' and y', have associated rotating base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ (fig. 16.6 and 16.7). So $\hat{\boldsymbol{\imath}}'$ is always in the x' direction and $\hat{\boldsymbol{\jmath}}'$ always in the y' direction. We will use these rotating coordinates and base vectors to keep track of some particle of interest P that is 'glued' to the object. To start, note that particle P which is glued to the object has x' and y' coordinates that don't change as the rotation progresses.

Example: A particle on the x' axis
If a particle P is fixed on the x'-axis at position $x' = 3\,\text{cm}$, then we have.

$$\vec{\boldsymbol{r}}_{\text{P}} = 3\,\text{cm}\,\hat{\boldsymbol{\imath}}'$$

for all time, even as the object rotates.

The position vector of a point P fixed to a rigid object hinged at O remains, as the rotation progresses,

$$\vec{\boldsymbol{r}}_{\text{P}} = x'\hat{\boldsymbol{\imath}}' + y'\hat{\boldsymbol{\jmath}}', \tag{16.1}$$

$$\tag{16.2}$$

with x' and y' both constant. These rotating coordinate system components, $[\vec{\boldsymbol{r}}]_{x'y'} = [x', y']$, are sometimes written as $[\vec{\boldsymbol{r}}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$.

You will see that much of the math for rotating $x'y'$ coordinates is reminiscent of that for polar coordinates. However, the spirit is a bit different. In polar coordinates the $\hat{\boldsymbol{e}}_R$ axis was picked to track a particular particle of interest. Here we pick axes that rotate with an extended object and use that one set of axes to track any and all particles of interest.

Note, even though neither x' nor y' change as θ changes, the point P they describe moves, in circles actually. How can the particle's position change if its coordinates don't change? Well, in eqn. (16.3) the change in position is represented by the base vectors changing as the object rotates. Thus we could write more explicitly that

$$\vec{\boldsymbol{r}}_{\text{P}} = x'\hat{\boldsymbol{\imath}}'(\theta) + y'\hat{\boldsymbol{\jmath}}'(\theta). \tag{16.3}$$

Here we show more explicitly that the base vectors $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$ depend on θ. Just like for polar base vectors (see eqn. (15.5) on page 727) we can express the rotating base vectors in terms of the fixed base vectors and θ.

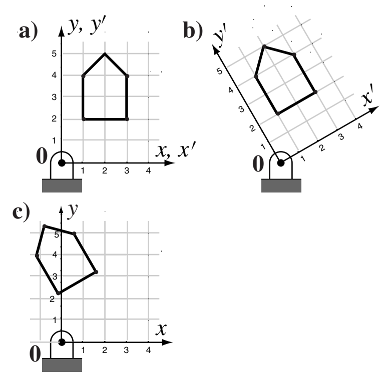


Figure 16.5: a) A house is drawn by connecting lines between 6 points, b) the house and coordinate system are rotated, thus its coordinates in the rotating system do not change c) But the coordinates in the original system do change

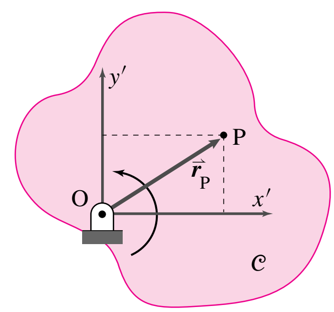


Figure 16.6: A rotating rigid object $\mathcal{C}$ with rotating coordinates $x'y'$ rigidly attached.

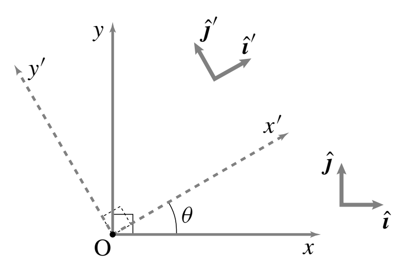


Figure 16.7: Fixed coordinate axes and rotating coordinate axes.

