15.2 Dynamics of a particle in circular motion

The simplest examples of circular motion concern the motion of a particle constrained by a massless connection to be a fixed distance from a support point.

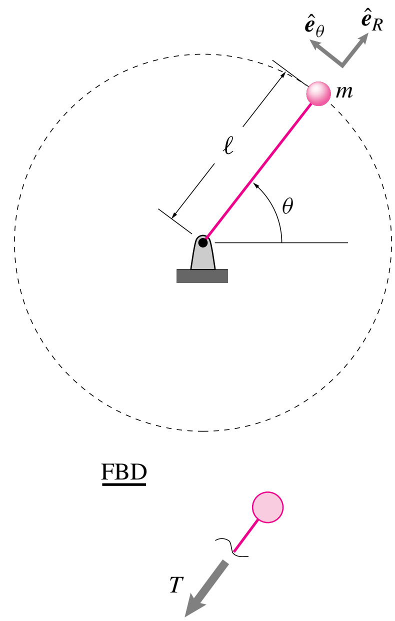


Figure 15.17: Point mass spinning in circles. Sketch of system and a free-body diagram.

Example: Rock spinning on a string
Neglecting gravity, we can now deal with the familiar problem of a point mass being held in constant circular-rate motion by a massless string or rod. Linear momentum balance for the mass gives:

$$\begin{aligned}
\sum \vec{\boldsymbol{F}}_i &= \dot{\vec{\boldsymbol{L}}} \\
\Rightarrow -T\hat{\boldsymbol{e}}_R &= m\vec{\boldsymbol{a}} \\
\{-T\hat{\boldsymbol{e}}_R &= m(-\dot{\theta}^2\ell\hat{\boldsymbol{e}}_R)\} \\
\{\}\cdot\hat{\boldsymbol{e}}_R \Rightarrow \qquad & T = \dot{\theta}^2\ell m = (v^2/\ell)m
\end{aligned}$$

The force required to keep a mass in constant rate circular motion is mv^2/ℓ (sometimes remembered as mv^2/R).

The simplest example of 'celestial mechanics' is also circular motion.

Example: Geosynchronous orbit
Assuming a spherical earth, the centrally acting force of earth's gravity on a satellite is mg at the earth's surface and decays with radius squared so is

$$F = mg\frac{R_e^2}{r^2}$$

where R_e is the radius of the earth and r is the distance of the satellite from the center of the earth. Linear momentum balance for the mass gives:

$$\begin{aligned}
\sum \vec{\boldsymbol{F}}_i &= \dot{\vec{\boldsymbol{L}}} \\
\Rightarrow -mg\frac{R_e^2}{r^2}\hat{\boldsymbol{e}}_R &= m\vec{\boldsymbol{a}} \\
\left\{-mg\frac{R_e^2}{r^2}\hat{\boldsymbol{e}}_R\right. &= m(-\dot{\theta}^2 r\hat{\boldsymbol{e}}_R)\} \\
\{\}\cdot\hat{\boldsymbol{e}}_R \Rightarrow \qquad & r = \left(\frac{gR_e^2}{\dot{\theta}^2}\right)^{1/3}.
\end{aligned}$$

Communication satellites in 'geosynchronous' orbits go around once a day (staying in the sights of millions of satellite dishes). So, using $g \approx 10\,\mathrm{m/s^2}$, $R_e \approx 6400\,\mathrm{km}$ and $\dot{\theta} \approx 1$ rev/day, we get $r = 42600\,\mathrm{km}$①.

①There are various errors (approximations) in this satellite calculation, of course. The earth doesn't rotate once per day, but a little more because it goes around once per day relative to a line connecting the earth and sun and that line is itself rotating relative to the 'fixed stars' (the period of a geosynchronous satellite is one part in 365.25 shorter than a day). And the force of gravity on a near-earth mass is a bit more than mg because 'g' actually measures the force it takes to hold up a mass on the earth's surface, which is the gravity force less the acceleration from going in circles on the surface of the earth (the actual acceleration of gravity at the earth's surface is about 0.5% more than g). And the earth isn't exactly spherical, and so on. The actual geosynchronous radius is close to 42164 km. So the example calculation is off by 436 kmor about 1%.

Similar calculations can find the motion of low altitude satellites, the motion of the moon around the earth and of the earth around the sun.

Centripetal and centrifugal forces

Because the centrally directed part of a particle's acceleration is called the 'centripetal' acceleration, the centrally directed force needed to keep a particle in circular motion is sometimes called the 'centripetal' force. Thus, in the first example above the tension in the string is a centripetal force, and in the satellite problem the gravity force is a centripetal force. On the other hand, the 'centrifugal' force outwards is not really a force at all and is best dropped as a concept, at least for beginners.

Non-constant rate circular motion

Situations in which the circular rate is not constant are just slightly more complex. In these cases the part of the acceleration tangent to the circular motion is non-zero,

$$a_\theta = \ddot{\theta} r = \dot{v}_\theta,$$

so the net force on the particle has a component tangent to the circle.

Linear momentum balance in polar coordinates

The equation of linear momentum balance for a particle $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ in polar coordinates can be written as follows:

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ \sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta &= m\left(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta\right). \qquad (15.11) \\ & \qquad (15.12) \end{aligned}$$

For circular motion we have, from Section 15.1, that

$$a_r = -\dot{\theta}^2 r = -\frac{v^2}{r} \qquad \text{and} \qquad a_\theta = \ddot{\theta} r = \dot{v}.$$

Angular momentum

In general one can use angular momentum balance with respect to any point you like. But for circular motion with the circle center at 0 one is almost always concerned with angular momentum balance about 0. In this case the various torque and angular momentum expressions are particularly simple, for example

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_{/0} &= \dot{\vec{\boldsymbol{H}}}_{/0} \\ \Rightarrow \quad \vec{\boldsymbol{r}} \times \left(\sum \vec{\boldsymbol{F}}\right) &= \vec{\boldsymbol{r}} \times (m\vec{\boldsymbol{a}}) \\ \Rightarrow \quad r\hat{\boldsymbol{e}}_R \times \left(\sum F_r \hat{\boldsymbol{e}}_R + \sum F_\theta \hat{\boldsymbol{e}}_\theta\right) &= r\hat{\boldsymbol{e}}_R \times m(a_r \hat{\boldsymbol{e}}_R + a_\theta \hat{\boldsymbol{e}}_\theta) \\ \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_R = \vec{\boldsymbol{0}}, \hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}} \quad \Rightarrow \quad \left\{ r \sum F_\theta \hat{\boldsymbol{k}} \right. &= \left. rma_\theta \hat{\boldsymbol{k}} \right\} \\ \{\} \cdot \hat{\boldsymbol{k}} \quad \Rightarrow \quad r \sum F_\theta &= rma_\theta. \qquad (15.13) \end{aligned}$$

Energy

Kinetic energy is also particularly simple in polar coordinates for circular motion because there is only one degree of freedom:

$$\begin{aligned} E_\mathrm{K} &= \frac{mv^2}{2} = \frac{mv_\theta^2}{2} \\ &= m\dot{\theta}^2 r^2/2. \qquad (15.14) \end{aligned}$$

The simple pendulum

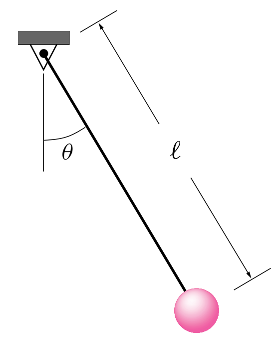


Figure 15.18: The ideal simple pendulum.

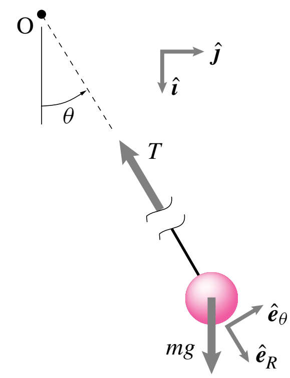


Figure 15.19: Free-body diagram of the mass in a simple pendulum.

Perhaps the most famous mechanics example of circular motion at non-constant rate is a simple pendulum. As a child's swing, the inside of a grandfather clock, a hypnotist's device, or a gallows, the motion of a simple pendulum is a clear image to all of us. Galileo studied the simple pendulum before Newton created Newton's laws, and the pendulum is a core topic in high-school and freshman physics.

For starters, we consider a 2-D pendulum of fixed length with no forcing other than gravity. All mass is concentrated at a point. Of primary interest is the motion of the pendulum and the tension in the string. First we find governing differential equations (the equations of motion).

