SAMPLE 15.6 **Circular motion in 2-D.** Two bars, each of negligible mass and length $\ell = 3\,\mathrm{ft}$, are welded together at right angles to form an 'L' shaped structure. The structure supports a $3.2\,\mathrm{lbf}$ $(= mg)$ ball at one end and is connected to a motor on the other end (see Fig. 15.21). The motor rotates the structure in the vertical plane at a constant rate $\dot{\theta} = 10\,\mathrm{rad/s}$ in the counter-clockwise direction. Take $g = 32\,\mathrm{ft/s^2}$. At the instant shown in Fig. 15.21, find

1. the velocity of the ball,
2. the acceleration of the ball, and
3. the net force and moment applied by the motor and the support at O on the structure.

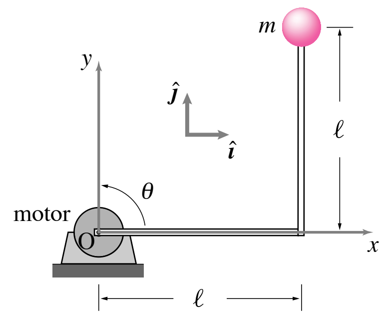

Figure 15.21: The motor rotates the structure at a constant angular speed in the counterclockwise direction.

Solution The motor rotates the structure at a constant rate. Therefore, the ball is going in circles with angular velocity $\vec{\omega} = \dot{\theta}\hat{\boldsymbol{k}} = 10\,\mathrm{rad/s}\hat{\boldsymbol{k}}$. The radius of the circle is $R = \sqrt{\ell^2 + \ell^2} = \ell\sqrt{2}$. Since the motion is in the xy plane, we use the following formulae to find the velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$.

$$\vec{\boldsymbol{v}} = \dot{R}\hat{\boldsymbol{e}}_R + R\dot{\theta}\hat{\boldsymbol{e}}_\theta$$

$$\vec{\boldsymbol{a}} = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta,$$

where $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are the polar basis vectors shown in Fig. 15.22. In Fig. 15.22, we note that $\theta = 45^\circ$. Therefore,

$$\begin{aligned}
\hat{\boldsymbol{e}}_R &= \cos\theta\,\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \\
&= \frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}), \\
\hat{\boldsymbol{e}}_\theta &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} \\
&= \frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

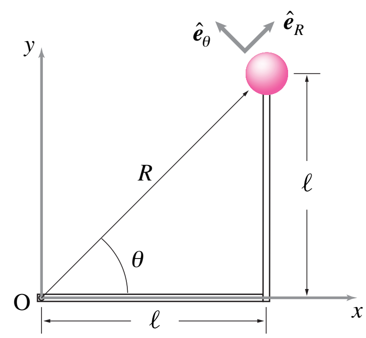

Figure 15.22: The ball follows a circular path of radius R. The position, velocity, and acceleration of the ball can be expressed in terms of the polar basis vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$.

Here, $R = L\sqrt{2} = 3\sqrt{2}\,\mathrm{ft}$ is constant, and $\ddot{\theta} = 0$ because $\dot{\theta} = 10\,\mathrm{rad/s} = $ constant. Thus,

1. the velocity of the ball is

$$\begin{aligned}
\vec{\boldsymbol{v}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\
&= 3\sqrt{2}\,\mathrm{ft}\cdot 10\,\mathrm{rad/s}\hat{\boldsymbol{e}}_\theta \\
&= 30\sqrt{2}\,\mathrm{ft/s}\cdot\frac{1}{\sqrt{2}}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
&= 30\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = 30\,\mathrm{ft/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

2. The acceleration of the ball is

$$\begin{aligned}
\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R \\
&= -3\sqrt{2}\,\mathrm{ft}\cdot(10\,\mathrm{rad/s})^2\hat{\boldsymbol{e}}_R \\
&= -300\sqrt{2}\,\mathrm{ft/s^2}\cdot\frac{1}{\sqrt{2}}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}) \\
&= -300\,\mathrm{ft/s^2}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = -300\,\mathrm{ft/s^2}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}$$

