15.2 Dynamics of a particle in circular motion

The simplest examples of circular motion concern the motion of a particle constrained by a massless connection to be a fixed distance from a support point.

Example: Rock spinning on a string
Neglecting gravity, we can now deal with the familiar problem of a point mass being held in constant circular-rate motion by a massless string or rod. Linear momentum balance for the mass gives:

$$\begin{aligned}
\sum \vec{F}_i &= \dot{\vec{L}} \\
\Rightarrow -T\hat{e}_R &= m\vec{a} \\
\left\{-T\hat{e}_R\right. &= \left. m(-\dot{\theta}^2 \ell \hat{e}_R)\right\} \\
\{\}\cdot\hat{e}_R \Rightarrow \qquad & T = \dot{\theta}^2\ell m = (v^2/\ell)m
\end{aligned}$$

The force required to keep a mass in constant rate circular motion is mv^2/ℓ (sometimes remembered as mv^2/R).

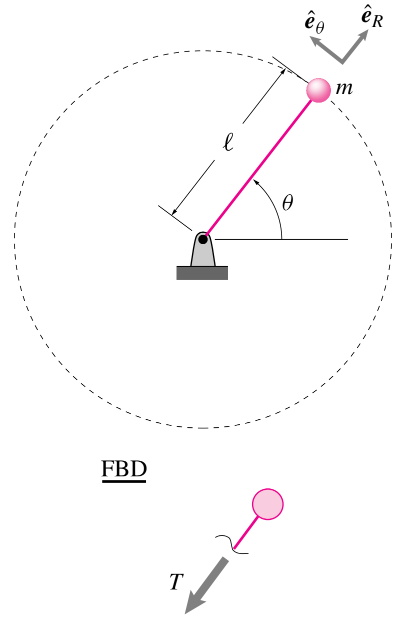

Figure 15.17: Point mass spinning in circles. Sketch of system and a free-body diagram.

The simplest example of ‘celestial mechanics’ is also circular motion.

Example: Geosynchronous orbit
Assuming a spherical earth, the centrally acting force of earth’s gravity on a satellite is mg at the earth’s surface and decays with radius squared so is

$$F = mg\frac{R_e^2}{r^2}$$

where R_e is the radius of the earth and r is the distance of the satellite from the center of the earth. Linear momentum balance for the mass gives:

$$\begin{aligned}
\sum \vec{F}_i &= \dot{\vec{L}} \\
\Rightarrow -mg\frac{R_e^2}{r^2}\hat{e}_R &= m\vec{a} \\
\left\{-mg\frac{R_e^2}{r^2}\hat{e}_R\right. &= \left. m(-\dot{\theta}^2 r\hat{e}_R)\right\} \\
\{\}\cdot\hat{e}_R \Rightarrow \qquad & r = \left(\frac{gR_e^2}{\dot{\theta}^2}\right)^{1/3}.
\end{aligned}$$

Communication satellites in ‘geosynchronous’ orbits go around once a day (staying in the sights of millions of satellite dishes). So, using $g \approx 10\,\mathrm{m/s^2}$, $R_e \approx 6400\,\mathrm{km}$ and $\dot{\theta} \approx 1$ rev/day, we get $r = 42600\,\mathrm{km}$[1].

[1] There are various errors (approximations) in this satellite calculation, of course. The earth doesn’t rotate once per day, but a little more because it goes around once per day relative to a line connecting the earth and sun and that line is itself rotating relative to the ‘fixed stars’ (the period of a geosynchronous satellite is one part in 365.25 shorter than a day). And the force of gravity on a near-earth mass is a bit more than mg because ‘g’ actually measures the force it takes to hold up a mass on the earth’s surface, which is the gravity force less the acceleration from going in circles on the surface of the earth (the actual acceleration of gravity at the earth’s surface is about 0.5% more than g). And the earth isn’t exactly spherical, and so on. The actual geosynchronous radius is close to 42164 km. So the example calculation is off by 436 kmor about 1%.