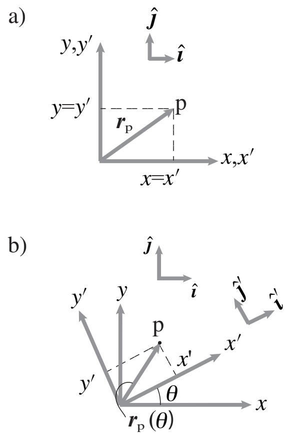


Figure 16.8: The x' and y' coordinates of a point fixed on a rotating object stay constant while the base vectors $\hat{\imath}'$ and $\hat{\jmath}'$ rotate with the object.

②Advanced aside. Sometimes a *reference frame* is defined as the set of all coordinate systems that could be attached to a rigid object. Two coordinate systems, even if rotated with respect to each other, then represent the same frame so long as they rotate together. Some of the results we will develop only depend on this more slack definition of frame, that the coordinates are glued to the object with no mind of their orientation in the reference configuration.

$$\begin{aligned} \hat{\imath}' &= \cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}, \\ \hat{\jmath}' &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \end{aligned} \tag{16.4}$$

Also we can express the fixed basis vectors in terms of the rotating vectors like this:

$$\begin{aligned} \hat{\imath} &= \cos\theta\,\hat{\imath}' - \sin\theta\,\hat{\jmath}' \\ \hat{\jmath} &= \sin\theta\,\hat{\imath}' + \cos\theta\,\hat{\jmath}'. \end{aligned} \tag{16.5}$$

Please review the section on dot products, 1.2, to see one derivation of these formulae.

We will use the phrase *reference frame* or just *frame* to mean "a coordinate system attached to a rigid object". One can think of the coordinate grid as like an invisible metal framework (hence the word 'frame') that rotates with the object. We refer to a calculation based on the rotating coordinates in fig. 16.6 variously as "in the frame $\mathcal{C}$" or "using the $x'y'$ frame" or "in the $\hat{\imath}'\,\hat{\jmath}'$ frame②.

In computer calculations we usually manipulate lists and arrays of numbers and not geometric vectors. So on a computer we keep track of vectors by keeping track of their lists of components. Let's look at a point fixed to the object and whose coordinates we know in the reference configuration:

$$\left[\vec{r}_P^{\,\text{ref}}\right]_{xy} = \begin{bmatrix} x^{\text{ref}} \\ y^{\text{ref}} \end{bmatrix}.$$

Assuming the object axes and fixed axes coincide in the reference configuration, the object coordinates of a point $[\vec{r}_P]_{x'y'}$ are equal to the space fixed coordinates of the point in the reference configuration $\left[\vec{r}_P^{\,\text{ref}}\right]_{xy}$. We can think of the point as defined either way, so

$$[\vec{r}_P]_{x'y'} = \left[\vec{r}_P^{\,\text{ref}}\right]_{xy}.$$

The rotation matrix $[R]$

Here is a question we often need to answer: What are the fixed basis coordinates of a point that has the rotating-frame coordinates $[\vec{r}]_{x'y'} = \begin{bmatrix} x' \\ y' \end{bmatrix}$? Here is one way to find the answer:

$$\begin{aligned} \vec{r}_P &= x'\hat{\imath}' + y'\hat{\jmath}' \\ &= x'(\cos\theta\,\hat{\imath} + \sin\theta\,\hat{\jmath}) + y'(-\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}) \\ &= \underbrace{\big((\cos\theta)x' - (\sin\theta)y'\big)}_{x}\,\hat{\imath} + \underbrace{\big((\sin\theta)x' + (\cos\theta)y'\big)}_{y}\,\hat{\jmath} \end{aligned} \tag{16.6}$$

so we can pull out the x and y coordinates compactly as,

$$[\vec{\boldsymbol{r}}_{\mathrm{P}}]_{xy} = \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \cos\theta\, x' + \sin\theta(-y') \\ \sin\theta\, x' + \cos\theta(y') \end{bmatrix}. \tag{16.7}$$