3. Let the net force and the moment applied by the motor-support system be $\vec{F}$ and $\vec{M}$ as shown in Fig. 15.23. From the linear momentum balance for the structure,

$$
\begin{aligned}
\sum \vec{F} &= m\vec{a}\\
\vec{F} - mg\hat{\jmath} &= m\vec{a}\\
\Rightarrow \quad \vec{F} &= m\vec{a} + mg\hat{\jmath}\\
&= \overbrace{\frac{3.2\,\text{lbf}}{32\,\text{ft/s}^2}}^{m}(-300\sqrt{2}\,\text{ft/s}^2)\hat{e}_R + \overbrace{3.2\,\text{lbf}}^{mg}\,\hat{\jmath}\\
&= -30\sqrt{2}\,\text{lbf}\,\hat{e}_R + 3.2\,\text{lbf}\,\hat{\jmath}.\\
&= -30\sqrt{2}\,\text{lbf}\,\frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath}) + 3.2\,\text{lbf}\,\hat{\jmath}\\
&= -30\,\text{lbf}\,\hat{\imath} - 26.8\,\text{lbf}\,\hat{\jmath}.
\end{aligned}
$$

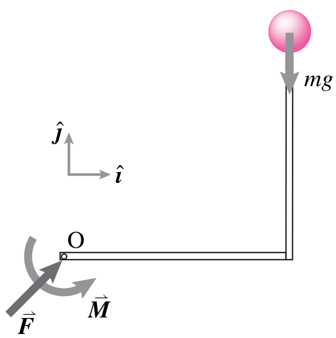

Figure 15.23: Free-body diagram of the structure.

Similarly, from the angular momentum balance for the structure,

$$
\begin{aligned}
\sum \vec{M}_O &= \dot{\vec{H}}_{/O},\\
\text{where} \quad \sum \vec{M}_O &= \vec{M} + \vec{r}_{/O} \times mg(-\hat{\jmath})\\
&= \vec{M} + \underbrace{R\hat{e}_R}_{\ell(\hat{\imath}+\hat{\jmath})} \times mg(-\hat{\jmath})\\
&= \vec{M} - mg\ell\hat{k},\\
\text{and} \quad \dot{\vec{H}}_{/O} &= \vec{r}_{/O} \times m\vec{a}\\
&= R\hat{e}_R \times m(-R\dot{\theta}^2\hat{e}_R)\\
&= -mR^2\dot{\theta}^2\underbrace{(\hat{e}_R \times \hat{e}_R)}_{\vec{0}}\\
&= \vec{0}.
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\vec{M} &= mg\ell\hat{k}\\
&= \underbrace{3.2\,\text{lbf}}_{mg} \cdot \underbrace{3\,\text{ft}}_{\ell}\,\hat{k}\\
&= 9.6\,\text{lbf}\cdot\text{ft}\,\hat{k}.
\end{aligned}
$$

$\vec{F} = -30\,\text{lbf}\,\hat{\imath} - 26.8\,\text{lbf}\,\hat{\jmath}, \quad \vec{M} = 9.6\,\text{lbf}\cdot\text{ft}\,\hat{k}$

Note: If there was no gravity, the moment applied by the motor would be zero.

SAMPLE 15.7 A 50 gm point mass executes circular motion with angular acceleration $\ddot{\theta} = 2\,\mathrm{rad/s^2}$. The radius of the circular path is 200 mm. If the mass starts from rest at $t = 0$, find

1. Its angular momentum $\vec{\boldsymbol{H}}$ about the center at $t = 5$ s.
2. Its rate of change of angular momentum $\dot{\vec{\boldsymbol{H}}}$ about the center.

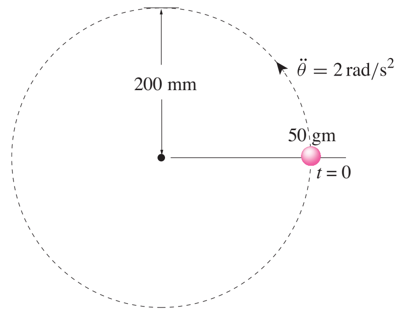

Figure 15.24

Solution

1. From the definition of angular momentum,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/O} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{v}} \\
&= R\hat{\boldsymbol{e}}_R \times m\dot{\theta}R\hat{\boldsymbol{e}}_\theta \\
&= mR^2\dot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\
&= mR^2\dot{\theta}\hat{\boldsymbol{k}}
\end{aligned}$$