First, the tension in the pendulum rod (or string) acts along the length because the rod is a massless two-force body. At least that is the idealization. For any real pendulum, where the rod is not precisely massless and where the mass is not precisely concentrated at a point, there is a small force transmitted that is not along the rod. We neglect this 'shear' force in the treatment of the ideal pendulum. One way to get the equation of motion is to use linear momentum balance in polar coordinates, eqn. (15.12), and dot both sides with $\hat{\boldsymbol{e}}_\theta$ to get

$$-T\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0}+mg\underbrace{\hat{\boldsymbol{\imath}}\cdot\hat{\boldsymbol{e}}_\theta}_{-\sin\theta}=m\left(\ell\ddot{\theta}\underbrace{\hat{\boldsymbol{e}}_\theta\cdot\hat{\boldsymbol{e}}_\theta}_{1}-\ell\dot{\theta}^2\underbrace{\hat{\boldsymbol{e}}_R\cdot\hat{\boldsymbol{e}}_\theta}_{0}\right)$$

$$\Rightarrow -mg\sin\theta = m\ell\ddot{\theta}$$

$$\text{so}\qquad \ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

Small angle approximation (linearization)

For small angles, $\sin\theta \approx \theta$, so we have

$$\ddot{\theta} = -\frac{g}{\ell}\theta$$

for small oscillations. This equation describes a harmonic oscillator with $\sqrt{\frac{g}{\ell}}$ replacing the $\sqrt{\frac{k}{m}}$ coefficient in a spring-mass system. Thus the general solution is

$$\theta = A\cos\sqrt{g/l}t + B\sin\sqrt{g/l}t \tag{15.15}$$

where $A = \theta_0$ and $B\sqrt{g/l} = \dot{\theta}_0$. This solution has the famous property, Galileo loved this, that the frequency is the same for big as for small oscillations. Thus, a pendulum of a given length that swings back and forth 1 degree makes about the same number of swings per minute as one that swings with an amplitude of 10 degrees. How big is the error in this constant frequency result? Well, something less than the error in the approximation that $\sin\theta = \theta$.

$$\%\ \text{error} = 100 \cdot \frac{\theta - \sin\theta}{\sin\theta} \approx 100 \cdot \frac{\theta^3/3}{\theta} \approx \frac{\theta^2}{3} \approx 1\%$$

for $\theta = 10° \approx 1/6$ rad. The actual error in the period is less than this, as you can find by numerically solving the non-linear pendulum equation.

The inverted pendulum

A pendulum with the mass-end up is called an inverted pendulum. By methods just like we used for the regular pendulum, we find the equation of motion to be

$$\ddot{\theta} = \frac{g}{\ell}\sin\theta$$

which, for small θ, is well approximated by

$$\ddot{\theta} = \frac{g}{\ell}\theta.$$

As opposed to the simple pendulum, which has oscillatory solutions, this differential equation has exponential solutions

$$\theta = C_1 e^{\sqrt{g/\ell}\,t} + C_2 e^{-\sqrt{g/\ell}\,t},$$

one term of which has exponential growth (the implicit "+" in front of the argument of the $\sqrt{g/\ell}\,t$), indicating the inherent *instability* of the inverted pendulum. That is, as is intuitively obvious, an inverted pendulum has tendency to fall over when slightly disturbed from the vertical position②.

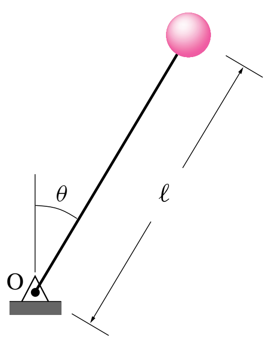


Figure 15.20: The inverted pendulum

②After the pendulum falls a ways, say past 30 degrees from vertical, the exponential solution is not an accurate description, but the actual motion (as viewed by an experiment, a computer simulation, or the exact elliptic integral solution of the equations) shows that the pendulum keeps falling.

More about pendula

Pendula are useful as models of many phenomena from the swing of a leg in walking to the tipping of a chimney in an earthquake. Pendula also serve as a simple example for many more general concepts in mechanics. For example, the pendulum is popular as an example of "chaos"; if you push a pendulum periodically its motions can be wild.

15.3 Other derivations of the pendulum equation

The simplest derivation of the pendulum differential equation is to use linear momentum balance in polar coordinates. Here are two other derivations.

Method one: linear momentum balance in cartesian coordinates

The equation of linear momentum balance is

$$\sum \vec{F} = \overbrace{\dot{\vec{L}}}^{m\vec{a}}$$

Evaluating the left side (using the free-body diagram) and right side (using the kinematics of circular motion), we get

$$-T\hat{\boldsymbol{e}}_R + mg\hat{\boldsymbol{\imath}} = m[\ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R] \tag{15.16}$$

From the picture (or recalling) we see that $\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{e}}_\theta = \cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}$. So, upon substitution into the equation above, we get

$$\begin{aligned} &-T(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) + mg\hat{\boldsymbol{\imath}} \\ &\qquad = m\left[\ell\ddot{\theta}(\cos\theta\hat{\boldsymbol{\jmath}} - \sin\theta\hat{\boldsymbol{\imath}}) - \ell\dot{\theta}^2(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}})\right] \end{aligned}$$

Breaking this equation into its x and y components (by dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, respectively) gives

$$\begin{aligned} -T\cos\theta + mg &= -m\ell\left(\ddot{\theta}\sin\theta + \dot{\theta}^2\cos\theta\right) \text{ and} \\ -T\sin\theta &= m\ell\left(\ddot{\theta}\cos\theta - \dot{\theta}^2\sin\theta\right). \end{aligned} \tag{15.17}$$

Note, when deriving equations of motion, we think of both positions and the rates and velocities as knowns. For example, we take θ and $\dot{\theta}$ as known. But how do we know them? We don't. But thinking of them as known helps us write a set of differential equations from which we can eventually find them. Thus the equations above are two simultaneous equations that we can solve for the two unknowns T and $\ddot{\theta}$ to get

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta \tag{15.18}$$

$$T = m[\ell\dot{\theta}^2 + g\cos\theta]. \tag{15.19}$$

The first equation is the familiar pendulum differential equation, the second allows us to find the tension in the pendulum string.

Method two: angular momentum balance

Using angular momentum balance, we can 'kill' (eliminate) the tension term at the start. Taking angular momentum balance about the point O, we get

$$\begin{aligned} \sum \vec{\boldsymbol{M}}_O &= \dot{\vec{\boldsymbol{H}}}_{/O} \\ -mg\ell\sin\theta\hat{\boldsymbol{k}} &= \underbrace{\vec{\boldsymbol{r}}_{/O}}_{\boxed{\ell\hat{\boldsymbol{e}}_R}} \times \underbrace{\vec{\boldsymbol{a}}}_{\boxed{\ell\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \ell\dot{\theta}^2\hat{\boldsymbol{e}}_R}} m \\ -mg\ell\sin\theta\hat{\boldsymbol{k}} &= m\ell^2\ddot{\theta}\hat{\boldsymbol{k}} \\ \Rightarrow \ddot{\theta} &= -\frac{g}{\ell}\sin\theta \end{aligned}$$

since $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_R = 0$ and $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$. So, the governing equation for a simple pendulum is

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

Method three: Conservation of energy

The string tension is always orthogonal to the velocity so does no work. The gravity force is conservative. So energy is conserved.

$$\begin{aligned} \textit{constant} &= E_T \\ \Rightarrow \quad \textit{constant} &= E_K + E_P \\ \Rightarrow \quad 0 &= \dot{E}_K + \dot{E}_P \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}mv^2\right) + \frac{d}{dt}(mgh) \\ \Rightarrow \quad 0 &= \frac{d}{dt}\left(\frac{1}{2}m(\ell\dot{\theta})^2\right) + \frac{d}{dt}(-mg\ell\cos\theta) \\ \Rightarrow \quad 0 &= m\ell^2\ddot{\theta}\dot{\theta} + mg\ell(\sin\theta)\dot{\theta} \end{aligned}$$

Now m cancels from both sides and we can divide through by ℓ^2. We can also divide through by $\dot{\theta}$, but for exceptional instants in time when $\dot{\theta} = 0$. Thus

$$\ddot{\theta} + \frac{g}{\ell}\sin\theta = 0$$

which is the familiar differential equation for a pendulum. This method lacks some rigor in that the cancelation of $\dot{\theta}$ is not valid at exactly every instant in time. However, it is valid for all but those instants, and happens to give the right answer at the exceptional instants as well.