But this can, in turn be written in matrix notation as

$$\begin{aligned} \begin{bmatrix} x \\ y \end{bmatrix} &= \underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{[R]} \begin{bmatrix} x' \\ y' \end{bmatrix}, \quad \text{or} \\ [\vec{\boldsymbol{r}}_{\mathrm{P}}]_{xy} &= [R]\,[\vec{\boldsymbol{r}}_{\mathrm{P}}]_{x'y'}, \quad \text{or} \\ [\vec{\boldsymbol{r}}_{\mathrm{P}}]_{xy} &= [R]\left[\vec{\boldsymbol{r}}_{\mathrm{P}}^{\,\mathrm{ref}}\right]_{xy}. \end{aligned} \tag{16.8}$$

The matrix $[R]$ or $[R(\theta)]$ is the *rotation matrix* for counterclockwise rotations by θ. As shown above, if you know the coordinates of a point fixed on an object before rotation, you can find its coordinates after rotation by multiplying the coordinate column vector by the matrix $[R]$. You can remember what $[R]$ is by remembering its components or by remembering that

the first and second column of $[R]$ are the components of $\hat{\boldsymbol{\imath}}'$ and $\hat{\boldsymbol{\jmath}}'$, respectively, in the fixed coordinate system.

For example, the first column of $[R]$ consists of the x and y components of $\hat{\boldsymbol{\imath}}'$. A feature of eqn. (16.8) is that the same matrix $[R]$ prescribes the coordinate change for every different point on the object. Thus for points called 1, 2 and 3 we have

$$\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} = [R]\begin{bmatrix} x_1' \\ y_1' \end{bmatrix}, \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} = [R]\begin{bmatrix} x_2' \\ y_2' \end{bmatrix} \quad \& \quad \begin{bmatrix} x_3 \\ y_3 \end{bmatrix} = [R]\begin{bmatrix} x_3' \\ y_3' \end{bmatrix}.$$

A more compact way to write a matrix times a list of column vectors is to arrange the column vectors one next to the other in a matrix. By multiplying this matrix by $[R]$ we get a new matrix whose columns are the new coordinates of various points. For example,

$$\begin{bmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} = [R]\begin{bmatrix} x_1' & x_2' & x_3' \\ y_1' & y_2' & y_3' \end{bmatrix}. \tag{16.9}$$

Eqn. 16.9 is useful for computer animation of rotating things in video games (and in dynamics simulations too) where points 1,2, and 3 are points on an object.

Example: Rotate a picture

If a simple picture of a house is drawn by connecting the six points (fig. 16.5a) with the first point at $(x, y) = (1, 2)$, the second at $(x, y) = (3, 2)$, etc., and the sixth point on top of the first, we have,

$$[xy \text{ points BEFORE}] \equiv \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix}.$$

After a 30° counter-clockwise rotation about O, the coordinates of the house, in a coordinate system that rotates with the house, are unchanged (fig. 16.5b). But in the fixed (non-rotating, Newtonian) coordinate system the new coordinates of the rotated house points are,

$$\begin{aligned}
[xy \text{ points AFTER}] &= [R]\,[xy \text{ points BEFORE}] = [R]\,[x'y' \text{ points}] \\
&= \begin{bmatrix} \sqrt{3}/2 & -.5 \\ .5 & \sqrt{3}/2 \end{bmatrix} \begin{bmatrix} 1 & 3 & 3 & 2 & 1 & 1 \\ 2 & 2 & 4 & 5 & 4 & 2 \end{bmatrix} \\
&\approx \begin{bmatrix} -0.1 & 1.6 & 0.6 & -0.8 & -1.1 & -0.1 \\ 2.2 & 3.2 & 5.0 & 5.3 & 4.0 & 2.2 \end{bmatrix}
\end{aligned}$$

as shown in fig. 16.5c.