On the right hand side of this equation, the only unknown is $\dot{\theta}$. Thus to find $\vec{\boldsymbol{H}}_{/O}$ at $t = 5$ s, we need to find $\dot{\theta}$ at $t = 5$ s. Now,

$$\begin{aligned}
\ddot{\theta} &= \frac{d\dot{\theta}}{dt} \\
d\dot{\theta} &= \ddot{\theta}\,dt \\
\int_{\dot{\theta}_0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t \ddot{\theta}\,dt \\
\dot{\theta}(t) - \dot{\theta}_0 &= \ddot{\theta}(t_t - t_0) \\
\dot{\theta} &= \dot{\theta}_0 + \ddot{\theta}(t - t_0)
\end{aligned}$$

Writing α for $\ddot{\theta}$ and substituting $t_0 = 0$ in the above expression, we get $\dot{\theta}(t) = \dot{\theta}_0 + \alpha t$, which is the angular speed version of the linear speed formula $v(t) = v_0 + at$. (1) Substituting $t = 5$ s, $\dot{\theta}_0 = 0$, and $\alpha = 2\,\mathrm{rad/s^2}$ we get $\dot{\theta} = 2\,\mathrm{rad/s^2} \cdot 5\,\mathrm{s} = 10\,\mathrm{rad/s}$. Therefore,

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/O} &= 0.05\,\mathrm{kg} \cdot (0.2\,\mathrm{m})^2 \cdot 10\,\mathrm{rad/s}\hat{\boldsymbol{k}} \\
&= 0.02\,\mathrm{kg.\,m^2/\,s} = 0.02\,\mathrm{N{\cdot}m \cdot\ s.}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{H}}_{/O} = 0.02\,\mathrm{N{\cdot}m \cdot\ s.}}$$

(1) Be warned that these formulae are valid only for constant rate of change of speed.

2. Similarly, we can calculate the rate of change of angular momentum:

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_{/O} &= \vec{\boldsymbol{r}}_{/0} \times m\vec{\boldsymbol{a}} \\
&= R\hat{\boldsymbol{e}}_R \times m(R\ddot{\theta}\hat{\boldsymbol{e}}_\theta - \dot{\theta}^2 R\hat{\boldsymbol{e}}_R) \\
&= mR^2\ddot{\theta}(\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta) \\
&= mR^2\ddot{\theta}\hat{\boldsymbol{k}} \\
&= 0.05\,\mathrm{kg} \cdot (0.2m)^2 \cdot 2\,\mathrm{rad/s^2}\hat{\boldsymbol{k}} \\
&= 0.004\,\mathrm{kg \cdot\ m^2/\,s^2} = 0.004\,\mathrm{N{\cdot}m}
\end{aligned}$$

$$\boxed{\dot{\vec{\boldsymbol{H}}}_{/O} = 0.004\,\mathrm{N{\cdot}m}}$$

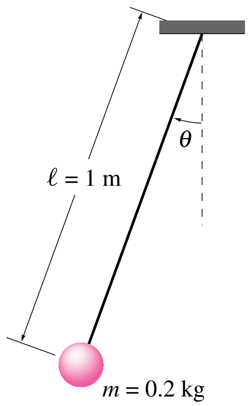

Figure 15.25

SAMPLE 15.8 The simple pendulum. A simple pendulum swings about its vertical equilibrium position (2-D motion) with amplitude $\theta_{max} = 10^\circ$. Find

1. the magnitude of the maximum angular acceleration,
2. the maximum tension in the string.

Solution

1. The equation of motion of the pendulum is given by (see eqn. (15.18) in the text):

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

We are given that $|\theta| \leq \theta_{max}$. For $\theta_{max} = 10^\circ = 0.1745\,\text{rad}$, $\sin\theta_{max} = 0.1736$. Thus we see that $\sin\theta \approx \theta$ even when θ is maximum. Therefore, we can safely use linear approximation (although we could solve this problem without it); i.e.,

$$\ddot{\theta} = -\frac{g}{\ell}\theta.$$

Clearly, $|\ddot{\theta}|$ is maximum when θ is maximum. Thus,

$$|\ddot{\theta}|_{max} = \frac{g}{\ell}\theta_{max} = \frac{9.81\,\text{m/s}^2}{1\,\text{m}}\cdot(0.1745\,\text{rad}) = 1.71\,\text{rad/s}^2.$$