SAMPLE 15.6 **Circular motion in 2-D.** Two bars, each of negligible mass and length $\ell = 3\,\text{ft}$, are welded together at right angles to form an 'L' shaped structure. The structure supports a 3.2 lbf ($= mg$) ball at one end and is connected to a motor on the other end (see Fig. 15.21). The motor rotates the structure in the vertical plane at a constant rate $\dot{\theta} = 10\,\text{rad/s}$ in the counter-clockwise direction. Take $g = 32\,\text{ft/s}^2$. At the instant shown in Fig. 15.21, find

1. the velocity of the ball,
2. the acceleration of the ball, and
3. the net force and moment applied by the motor and the support at O on the structure.

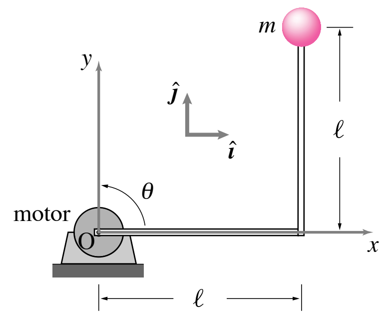


Figure 15.21: The motor rotates the structure at a constant angular speed in the counterclockwise direction.

Solution The motor rotates the structure at a constant rate. Therefore, the ball is going in circles with angular velocity $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}} = 10\,\text{rad/s}\hat{\boldsymbol{k}}$. The radius of the circle is $R = \sqrt{\ell^2 + \ell^2} = \ell\sqrt{2}$. Since the motion is in the xy plane, we use the following formulae to find the velocity $\vec{v}$ and acceleration $\vec{a}$.

$$\vec{v} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$$

$$\vec{a} = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta,$$

where $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are the polar basis vectors shown in Fig. 15.22. In Fig. 15.22, we note that $\theta = 45^\circ$. Therefore,

$$\begin{aligned}
\hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} \\
&= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \\
\hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \\
&= \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

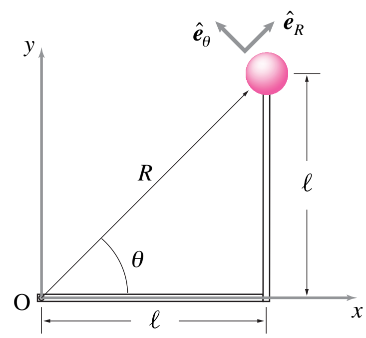


Figure 15.22: The ball follows a circular path of radius R. The position, velocity, and acceleration of the ball can be expressed in terms of the polar basis vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

Here, $R = L\sqrt{2} = 3\sqrt{2}\,\text{ft}$ is constant, and $\ddot{\theta} = 0$ because $\dot{\theta} = 10\,\text{rad/s} = \text{constant}$. Thus,

1. the velocity of the ball is

$$\begin{aligned}
\vec{v} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\
&= 3\sqrt{2}\,\text{ft}\cdot 10\,\text{rad/s}\hat{\boldsymbol{e}}_\theta \\
&= 30\sqrt{2}\,\text{ft/s}\cdot\frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
&= 30\,\text{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

$$\boxed{\vec{v} = 30\,\text{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

2. The acceleration of the ball is

$$\begin{aligned}
\vec{a} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R \\
&= -3\sqrt{2}\,\text{ft}\cdot(10\,\text{rad/s})^2\hat{\boldsymbol{e}}_R \\
&= -300\sqrt{2}\,\text{ft/s}^2\cdot\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
&= -300\,\text{ft/s}^2(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

$$\boxed{\vec{a} = -300\,\text{ft/s}^2(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

3. Let the net force and the moment applied by the motor-support system be $\vec{F}$ and $\vec{M}$ as shown in Fig. 15.23. From the linear momentum balance for the structure,

$$\begin{aligned}
\sum \vec{F} &= m\vec{a} \\
\vec{F} - mg\hat{\jmath} &= m\vec{a} \\
\Rightarrow \quad \vec{F} &= m\vec{a} + mg\hat{\jmath} \\
&= \overbrace{\frac{3.2\,\text{lbf}}{32\,\text{ft/s}^2}}^{m}(-300\sqrt{2}\,\text{ft/s}^2)\hat{e}_R + \overbrace{3.2\,\text{lbf}}^{mg}\,\hat{\jmath} \\
&= -30\sqrt{2}\,\text{lbf}\,\hat{e}_R + 3.2\,\text{lbf}\,\hat{\jmath}. \\
&= -30\sqrt{2}\,\text{lbf}\,\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) + 3.2\,\text{lbf}\,\hat{\jmath} \\
&= -30\,\text{lbf}\,\hat{\imath} - 26.8\,\text{lbf}\,\hat{\jmath}.
\end{aligned}$$

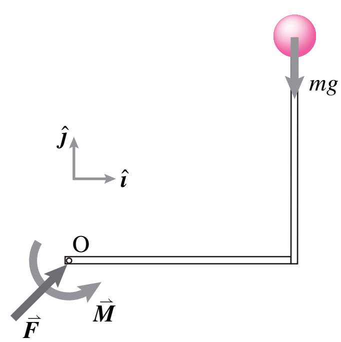


Figure 15.23: Free-body diagram of the structure.

Similarly, from the angular momentum balance for the structure,

$$\begin{aligned}
\sum \vec{M}_O &= \dot{\vec{H}}_{/O}, \\
\text{where} \quad \sum \vec{M}_O &= \vec{M} + \vec{r}_{/O} \times mg(-\hat{\jmath}) \\
&= \vec{M} + \underbrace{R\hat{e}_R}_{\ell(\hat{\imath}+\hat{\jmath})} \times mg(-\hat{\jmath}) \\
&= \vec{M} - mg\ell\hat{k}, \\
\text{and} \quad \dot{\vec{H}}_{/O} &= \vec{r}_{/O} \times m\vec{a} \\
&= R\hat{e}_R \times m(-R\dot{\theta}^2\hat{e}_R) \\
&= -mR^2\dot{\theta}^2 \underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}} \\
&= \vec{0}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\vec{M} &= mg\ell\hat{k} \\
&= \underbrace{3.2\,\text{lbf}}_{mg} \cdot \underbrace{3\,\text{ft}}_{\ell}\,\hat{k} \\
&= 9.6\,\text{lbf}\cdot\text{ft}\,\hat{k}.
\end{aligned}$$

$\vec{F} = -30\,\text{lbf}\,\hat{\imath} - 26.8\,\text{lbf}\,\hat{\jmath}, \quad \vec{M} = 9.6\,\text{lbf}\cdot\text{ft}\,\hat{k}$

Note: If there was no gravity, the moment applied by the motor would be zero.

SAMPLE 15.7 A 50 gm point mass executes circular motion with angular acceleration $\ddot{\theta} = 2\,\text{rad/s}^2$. The radius of the circular path is 200 mm. If the mass starts from rest at $t = 0$, find

1. Its angular momentum $\vec{\boldsymbol{H}}$ about the center at $t = 5$ s.
2. Its rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$ about the center.

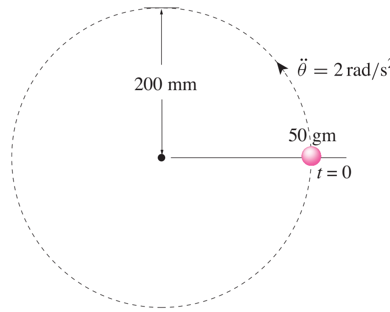


Figure 15.24

Solution

1. From the definition of angular momentum,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/O} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{v}} \\
&= R\hat{\boldsymbol{e}}_R \times m\dot{\theta}R\hat{\boldsymbol{e}}_\theta \\
&= mR^2\dot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\
&= mR^2\dot{\theta}\hat{\boldsymbol{k}}
\end{aligned}$$

On the right hand side of this equation, the only unknown is $\dot{\theta}$. Thus to find $\vec{\boldsymbol{H}}_{/O}$ at $t = 5$ s, we need to find $\dot{\theta}$ at $t = 5$ s. Now,

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\
d\dot{\theta} &= \ddot{\theta}\,dt \\
\int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t \ddot{\theta}\,dt \\
\dot{\theta}(t) - \dot{\theta}_0 &= \ddot{\theta}(t_t - t_0) \\
\dot{\theta} &= \dot{\theta}_0 + \ddot{\theta}(t - t_0)
\end{aligned}$$

Writing α for $\ddot{\theta}$ and substituting $t_0 = 0$ in the above expression, we get $\dot{\theta}(t) = \dot{\theta}_0 + \alpha t$, which is the angular speed version of the linear speed formula $v(t) = v_0 + at$. ① Substituting $t = 5$ s, $\dot{\theta}_0 = 0$, and $\alpha = 2\,\text{rad/s}^2$ we get $\dot{\theta} = 2\,\text{rad/s}^2 \cdot 5\,\text{s} = 10\,\text{rad/s}$. Therefore,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/O} &= 0.05\,\text{kg} \cdot (0.2\,\text{m})^2 \cdot 10\,\text{rad/s}\hat{\boldsymbol{k}} \\
&= 0.02\,\text{kg.}\,\text{m}^2/\text{s} = 0.02\,\text{N·m} \cdot \text{s.}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{/O} = 0.02\,\text{N·m} \cdot \text{s.}}$$

① Be warned that these formulae are valid only for constant rate of change of speed.