Similar calculations can find the motion of low altitude satellites, the motion of the moon around the earth and of the earth around the sun.

Centripetal and centrifugal forces

Because the centrally directed part of a particle’s acceleration is called the ‘centripetal’ acceleration, the centrally directed force needed to keep a particle in circular motion is sometimes called the ‘centripetal’ force. Thus, in the first example above the tension in the string is a centripetal force, and in the satellite problem the gravity force is a centripetal force. On the other hand, the 'centrifugal' force outwards is not really a force at all and is best dropped as a concept, at least for beginners.

Non-constant rate circular motion

Situations in which the circular rate is not constant are just slightly more complex. In these cases the part of the acceleration tangent to the circular motion is non-zero,

$$a_\theta = \ddot{\theta} r = \dot{v}_\theta,$$

so the net force on the particle has a component tangent to the circle.

Linear momentum balance in polar coordinates

The equation of linear momentum balance for a particle $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ in polar coordinates can be written as follows:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ \sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta &= m\left(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta\right). \qquad (15.11) \\ & \qquad (15.12) \end{aligned}$$

For circular motion we have, from Section 15.1, that

$$a_r = -\dot{\theta}^2 r = -\frac{v^2}{r} \qquad \text{and} \qquad a_\theta = \ddot{\theta} r = \dot{v}.$$

Angular momentum

In general one can use angular momentum balance with respect to any point you like. But for circular motion with the circle center at 0 one is almost always concerned with angular momentum balance about 0. In this case the various torque and angular momentum expressions are particularly simple, for example

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{/0} &= \dot{\vec{\boldsymbol{H}}}_{/0} \\ \Rightarrow \quad \vec{\boldsymbol{r}} \times \left(\sum \vec{\boldsymbol{F}}\right) &= \vec{\boldsymbol{r}} \times (m\vec{\boldsymbol{a}}) \\ \Rightarrow \quad r\hat{\boldsymbol{e}}_R \times \left(\sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta\right) &= r\hat{\boldsymbol{e}}_R \times m(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta) \\ \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_R = \vec{\boldsymbol{0}}, \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}} \quad \Rightarrow \quad \left\{ r \sum F_\theta \hat{\boldsymbol{k}} \right. &= \left. r m a_\theta \hat{\boldsymbol{k}} \right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad \Rightarrow \quad r \sum F_\theta &= r m a_\theta. \qquad (15.13) \end{aligned}$$

Energy

Kinetic energy is also particularly simple in polar coordinates for circular motion because there is only one degree of freedom:

$$\begin{aligned} E_{\mathrm{K}} &= \frac{mv^2}{2} = \frac{mv_\theta^2}{2} \\ &= m\dot{\theta}^2 r^2 / 2. \qquad (15.14) \end{aligned}$$

The simple pendulum

Perhaps the most famous mechanics example of circular motion at non-constant rate is a simple pendulum. As a child's swing, the inside of a grandfather clock, a hypnotist's device, or a gallows, the motion of a simple pendulum is a clear image to all of us. Galileo studied the simple pendulum before Newton created Newton's laws, and the pendulum is a core topic in high-school and freshman physics.

For starters, we consider a 2-D pendulum of fixed length with no forcing other than gravity. All mass is concentrated at a point. Of primary interest is the motion of the pendulum and the tension in the string. First we find governing differential equations (the equations of motion).

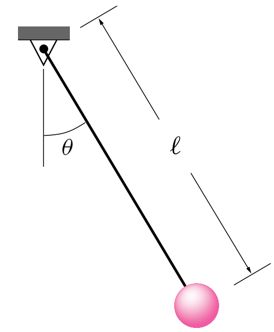

Figure 15.18: The ideal simple pendulum.