$$\boxed{|\ddot{\theta}|_{max} = 1.71\,\text{rad/s}^2}$$

2. The tension in the string is given by (see equation 15.19 of text):

$$T = m(\ell\dot{\theta}^2 + g\cos\theta).$$

This time, we will not make the small angle assumption. We can find T_{max} and the corresponding θ using conservation of energy. Let the position of maximum amplitude be position 1 and the position at any θ be position 2. When $\theta = \theta_{max}$, the mass comes to rest and switches its direction of motion. Thus, its angular velocity and, hence, its kinetic energy is zero at θ_{max}. Using conservation of energy, we have

$$\begin{aligned} E_{K_1} + E_{P_1} &= E_{K_2} + E_{P_2} \\ 0 + mg\ell(1-\cos\theta_{max}) &= \frac{1}{2}m(\ell\dot{\theta})^2 + mg\ell(1-\cos\theta). \end{aligned} \tag{15.20}$$

and solving for $\dot{\theta}$, we get,

$$\dot{\theta} = \sqrt{\frac{2g}{\ell}(\cos\theta - \cos\theta_{max})}.$$

Therefore, the tension at any θ is

$$T(\theta) = m(\ell\dot{\theta}^2 + g\cos\theta) = mg(3\cos\theta - 2\cos\theta_{max}).$$

To find the maximum tension, we set $\frac{dT}{d\theta} = 0$, and find that, for $0 \leq \theta \leq \theta_{max}$, T is maximum when $\theta = 0$. Now, substituting $\theta = 0$ in $T(\theta)$, we get,

$$T_{max} = mg\big(3\cos(0) - 2\cos(\theta_{max})\big) = 0.2\,\text{kg}\cdot 9.81\,\text{m/s}^2(3 - 1.97) = 2.02\,\text{N}.$$

The maximum tension corresponds to maximum speed which occurs at the bottom of the swing where all of the potential energy is converted to kinetic energy.

$$\boxed{T_{max} = 2.02\,\text{N}}$$

SAMPLE 15.9 **The nonlinear pendulum:** Consider the simple pendulum of Sample 15.8 again. Let the mass be m and the length of the pendulum ℓ. The equation of motion of the pendulum is $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$ as derived in the text (see eqn. (15.18)). This is a nonlinear ordinary differential equation but it can be solved easily numerically. Write a computer code using some ODE solver to solve the equation. Take g and ℓ such that $\lambda = \sqrt{g/\ell} = 2\pi$ (this makes the time period of the pendulum $T = 2\pi/\lambda = 1\,\mathrm{s}$). Using the code, do the following calculations.

1. Solve the equation over a time interval of $t = 0$ to 4 seconds using the initial conditions $\theta(0) = 6°$ and $\dot{\theta}(0) = 0$, and plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ. How do these plots compare with the solution of the linear equation $\ddot{\theta} = -\frac{g}{\ell}\theta$?
2. Solve the equation again over the same time interval using the initial conditions $[\theta(0), \dot{\theta}(0)] = [18°, 0]$, and $[30°, 0]$. Plot $\theta(t)$ starting with all the three initial conditions used so far on the same graph and comment on the time period of oscillations.
3. Solve the equation again over $t \in [0, 4\,\mathrm{s}]$ using $\theta(0) = \pi/2$ and $\pi/1.02$ while keeping $\dot{\theta}(0) = 0$. Again plot θ vs t, $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ, for the three solutions obtained with $\theta(0) = \pi/6, \pi/2$, and $\pi/1.02$. Comment on the plots.
4. For the last three initial conditions, compute E_P, E_K, and $E_\mathrm{T} = E_\mathrm{P} + E_\mathrm{K}$ from the solutions obtained. For each initial condition, plot E_P, E_K, and E_T on the same graph and show that the total energy in each case remains constant irrespective of the nature of oscillations.