2. Similarly, we can calculate the rate of change of angular momentum:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/O} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{a}} \\
&= R\hat{\boldsymbol{e}}_R \times m(R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2R\hat{\boldsymbol{e}}_R) \\
&= mR^2\ddot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\
&= mR^2\ddot{\theta}\hat{\boldsymbol{k}} \\
&= 0.05\,\text{kg} \cdot (0.2m)^2 \cdot 2\,\text{rad/s}^2\hat{\boldsymbol{k}} \\
&= 0.004\,\text{kg} \cdot \text{m}^2/\text{s}^2 = 0.004\,\text{N·m}
\end{aligned}$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{/O} = 0.004\,\text{N·m}}$$

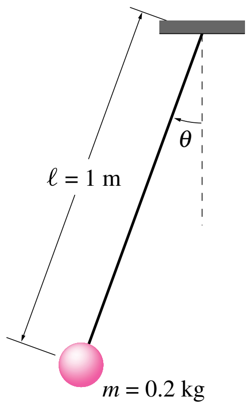


Figure 15.25

SAMPLE 15.8 The simple pendulum. A simple pendulum swings about its vertical equilibrium position (2-D motion) with amplitude $\theta_{max} = 10^\circ$. Find

1. the magnitude of the maximum angular acceleration,
2. the maximum tension in the string.

Solution

1. The equation of motion of the pendulum is given by (see eqn. (15.18) in the text):

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

We are given that $|\theta| \leq \theta_{max}$. For $\theta_{max} = 10^\circ = 0.1745\,\text{rad}$, $\sin\theta_{max} = 0.1736$. Thus we see that $\sin\theta \approx \theta$ even when θ is maximum. Therefore, we can safely use linear approximation (although we could solve this problem without it); i.e.,

$$\ddot{\theta} = -\frac{g}{\ell}\theta.$$

Clearly, $|\ddot{\theta}|$ is maximum when θ is maximum. Thus,

$$|\ddot{\theta}|_{max} = \frac{g}{\ell}\theta_{max} = \frac{9.81\,\text{m/s}^2}{1\,\text{m}}\cdot(0.1745\,\text{rad}) = 1.71\,\text{rad/s}^2.$$

$$\boxed{|\ddot{\theta}|_{max} = 1.71\,\text{rad/s}^2}$$

2. The tension in the string is given by (see equation 15.19 of text):

$$T = m(\ell\dot{\theta}^2 + g\cos\theta).$$

This time, we will not make the small angle assumption. We can find T_{max} and the corresponding θ using conservation of energy. Let the position of maximum amplitude be position 1 and the position at any θ be position 2. When $\theta = \theta_{max}$, the mass comes to rest and switches its direction of motion. Thus, its angular velocity and, hence, its kinetic energy is zero at θ_{max}. Using conservation of energy, we have

$$\begin{aligned} E_{K_1} + E_{P_1} &= E_{K_2} + E_{P_2} \\ 0 + mg\ell(1-\cos\theta_{max}) &= \frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1-\cos\theta). \end{aligned} \tag{15.20}$$

and solving for $\dot{\theta}$, we get,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_{max})}.$$

Therefore, the tension at any θ is

$$T(\theta) = m(\ell\dot{\theta}^2 + g\cos\theta) = mg(3\cos\theta - 2\cos\theta_{max}).$$

To find the maximum tension, we set $\frac{dT}{d\theta} = 0$, and find that, for $0 \leq \theta \leq \theta_{max}$, T is maximum when $\theta = 0$. Now, substituting $\theta = 0$ in $T(\theta)$, we get,

$$T_{max} = mg\big(3\cos(0) - 2\cos(\theta_{max})\big) = 0.2\,\text{kg}\cdot 9.81\,\text{m/s}^2(3-1.97) = 2.02\,\text{N}.$$

The maximum tension corresponds to maximum speed which occurs at the bottom of the swing where all of the potential energy is converted to kinetic energy.

$$\boxed{T_{max} = 2.02\,\text{N}}$$

SAMPLE 15.9 **The nonlinear pendulum:** Consider the simple pendulum of Sample 15.8 again. Let the mass be m and the length of the pendulum ℓ. The equation of motion of the pendulum is $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$ as derived in the text (see eqn. (15.18)). This is a nonlinear ordinary differential equation but it can be solved easily numerically. Write a computer code using some ODE solver to solve the equation. Take g and ℓ such that $\lambda = \sqrt{g/\ell} = 2\pi$ (this makes the time period of the pendulum $T = 2\pi/\lambda = 1\,\mathrm{s}$). Using the code, do the following calculations.

1. Solve the equation over a time interval of $t = 0$ to 4 seconds using the initial conditions $\theta(0) = 6°$ and $\dot{\theta}(0) = 0$, and plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ. How do these plots compare with the solution of the linear equation $\ddot{\theta} = -\frac{g}{\ell}\theta$?
2. Solve the equation again over the same time interval using the initial conditions $[\theta(0), \dot{\theta}(0)] = [18°, 0]$, and $[30°, 0]$. Plot $\theta(t)$ starting with all the three initial conditions used so far on the same graph and comment on the time period of oscillations.
3. Solve the equation again over $t \in [0, 4\,\mathrm{s}]$ using $\theta(0) = \pi/2$ and $\pi/1.02$ while keeping $\dot{\theta}(0) = 0$. Again plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ, for the three solutions obtained with $\theta(0) = \pi/6, \pi/2$, and $\pi/1.02$. Comment on the plots.
4. For the last three initial conditions, compute E_P, E_K, and $E_\mathrm{T} = E_\mathrm{P} + E_\mathrm{K}$ from the solutions obtained. For each initial condition, plot E_P, E_K, and E_T on the same graph and show that the total energy in each case remains constant irrespective of the nature of oscillations.

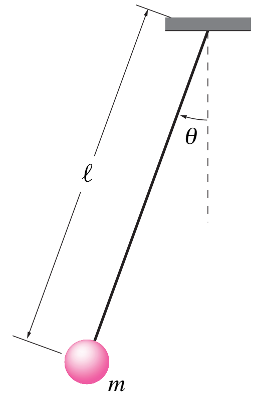


Figure 15.26

Solution The equation of motion of the pendulum is (as given)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

To solve this second order differential equation numerically, we need to first convert it into a set of two first order equations. Let $\omega = \dot{\theta}$. Then, we can write

$$\begin{aligned} \dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g}{\ell}\sin\theta. \end{aligned}$$

We are now ready to write a computer program to solve these equations numerically. We use the following pseudocode to accomplish the task.

```
ODEs = {thetadot = omega,
        omegadot = -g/l*sin(theta) }
ICs   = {theta(0) = pi/30, omega(0) = 0 }
Set  g = 1, l = g/(4*pi^2)
Solve ODEs with ICs for t=0 to t=4
plot theta vs t, and omega vs t; plot omega vs theta
```

1. **Small amplitude oscillations:** The solution obtained with $\theta(0) = 6° = \pi/30$, and $\dot{\theta}(0) = 0$ is shown in fig. 15.27. The plots of $\theta(t)$ and $\dot{\theta}(t)$ clearly show the initial conditions at $t = 0$. From the figure, we see that the motion is sinusoidal and the time period of oscillation is 1 second, as expected.
2. **Deviation from linear equation solution:** The new initial conditions involve larger initial angles ($\theta(0) = 18°$ and $30°$). That is the only difference. We use the

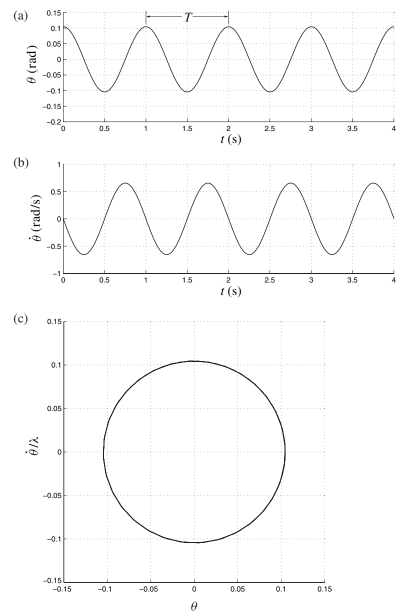


Figure 15.27: Numerical solution of the nonlinear pendulum equation, $\ddot{\theta} = -g/ell\sin\theta$, with the initial conditions, $\theta(0) = \pi/30$ and $\dot{\theta}(0) = 0$; (a) $\theta(t)$, (b) $\dot{\theta}(t)$, and (c) $\dot{\theta}$ vs θ (phase plane).