First, the tension in the pendulum rod (or string) acts along the length because the rod is a massless two-force body. At least that is the idealization. For any real pendulum, where the rod is not precisely massless and where the mass is not precisely concentrated at a point, there is a small force transmitted that is not along the rod. We neglect this 'shear' force in the treatment of the ideal pendulum. One way to get the equation of motion is to use linear momentum balance in polar coordinates, eqn. (15.12), and dot both sides with $\hat{\boldsymbol{e}}_\theta$ to get

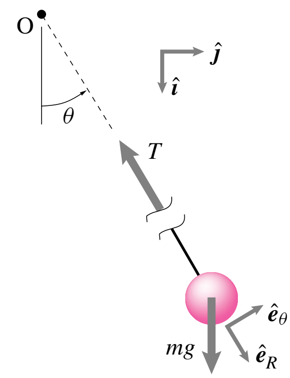

Figure 15.19: Free-body diagram of the mass in a simple pendulum.

$$
\begin{aligned}
-T\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0}+mg\underbrace{\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta}_{-\sin\theta} &= m\left(\ell\ddot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{e}}_\theta}_{1}-\ell\dot{\theta}^2\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0}\right)\\
\Rightarrow -mg\sin\theta &= m\ell\ddot{\theta}\\
\text{so}\qquad \ddot{\theta} &= -\frac{g}{\ell}\sin\theta .
\end{aligned}
$$

Small angle approximation (linearization)

For small angles, $\sin\theta \approx \theta$, so we have

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

for small oscillations. This equation describes a harmonic oscillator with $\sqrt{\frac{g}{\ell}}$ replacing the $\sqrt{\frac{k}{m}}$ coefficient in a spring-mass system. Thus the general solution is

$$\theta = A\cos\sqrt{g/l}t + B\sin\sqrt{g/l}t \tag{15.15}$$

where $A = \theta_0$ and $B\sqrt{g/l} = \dot{\theta}_0$. This solution has the famous property, Galileo loved this, that the frequency is the same for big as for small oscillations. Thus, a pendulum of a given length that swings back and forth 1 degree makes about the same number of swings per minute as one that swings with an amplitude of 10 degrees. How big is the error in this constant frequency result? Well, something less than the error in the approximation that $\sin\theta = \theta$.

$$\%\ \text{error} = 100 \cdot \frac{\theta - \sin\theta}{\sin\theta} \approx 100 \cdot \frac{\theta^3/3}{\theta} \approx \frac{\theta^2}{3} \approx 1\%$$

for $\theta = 10° \approx 1/6\,\text{rad}$. The actual error in the period is less than this, as you can find by numerically solving the non-linear pendulum equation.

The inverted pendulum

A pendulum with the mass-end up is called an inverted pendulum. By methods just like we used for the regular pendulum, we find the equation of motion to be

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

which, for small θ, is well approximated by

$$\ddot{\theta} = \frac{g}{\ell}\theta.$$

As opposed to the simple pendulum, which has oscillatory solutions, this differential equation has exponential solutions

$$\theta = C_1 e^{\sqrt{g/\ell}\ t} + C_2 e^{-\sqrt{g/\ell}\ t},$$

one term of which has exponential growth (the implicit "+" in front of the argument of the $\sqrt{g/\ell}\ t$), indicating the inherent *instability* of the inverted pendulum. That is, as is intuitively obvious, an inverted pendulum has tendency to fall over when slightly disturbed from the vertical position②.

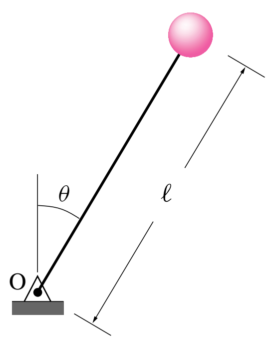

Figure 15.20: The inverted pendulum

②After the pendulum falls a ways, say past 30 degrees from vertical, the exponential solution is not an accurate description, but the actual motion (as viewed by an experiment, a computer simulation, or the exact elliptic integral solution of the equations) shows that the pendulum keeps falling.

More about pendula

Pendula are useful as models of many phenomena from the swing of a leg in walking to the tipping of a chimney in an earthquake. Pendula also serve as a simple example for many more general concepts in mechanics. For example, the pendulum is popular as an example of "chaos"; if you push a pendulum periodically its motions can be wild.