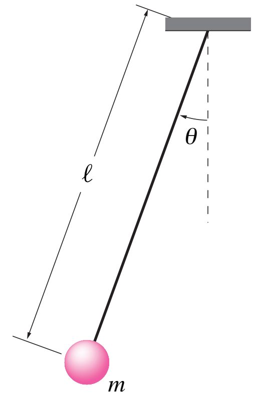

Figure 15.26

Solution The equation of motion of the pendulum is (as given)

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta.$$

To solve this second order differential equation numerically, we need to first convert it into a set of two first order equations. Let $\omega = \dot{\theta}$. Then, we can write

$$\begin{aligned}\dot{\theta} &= \omega \\ \dot{\omega} &= -\frac{g}{\ell}\sin\theta.\end{aligned}$$

We are now ready to write a computer program to solve these equations numerically. We use the following pseudocode to accomplish the task.

```
ODEs = {thetadot = omega,
        omegadot = -g/l*sin(theta) }
ICs   = {theta(0) = pi/30, omega(0) = 0 }
Set  g = 1, l = g/(4*pi^2)
Solve ODEs with ICs for t=0 to t=4
plot theta vs t, and omega vs t; plot omega vs theta
```

1. **Small amplitude oscillations:** The solution obtained with $\theta(0) = 6° = \pi/30$, and $\dot{\theta}(0) = 0$ is shown in fig. 15.27. The plots of $\theta(t)$ and $\dot{\theta}(t)$ clearly show the initial conditions at $t = 0$. From the figure, we see that the motion is sinusoidal and the time period of oscillation is 1 second, as expected.
2. **Deviation from linear equation solution:** The new initial conditions involve larger initial angles ($\theta(0) = 18°$ and $30°$). That is the only difference. We use the

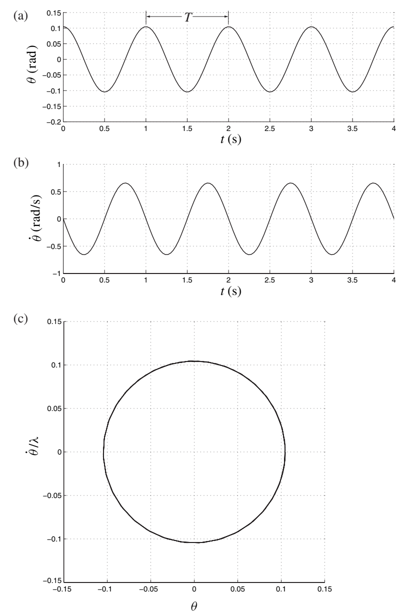

Figure 15.27: Numerical solution of the nonlinear pendulum equation, $\ddot{\theta} = -g/ell\sin\theta$, with the initial conditions, $\theta(0) = \pi/30$ and $\dot{\theta}(0) = 0$; (a) $\theta(t)$, (b) $\dot{\theta}(t)$, and (c) $\dot{\theta}$ vs θ (phase plane).

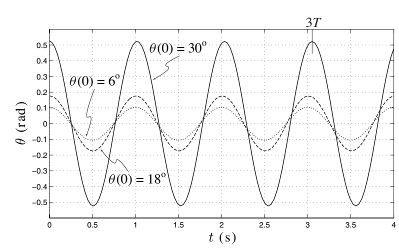

Figure 15.28: Comparison of $\theta(t)$ obtained from three different initial launch angles, $\theta(0) = 6°$, $18°$ and $30°$.

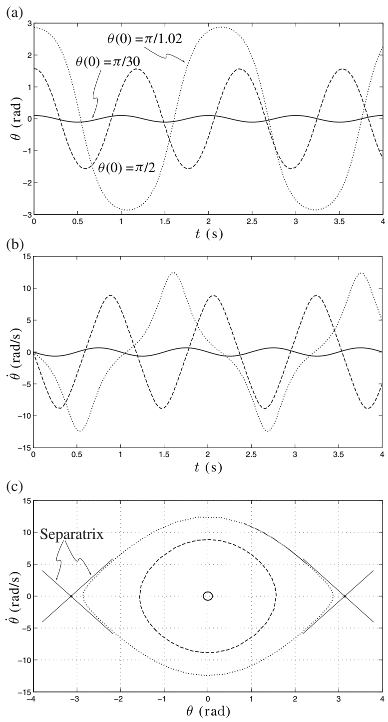

Figure 15.29: Comparison of pendulum motions when it is released from rest at small angle ($\theta(0) = \pi/30$), at horizontal position (($\theta(0) = \pi/2$), and at almost vertically upright position ($\theta(0) = \pi/1.02$); (a) θ vs t, (b) $\dot{\theta}$ vs t, and $\dot{\theta}$ vs θ (phase plane plot).