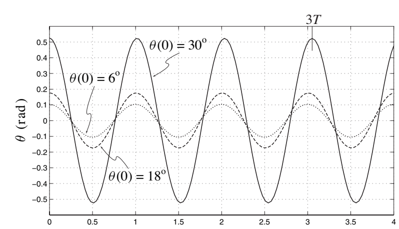


Figure 15.28: Comparison of $\theta(t)$ obtained from three different initial launch angles, $\theta(0) = 6°$, $18°$ and $30°$.

same program as used before and get the solutions with the new initial conditions. We plot $\theta(t)$ against t for all the three solutions on the same graph. The resulting plot is shown in fig. 15.28.

Now what we observe from this plot is that the three solutions, starting with the three different initial conditions, do not have the same time period of oscillations. The difference is not clearly visible between $\theta(0) = 6°$ and $\theta(0) = 18°$ solutions but it is much clearer for $\theta = 30°$ (see the third peak, marked with $3T$). As the initial angle, $\theta(0)$, increases, the period of oscillation seems to increase.

The dependence of time period (or frequency) of oscillations on the amplitude is the hallmark of nonlinear oscillators. In contrast, linear oscillators have a constant period of oscillation, irrespective of the amplitude of motion. For our pendulum, as long as the initial θ is so small that $\sin\theta \approx \theta$, the equation of motion can be replaced by the linear equation, $\ddot{\theta} = -g/\ell\theta$, and all solutions will have the same time period of oscillation. As $\theta(0)$ becomes larger, the approximation $\sin\theta \approx \theta$ breaks down, and the linear equation of motion is no longer valid.

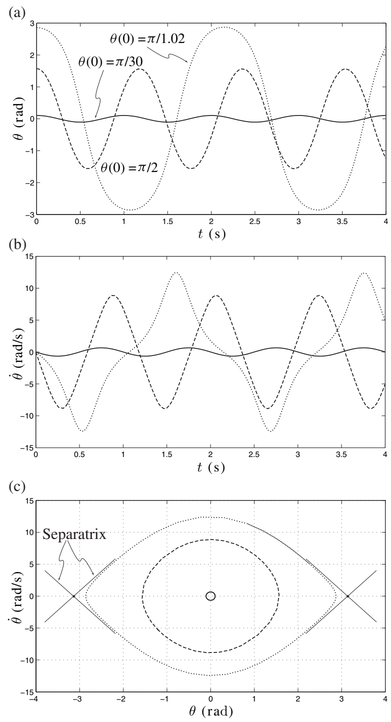


Figure 15.29: Comparison of pendulum motions when it is released from rest at small angle ($\theta(0) = \pi/30$), at horizontal position (($\theta(0) = \pi/2$), and at almost vertically upright position ($\theta(0) = \pi/1.02$); (a) θ vs t, (b) $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ (phase plane plot).

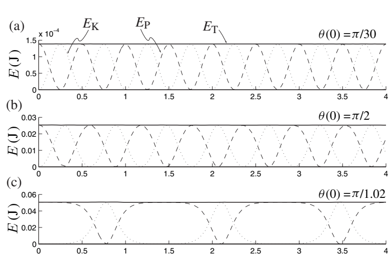


Figure 15.30: Plots of potential energy E_P, kinetic energy E_K, and total energy E_T during motion under three different initial conditions: (a) $\theta(0) = \pi/30$, $\dot{\theta}(0) = 0$; (b) $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$; and (c) $\theta(0) = \pi/1.02$, $\dot{\theta}(0) = 0$.

3. **Large amplitude oscillations:** We now run the program with large initial angles, $\theta(0) = \pi/2\,(90°)$ and $\theta(0) = \pi/1.02$ ($\approx 176°$, *i.e.,* close to the vertically upright position), and obtain the corresponding solutions. Plots of $\theta(t)$ and $\dot{\theta}(t)$ for three initial conditions, small θ ($\pi/30$), moderately large θ ($\pi/2$), and very large θ ($\pi/1.02$) are shown in fig. 15.29. From the plots it is clear that not only the period of oscillation increases drastically with larger amplitudes, but also the qualitative nature of oscillations changes. For small amplitude (small initial θ), oscillations are simple harmonic but for larger amplitudes (large initial θ) oscillations are no more simple harmonic. This fact is more evident from the velocity plot, fig. 15.29(b). The phase plot, fig. 15.29(c), shows how the three solution trajectories (also called *orbits*) look in the phase space. All simple harmonic motions lead to circular orbits (you can show that by writing the solution for $\theta(t)$ and $\dot{\theta}(t)$ and then showing that $\theta^2 + \dot{\theta}^2$ = constant) in this phase space. However, for large amplitude motion, the orbits become oblong and approach a rather strange looking trajectory, called the *separatrix*, as the amplitude of motion grows. This separatrix marks the boundary of all possible periodic motions of the pendulum. Outside this separatrix, solutions do exist but they correspond to whirling motion of the pendulum which is not periodic (because $\theta(t)$ keeps growing without bounds).

4. **Energy conservation:** Let θ^* and $\dot{\theta}^*$ be the values of angular displacement and angular speed of the pendulum at some instant t^*. Then, assuming $\theta = 0$ to be the datum for potential energy, we can write the expressions for potential energy and kinetic energy as

$$\begin{aligned} E_P &= mg\ell(1 - \cos\theta^*) \\ E_K &= \frac{1}{2}m\ell^2\dot{\theta}^{*2}. \end{aligned}$$

Therefore, the total energy at $t = t^*$ is,

$$E_T = E_P + E_K = mg\ell(1 - \cos\theta^*) + \frac{1}{2}m\ell^2\dot{\theta}^{*2}.$$

From the numerical solutions obtained for the three initial conditions, we have values of θ and $\dot{\theta}$ at different time instants. Now, using the formulas for E_P, E_K and E_T, we compute the values of these quantities and plot them as shown in fig. 15.30. We see that for each initial condition, the potential and kinetic energies vary differently with time. However, the total energy remains constant at all times. This is expected as there is no dissipation in the system (not present in our mathematical model). A given initial condition determines the initial energy of the pendulum which must be preserved throughout the motion.

15.2 Dynamics of a particle in circular motion

Preparatory Problems

15.2.1 Force on a person standing on the equator. Find the magnitude of the total force acting on a 150 lbm person standing on the equator. The total force is the gravity force plus the force of the ground on the person (note that these two do not exactly cancel). Neglect the motion of the earth around the sun and of the sun around the solar system, etc. The radius of the earth is 3963 mi. Give your solution in both pounds (lbf) and Newtons (N). *

15.2.2 Consider a mass m in circular motion. Let $\sum \vec{F} = \sum F_r \hat{e}_r + \sum F_\theta \hat{e}_\theta$. Using $\sum \vec{F} = m\vec{a}$, express $\sum F_r$ and $\sum F_\theta$ in terms of some or all of $\theta, \dot{\theta}, \ddot{\theta}, , r$, and m.

15.2.3 Using $\sum \vec{F} = m\vec{a}$, find the expressions for $\sum F_x$ and $\sum F_y$ in terms of $\ddot{\theta}$, $\dot{\theta}$, r, and θ. [Hint $\hat{e}_r = \cos\theta\hat{\imath} + \sin\theta\hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta\hat{\imath} + \cos\theta\hat{\jmath}$].

15.2.4 A bead of mass m goes around a circular path of radius R in the xy-plane with angular acceleration $\ddot{\theta} = ct^3$. The bead starts from rest at $\theta = 0$.

a) What is the angular momentum of the bead about the origin at $t = t_1$?

b) What is the rate of change of angular momentum about the origin at $t = t_1$?

c) What is the kinetic energy of the bead at $t = t_1$?

d) Does the kinetic energy increase, decrease, or remain constant with time? Why?

15.2.5 A 200 gm particle goes in circles about a fixed center at a constant speed $v = 1.5\,\text{m/s}$. It takes 7.5 s to go around the circle once.

a) Find the angular speed of the particle.

b) Find the magnitude of acceleration of the particle.

c) Take center of the circle to be the origin of a xy-coordinate system. Find the net force on the particle when it is at $\theta = 30^\circ$ from the x-axis.