15.3 Other derivations of the pendulum equation

The simplest derivation of the pendulum differential equation is to use linear momentum balance in polar coordinates. Here are two other derivations.

Method one: linear momentum balance in cartesian coordinates

The equation of linear momentum balance is

$$\sum \vec{F} = \overbrace{\dot{\vec{L}}}^{m\vec{a}}$$

Evaluating the left side (using the free-body diagram) and right side (using the kinematics of circular motion), we get

$$-T\hat{e}_R + mg\hat{\imath} = m[\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R] \tag{15.16}$$

From the picture (or recalling) we see that $\hat{e}_R = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = \cos\theta\hat{\jmath} - \sin\theta\hat{\imath}$. So, upon substitution into the equation above, we get

$$\begin{aligned} &-T(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}) + mg\hat{\imath} \\ &\quad = m\left[\ell\ddot{\theta}(\cos\theta\hat{\jmath} - \sin\theta\hat{\imath}) - \ell\dot{\theta}^2(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})\right] \end{aligned}$$

Breaking this equation into its x and y components (by dotting both sides with $\hat{\imath}$ and $\hat{\jmath}$, respectively) gives

$$\begin{aligned} -T\cos\theta + mg &= -m\ell\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \text{ and} \\ -T\sin\theta &= m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \end{aligned} \tag{15.17}$$

Note, when deriving equations of motion, we think of both positions and the rates and velocities as knowns. For example, we take θ and $\dot{\theta}$ as known. But how do we know them? We don't. But thinking of them as known helps us write a set of differential equations from which we can eventually find them. Thus the equations above are two simultaneous equations that we can solve for the two unknowns T and $\ddot{\theta}$ to get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{15.18}$$

$$T = m[\ell\dot{\theta}^2 + g\cos\theta]. \tag{15.19}$$

The first equation is the familiar pendulum differential equation, the second allows us to find the tension in the pendulum string.

Method two: angular momentum balance

Using angular momentum balance, we can 'kill' (eliminate) the tension term at the start. Taking angular momentum balance about the point O, we get

$$\begin{aligned} \sum \vec{M}_O &= \dot{\vec{H}}_{/O} \\ -mg\ell\sin\theta\hat{k} &= \underbrace{\vec{r}_{/O}}_{\ell\hat{e}_R} \times \underbrace{\vec{a}}_{\ell\ddot{\theta}\hat{e}_\theta - \ell\dot{\theta}^2\hat{e}_R}\, m \\ -mg\ell\sin\theta\hat{k} &= m\ell^2\ddot{\theta}\hat{k} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

since $\hat{e}_R \times \hat{e}_R = 0$ and $\hat{e}_R \times \hat{e}_\theta = \hat{k}$. So, the governing equation for a simple pendulum is

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

Method three: Conservation of energy

The string tension is always orthogonal to the velocity so does no work. The gravity force is conservative. So energy is conserved.

$$\begin{aligned} \textit{constant} &= E_T \\ \Rightarrow \quad \textit{constant} &= E_K + E_P \\ \Rightarrow \quad 0 &= \dot{E}_K + \dot{E}_P \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}(mgh) \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}m(\ell\dot{\theta})^2\right) + \frac{d}{dt}(-mg\ell\cos\theta) \\ \Rightarrow \quad 0 &= m\ell^2\ddot{\theta}\dot{\theta} + mg\ell(\sin\theta)\dot{\theta} \end{aligned}$$

Now m cancels from both sides and we can divide through by ℓ^2. We can also divide through by $\dot{\theta}$, but for exceptional instants in time when $\dot{\theta} = 0$. Thus

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

which is the familiar differential equation for a pendulum. This method lacks some rigor in that the cancelation of $\dot{\theta}$ is not valid at exactly every instant in time. However, it is valid for all but those instants, and happens to give the right answer at the exceptional instants as well.