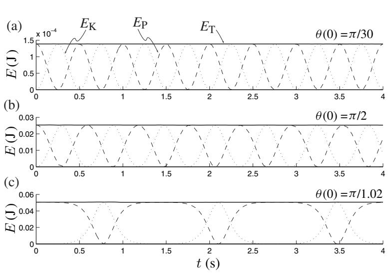

Figure 15.30: Plots of potential energy E_P, kinetic energy E_K, and total energy E_T during motion under three different initial conditions: (a) $\theta(0) = \pi/30$, $\dot{\theta}(0) = 0$; (b) $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$; and (c) $\theta(0) = \pi/1.02$, $\dot{\theta}(0) = 0$.

same program as used before and get the solutions with the new initial conditions. We plot $\theta(t)$ against t for all the three solutions on the same graph. The resulting plot is shown in fig. 15.28.

Now what we observe from this plot is that the three solutions, starting with the three different initial conditions, do not have the same time period of oscillations. The difference is not clearly visible between $\theta(0) = 6°$ and $\theta(0) = 18°$ solutions but it is much clearer for $\theta = 30°$ (see the third peak, marked with $3T$). As the initial angle, $\theta(0)$, increases, the period of oscillation seems to increase.

The dependence of time period (or frequency) of oscillations on the amplitude is the hallmark of nonlinear oscillators. In contrast, linear oscillators have a constant period of oscillation, irrespective of the amplitude of motion. For our pendulum, as long as the initial θ is so small that $\sin\theta \approx \theta$, the equation of motion can be replaced by the linear equation, $\ddot{\theta} = -g/\ell\theta$, and all solutions will have the same time period of oscillation. As $\theta(0)$ becomes larger, the approximation $\sin\theta \approx \theta$ breaks down, and the linear equation of motion is no longer valid.

3. **Large amplitude oscillations:** We now run the program with large initial angles, $\theta(0) = \pi/2\,(90°)$ and $\theta(0) = \pi/1.02$ ($\approx 176°$, *i.e.,* close to the vertically upright position), and obtain the corresponding solutions. Plots of $\theta(t)$ and $\dot{\theta}(t)$ for three initial conditions, small θ ($\pi/30$), moderately large θ ($\pi/2$), and very large θ ($\pi/1.02$) are shown in fig. 15.29. From the plots it is clear that not only the period of oscillation increases drastically with larger amplitudes, but also the qualitative nature of oscillations changes. For small amplitude (small initial θ), oscillations are simple harmonic but for larger amplitudes (large initial θ) oscillations are no more simple harmonic. This fact is more evident from the velocity plot, fig. 15.29(b). The phase plot, fig. 15.29(c), shows how the three solution trajectories (also called *orbits*) look in the phase space. All simple harmonic motions lead to circular orbits (you can show that by writing the solution for $\theta(t)$ and $\dot{\theta}(t)$ and then showing that $\theta^2 + \dot{\theta}^2$ = constant) in this phase space. However, for large amplitude motion, the orbits become oblong and approach a rather strange looking trajectory, called the *separatrix*, as the amplitude of motion grows. This separatrix marks the boundary of all possible periodic motions of the pendulum. Outside this separatrix, solutions do exist but they correspond to whirling motion of the pendulum which is not periodic (because $\theta(t)$ keeps growing without bounds).

4. **Energy conservation:** Let θ^* and $\dot{\theta}^*$ be the values of angular displacement and angular speed of the pendulum at some instant t^*. Then, assuming $\theta = 0$ to be the datum for potential energy, we can write the expressions for potential energy and kinetic energy as

$$\begin{aligned} E_\text{P} &= mg\ell(1-\cos\theta^*) \\ E_\text{K} &= \frac{1}{2}m\ell^2\dot{\theta}^{*2}. \end{aligned}$$

Therefore, the total energy at $t = t^*$ is,

$$E_\text{T} = E_\text{P} + E_\text{K} = mg\ell(1-\cos\theta^*) + \frac{1}{2}m\ell^2\dot{\theta}^{*2}.$$

From the numerical solutions obtained for the three initial conditions, we have values of θ and $\dot{\theta}$ at different time instants. Now, using the formulas for E_P, E_K and E_T, we compute the values of these quantities and plot them as shown in fig. 15.30. We see that for each initial condition, the potential and kinetic energies vary differently with time. However, the total energy remains constant at all times. This is expected as there is no dissipation in the system (not present in our mathematical model). A given initial condition determines the initial energy of the pendulum which must be preserved throughout the motion.