15.2.6 A race car cruises on a circular track at a constant speed of 120 mph. It goes around the track once in three minutes. Find the magnitude of the centripetal force on the car. What applies this force on the car? Does the driver have any control over this force?

15.2.7 A particle moves on a counterclockwise, origin-centered circular path in the xy-plane at a constant rate. The radius of the circle is r, the mass of the particle is m, and the particle completes one revolution in time τ.

a) Neatly draw the following things:

1. The path of the particle.
2. A dot on the path when the particle is at $\theta = 0°$, $90°$, and $210°$, where θ is measured from the x-axis (positive counter-clockwise).
3. Arrows representing $\hat{e}_R$, $\hat{e}_\theta$, $\vec{v}$, and $\vec{a}$ at each of these points.

b) Calculate all of the quantities in part (3) above at the points defined in part (2), (represent vector quantities in terms of the cartesian base vectors $\hat{\imath}$ and $\hat{\jmath}$). *

c) If this motion was imposed by the tension in a string, what would that tension be? *

d) Is radial tension enough to maintain this motion or is another force needed to keep the motion going (assuming no friction) ? *

e) Again, if this motion was imposed by the tension in a string, what is F_x, the x component of the force in the string, when $\theta = 210°$? Ignore gravity.

15.2.8 The velocity and acceleration of a 1 kg particle, undergoing constant rate circular motion, are known at some instant t:

$$\vec{v} = -10\,\text{m/s}(\hat{\imath} + \hat{\jmath}), \quad \vec{a} = 2\,\text{m/s}^2(\hat{\imath} - \hat{\jmath}).$$

a) Write the position of the particle at time t using $\hat{e}_R$ and $\hat{e}_\theta$ base vectors.

b) Find the net force on the particle at time t.

c) At some later time t^*, the net force on the particle is in the $-\hat{\jmath}$ direction. Find the elapsed time $t - t^*$.

d) After how much time does the force on the particle reverse its direction.

15.2.9 A particle of mass 3 kg moves in the xy-plane so that its position is given by

$$\vec{r}(t) = 4\,\text{m}\left[\cos\left(\frac{2\pi t}{s}\right)\hat{\imath} + \sin\left(\frac{2\pi t}{s}\right)\hat{\jmath}\right]$$

with respect to point O, the origin of a fixed cartesian coordinate system.

a) What is the path of the particle? Show how you know what the path is.

b) What is the angular velocity of the particle? Is it constant? Show how you know if it is constant or not.

c) What is the velocity of the particle in polar coordinates?

d) What is the speed of the particle at $t = 3\,\text{s}$?

e) What net force does it exert on its surroundings at $t = 0\,\text{s}$? Assume the x and y axes are fixed.

f) What is the angular momentum of the particle at $t = 3\,\text{s}$ about point O?

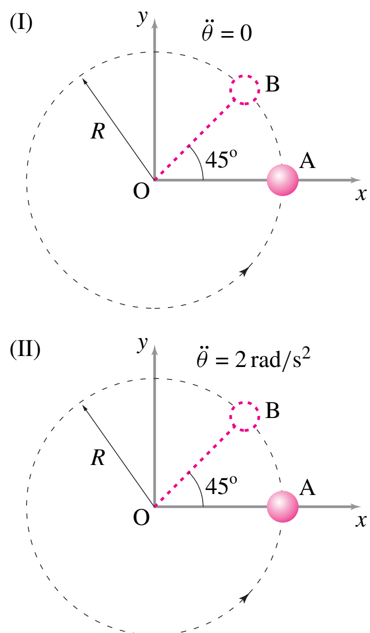
15.2.10 A comparison of constant and nonconstant rate circular motion. A 100 gm mass is going in circles of radius $R = 20\,\text{cm}$ at a constant rate $\dot{\theta} = 3\,\text{rad/s}$. Another identical mass is going in circles of the same radius but at a non-constant rate. The second mass is accelerating at $\ddot{\theta} = 2\,\text{rad/s}^2$ and at position A, it happens to have the same angular speed as the first mass.

a) Find and draw the accelerations of the two masses (call them I and II) at position A.

b) Find $\dot{\vec{H}}_{/O}$ for both masses at position A. *

c) Find $\vec{H}_{/O}$ for both masses at positions A and B. Do the changes in $\vec{H}_{/O}$ between the two positions reflect (qualitatively) the results obtained in (b)? *

d) If the masses are pinned to the center O by massless rigid rods, is tension in the rods enough to keep the two motions going? Explain.



Problem 15.2.10

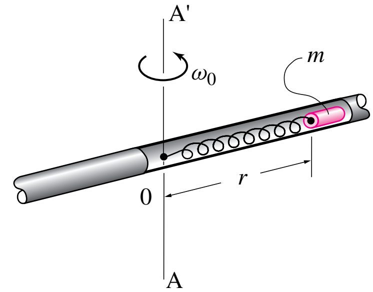
More-Involved Problems

15.2.11 A small mass m is connected to one end of a spring. The other end of the spring is fixed to the center of a circular track. The radius of the track is R, the unstretched length of the spring is ℓ_0 (with $\ell_0 < R$), and the spring constant is k.

a) With what speed should the mass be launched in the track so that it keeps going at a constant speed?

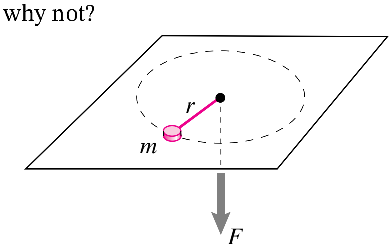
b) If the spring is replaced by another spring of same relaxed length but twice the stiffness, what will be the new required launch speed of the particle?

15.2.12 A bead of mass m is attached to a spring of stiffness k. The bead slides without friction in the tube shown. The tube is driven at a constant angular rate $\dot{\theta}_0$ about axis AA' by a motor (not pictured). There is no gravity. The unstretched spring length is r_0. Find the radial position r of the bead if it is stationary with respect to the rotating tube. *



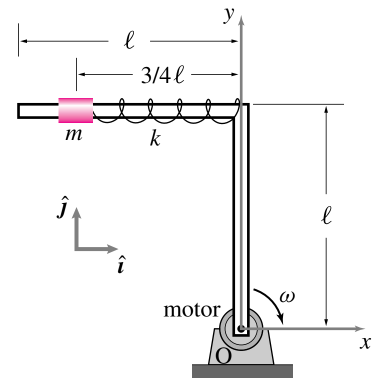
Problem 15.2.12

15.2.13 A particle of mass m is restrained by a string to move with a constant angular speed ω around a circle of radius R on a horizontal frictionless table. If the radius of the circle is reduced slowly to r, by pulling the string with a slowly varying force F through a hole in the table, what will the particle's angular velocity be in the final circular motion? Is kinetic energy changed in moving from circular motion at R to circular motion at r? Why or why not?



Problem 15.2.13

15.2.14 An 'L' shaped rigid, massless, and frictionless bar is made up of two uniform segments of length $\ell = 0.4\,\text{m}$ each. A collar of mass $m = 0.5\,\text{kg}$, attached to a spring at one end, slides frictionlessly on one of the arms of the 'L'. The spring is fixed to the elbow of the 'L' and has a spring constant $k = 6\,\text{N/m}$. The structure rotates clockwise at a constant rate $\omega = 2\,\text{rad/s}$. If the collar is steady at a distance $\frac{3}{4}\ell = 0.3\,\text{m}$ away from the elbow of the 'L', find the relaxed length of the spring, ℓ_0. Neglect gravity. *

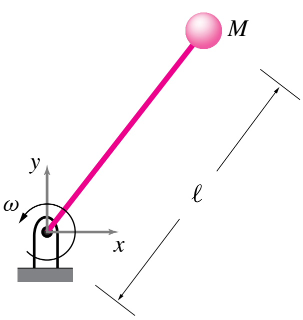


Problem 15.2.14: **Forces in constant rate circular motion.**

15.2.15 A massless rigid rod with length ℓ attached to a ball of mass M spins at a constant angular rate ω which is maintained by a motor (not shown) at the hinge point. The rod can only withstand a tension of T_{cr} before breaking. Find the maximum angular speed of the ball so that the rod does not break assuming

a) there is no gravity, and

b) there is gravity (neglect bending stresses).



Problem 15.2.15

15.2.16 A 1 m long massless string has a particle of 10 grams mass at one end and is tied to a stationary point O at the other end. The particle rotates counter-clockwise in circles on a frictionless horizontal plane. The rotation rate is 2π rev/sec. Assume an xy-coordinate system in the plane with its origin at O.

a) Make a clear sketch of the system.

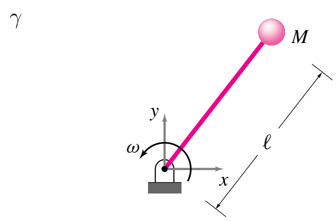
b) What is the tension in the string (in Newtons)? *

c) What is the angular momentum of the mass about O? *

d) When the string makes a 45° angle with the positive x and y axis on the plane, the string is quickly and cleanly cut. What is the position of the mass 1 sec later? Make a sketch of the particle's trajectory. *

15.2.17 A ball of mass M fixed to an inextensible rod of length ℓ and negligible mass rotates about a frictionless hinge as shown in the figure. A motor (not shown) at the hinge point accelerates the mass-rod system from rest by applying a constant torque M_O. The rod is initially lined up with the positive x-axis. The rod can only withstand a tension of T_{cr} before breaking. At what time will the rod break and after how many revolutions? Neglect bending stresses.

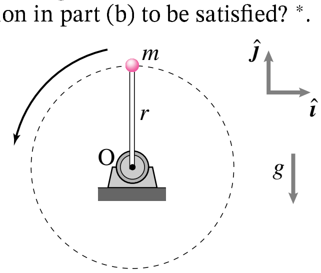
a) Neglect gravity.
b) Include gravity.



Problem 15.2.17

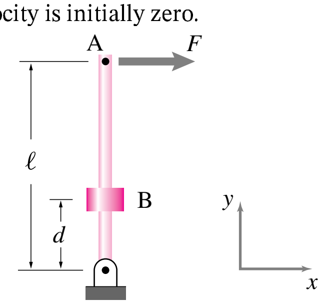
15.2.18 A particle of mass m, tied to one end of a rod whose other end is fixed at point O to a motor, moves in a circular path in the vertical plane at a constant rate. Gravity acts in the $-\hat{\jmath}$ direction.

a) Find the difference between the maximum and minimum tension in the rod. *
b) Find the ratio $\frac{\Delta T}{T_{max}}$ where $\Delta T = T_{max} - T_{min}$. A criterion for ignoring gravity might be if the variation in tension is less than 2% of the maximum tension; i.e., when $\frac{\Delta T}{T_{max}} < 0.02$. For a given length r of the rod, find the rotation rate ω for which this condition is met. *.
c) For $\omega = 300$ rpm, what would be the length of the rod for the condition in part (b) to be satisfied? *.



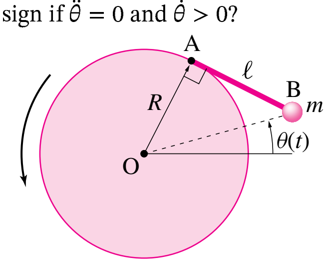
Problem 15.2.18

15.2.19 A massless rigid bar of length L is hinged at the bottom. A force F is applied at point A at the end of the bar. A mass m is glued to the bar at point B, a distance d from the hinge. There is no gravity. What is the acceleration of point A at the instant shown? Assume the angular velocity is initially zero.



Problem 15.2.19

15.2.20 The mass m is attached rigidly to the rotating disk by the light rod AB of length ℓ. Neglect gravity. Find M_A (the moment on the rod AB from its support point at A) in terms of $\dot{\theta}$ and $\ddot{\theta}$. What is the sign of M_A if $\dot{\theta} = 0$ and $\ddot{\theta} > 0$? What is the sign if $\ddot{\theta} = 0$ and $\dot{\theta} > 0$?



Problem 15.2.20

Pendulum problems

15.2.21 Simple pendulum, comprehensive version. This problem covers many aspects of a simple pendulum. A point mass M hangs on a massless string or rod of length L. The gravitational force is Mg. The pendulum is in a vertical plane. At any time t, the angle between the straight down line and the pendulum, measured counter clockwise, is $\theta(t)$. Neglect air friction. When numbers are called for use $M = 1\,\text{kg}$, $L = 1\,\text{m}$ and $g = 10\,\text{m/s}^2$.

a) **Find the equations of motion.** That is, assume that you know both θ and $\dot{\theta}$, find $\ddot{\theta}$. There are several ways to do this problem.* Find the equations using
   1. Linear momentum balance
   2. Angular momentum balance
   3. Conservation of energy
b) **Tension.** Assuming that you know θ and $\dot{\theta}$, find the tension T in the string.
c) **Reaction components.** Assuming you know θ and $\dot{\theta}$, find the x and y components of the force that the hinge support causes on the pendulum. Clearly define the directions of positive x and y with a sketch.
d) **Reduction to first order equations.** The equation that you found in (a) is a nonlinear second order ordinary differential equation. It can be changed to a pair of first order equations by defining a new variable $\omega \equiv \dot{\theta}$. Write the equation from (a) as a pair of first order equations. *
e) **Numerical solution.** Given the initial conditions $\theta(t=0) = \pi/2$ and $\omega(t=0) = \dot{\theta}(t=0) = 0$. Using numerical integration, find: $\theta(t)$, $\dot{\theta}(t)$ & $T(t)$. Make a single plot, or three vertically aligned plots, of these variables for one full oscillation of the pendulum.
f) **Maximum tension.** Using your numerical solutions, find the maximum value of the tension in the rod as the mass swings. *
g) Plot the x and y reaction components as a function of time.
h) **Period of oscillation.** How long does it take to make one oscillation?
i) **Other observations.** Some questions:
   1. Does the solution to (f) depend on the length of the string?
   2. Is the solution to (f) exactly 30 or just a number near 30? If it is exact can you find the result analytically?
   3. Is the period found in (h) longer or shorter than the period found by solving the linear equation $\ddot{\theta} + (g/l)\theta = 0$, based on the (inappropriate-to-use in this case) small angle approximation $\sin\theta = \theta$? Explain intuitively why you expect the period to be longer or shorter?

15.2.22 Tension in a simple pendulum string. A simple pendulum of length 2 m with mass 3 kg is released from rest at an initial angle of 60° from the vertically down position.

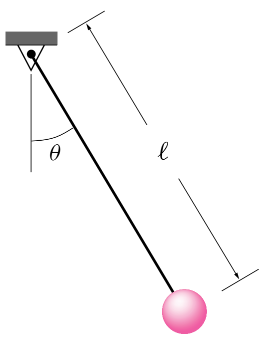
a) What is the tension in the string just after the pendulum is released?

b) What is the tension in the string when the pendulum has reached 30° from the vertical?

15.2.23 Cartesian coordinates Find the nonlinear governing differential equation for a simple pendulum

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

using linear momentum in Cartesian coordinates and without using the polar coordinate formulas for velocity and acceleration. Of course you can use that $x = \ell\cos\theta$ and $y = \ell\sin\theta$ for x pointing down.

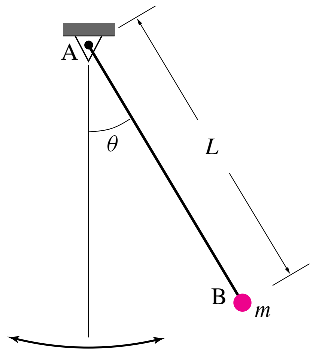


Problem 15.2.23

15.2.24 Tension in a rope-swing rope. Model a swinging person as a point mass. The swing starts from rest at an angle $\theta = 90°$. When the rope passes through vertical the tension in the rope is higher (it is hard to hang on). A person wants to know ahead of time if she is strong enough to hold on. How hard does she have to hang on compared, say, to her own weight? You are to find the solution two ways. Use the same m, g, and L for both solutions.

a) Find $\ddot{\theta}$ as a function of g, L, θ, and m. This equation is the governing differential equation. Write it as a system of first order equations. Solve them numerically. Once you know $\dot{\theta}$ at the time the rope is vertical you can use other mechanics relations to find the tension. If you like, you can plot the tension as a function of time as the mass falls.

b) Use conservation of energy to find $\dot{\theta}$ at $\theta = 0$. Then use other mechanics relations to find the tension. *



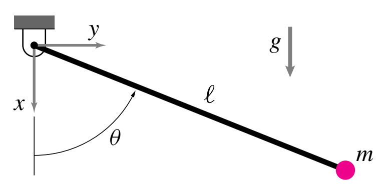
Problem 15.2.24

15.2.25 Pendulum. A pendulum with a negligible-mass rod and point mass m is released from rest at the horizontal position $\theta = \pi/2$.

a) Find the acceleration (a vector) of the mass just after it is released at $\theta = \pi/2$ in terms of ℓ, m, g and any base vectors you define clearly.

b) Find the acceleration (a vector) of the mass when the pendulum passes through the vertical at $\theta = 0$ in terms of ℓ, m, g and any base vectors you define clearly

c) Find the string tension when the pendulum passes through the vertical at $\theta = 0$ (in terms of ℓ, m and g).



Problem 15.2.25

15.2.26 Write a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, over a given time interval $(0, t)$, and initial conditions $\theta(0)$, and $\dot{\theta}(0)$. The output should be a vector of time instants, t_i, in the given time interval and the corresponding θ_i and $\dot{\theta}_i$.

Now use your computer program to find the solution of

$$\ddot{\theta} = -\sin\theta, \qquad \theta(0) = \pi/4,\ \dot{\theta}(0) = 0.$$

Compare the solution obtained with the analytical solution of the corresponding simple pendulum equation, $\ddot{\theta} = -\theta$ with the same initial conditions. In particular,

a) Find the difference in the time period of oscillations of the two systems.

b) Plot $\dot{\theta}$ obtained from the two solutions against time and comment on the differences.

c) Plot $\dot{\theta}$ against θ from the two solutions on the same plot and compare the two phase portraits. Comment on the differences.

15.2.27 Solve the nonlinear pendulum equation numerically taking 20 different initial angular positions between $\theta = 0$ and $\theta = \pi$, each time releasing the pendulum gently from rest. Find the time period of oscillation, T, from each solution and plot it against the amplitude of motion, *i.e.,* $\theta(0)$.

a) How does the period of oscillation depend on the amplitude for small amplitudes?

b) What is the limiting value of the time period for large amplitudes, *i.e.,* $\theta(0) \to \pi$?

c) How does T depend on the amplitude over the entire range?

15.2.28 A pendulum of mass m and length ℓ is released from rest at $\theta(0) = 60°$. It executes oscillatory motion. If the pendulum were to be released from two different positions, $\theta(0) = 45°$ and $\theta(0) = 0°$, with some corresponding initial angular speed such that the ensuing motion were exactly the same as that with $\theta(0) = 60°$ and $\dot{\theta}(0) = 0$, find the required initial angular speeds.

a) First, find the corresponding $\dot{\theta}(0)$ without any computer simulation.

b) Verify your answer by plotting computer generated solutions for the three different initial conditions.

c) What is the general relationship between $\theta(0)$ and $\dot{\theta}(0)$ that produces a predefined motion generated by, say, a given set of $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

15.2.29 Use a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} + \lambda^2 \sin\theta = 0$, where $\lambda^2 = 1.56/\mathrm{s}^2$, with the following 11 initial conditions: $[\theta(0), \dot{\theta}(0)]$ = [1, 0], [2, 0], [3, 0], [4, -1], [-4, 1], [4, -1.02], [-4, 1.02], [4, -1.1], [-4, 1.1], [4, 1.4], and [-4, 1.4], where θ is in radians and $\dot{\theta}$ in rad/s. Obtain each solution over the time interval $t = 0$ to $t = 20\,\mathrm{s}$.

a) Plot all solutions in the phase space (*i.e.*, θ vs $\dot{\theta}$) in a single graph.

b) What does the extension of the plot beyond $\theta = \pm\pi$ mean?

c) Which initial conditions give solutions outside the separatrix? What do these solutions mean? Are these solutions periodic?

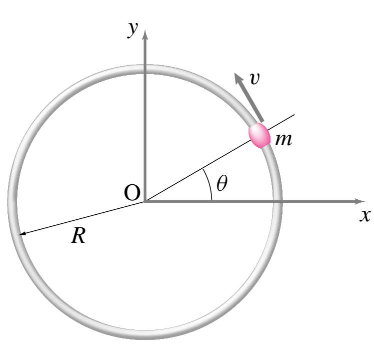
d) If you added a little bit of viscous damping to the pendulum motion, can you guess what will happen to the solutions inside the separatrix? [Hint: think about energy associated with these solutions.]

More circular motion problems

15.2.30 Bead on a hoop with friction. A bead slides on a rigid, stationary, circular wire. The coefficient of friction between the bead and the wire is μ. The bead is loose on the wire (not a tight fit but not so loose that you have to worry about rattling). Assume gravity is negligible.

a) Given v, m, R, & μ; what is $\dot{v}$? *

b) If $v(\theta = 0) = v_0$, how does v depend on θ, μ, v_0 and m? *

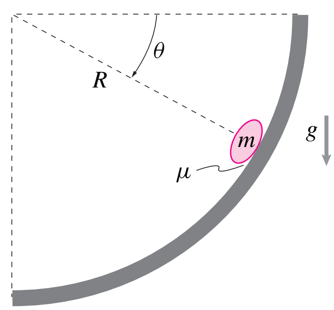


Problem 15.2.30

15.2.31 Particle in a chute. One of a million non-interacting rice grains is sliding in a circular chute with radius R. Its mass is m and it slides with coefficient of friction μ (Actually it slides, rolls and tumbles — μ is just the effective coefficient of friction from all of these interactions.) Gravity g acts downwards.

a) Find a differential equation that is satisfied by θ that governs the speed of the rice as it slides down the hoop. Parameters in this equation can be m, g, R and μ [Hint: Draw FBD, write eqs of mechanics, express as ODE.]

b) Find the particle speed at the bottom of the chute if $R = 0.5m$, $m = 0.1$ grams, $g = 10\,\mathrm{m/s^2}$, and $\mu = 0.2$ as well as the initial values of $\theta_0 = 0$ and its initial downward speed is $v_0 = 10\,\mathrm{m/s}$. [Hint: you are probably best off using a numerical solution.]

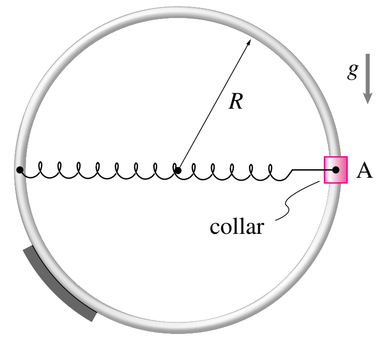


Problem 15.2.31

15.2.32 Due to a push which happened in the past, the collar with mass m is sliding up at speed v_0 on the circular ring when it passes through the point A. The ring is frictionless. A spring of constant k and unstretched length R is also pulling on the collar.

a) What is the acceleration of the collar at A. Solve in terms of R, v_0, m, k, g and any base vectors you define.

b) What is the force on the collar from the ring when it passes point A? Solve in terms of R, v_0, m, k, g and any base vectors you define.

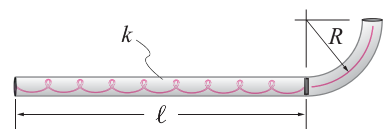


Problem 15.2.32

15.2.33 A toy used to shoot pellets is made out of a thin tube which has a spring of spring constant k on one end. The spring is placed in a straight section of length ℓ; it is unstretched when its length is ℓ. The straight part is attached to a (quarter) circular tube of radius R, which points up in the air.

a) A pellet of mass m is placed in the device and the spring is pulled to the left by an amount $\Delta\ell$. Ignoring friction along the travel path, what is the pellet's velocity $\bar{\boldsymbol{v}}$ as it leaves the tube? *

b) What force acts on the pellet just prior to its departure from the tube? What about just after? *

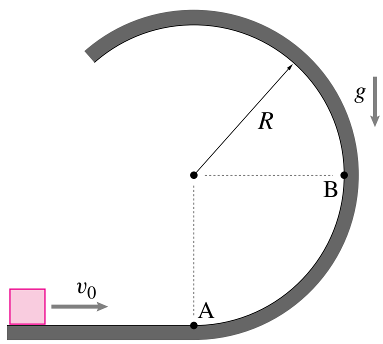


Problem 15.2.33

15.2.34 A block with mass m is moving to the right at speed v_0 when it reaches a circular frictionless portion of the ramp.

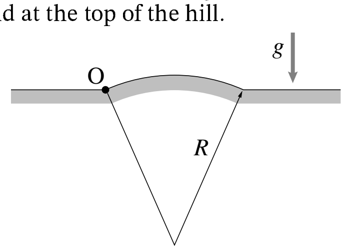
a) What is the speed of the block when it reaches point B? Solve in terms of R, v_0, m and g.

b) What is the force on the block from the ramp just after it gets onto the ramp at point A? Solve in terms of R, v_0, m and g. Remember, force is a vector.



Problem 15.2.34

15.2.35 A car moves with speed v along the surface of the hill shown which can be approximated as a circle of radius R. The car starts at a point on the hill at point O. Compute the magnitude of the speed v such that the car just leaves the ground at the top of the hill.



Problem 15.2.35