15.2 Dynamics of a particle in circular motion

Preparatory Problems

15.2.1 Force on a person standing on the equator. Find the magnitude of the total force acting on a 150 lbm person standing on the equator. The total force is the gravity force plus the force of the ground on the person (note that these two do not exactly cancel). Neglect the motion of the earth around the sun and of the sun around the solar system, etc. The radius of the earth is 3963 mi. Give your solution in both pounds (lbf) and Newtons (N). *

15.2.2 Consider a mass m in circular motion. Let $\sum \vec{F} = \sum F_r \hat{e}_r + \sum F_\theta \hat{e}_\theta$. Using $\sum \vec{F} = m\vec{a}$, express $\sum F_r$ and $\sum F_\theta$ in terms of some or all of $\theta, \dot{\theta}, \ddot{\theta}, , r$, and m.

15.2.3 Using $\sum \vec{F} = m\vec{a}$, find the expressions for $\sum F_x$ and $\sum F_y$ in terms of $\ddot{\theta}$, $\dot{\theta}$, r, and θ. [Hint $\hat{e}_r = \cos\theta \hat{\imath} + \sin\theta \hat{\jmath}$ and $\hat{e}_\theta = -\sin\theta \hat{\imath} + \cos\theta \hat{\jmath}$].

15.2.4 A bead of mass m goes around a circular path of radius R in the xy-plane with angular acceleration $\ddot{\theta} = ct^3$. The bead starts from rest at $\theta = 0$.

a) What is the angular momentum of the bead about the origin at $t = t_1$?

b) What is the rate of change of angular momentum about the origin at $t = t_1$?

c) What is the kinetic energy of the bead at $t = t_1$?

d) Does the kinetic energy increase, decrease, or remain constant with time? Why?

15.2.5 A 200 gm particle goes in circles about a fixed center at a constant speed $v = 1.5\,\text{m/s}$. It takes 7.5 s to go around the circle once.

a) Find the angular speed of the particle.

b) Find the magnitude of acceleration of the particle.

c) Take center of the circle to be the origin of a xy-coordinate system. Find the net force on the particle when it is at $\theta = 30^\circ$ from the x-axis.

15.2.6 A race car cruises on a circular track at a constant speed of 120 mph. It goes around the track once in three minutes. Find the magnitude of the centripetal force on the car. What applies this force on the car? Does the driver have any control over this force?

15.2.7 A particle moves on a counter-clockwise, origin-centered circular path in the xy-plane at a constant rate. The radius of the circle is r, the mass of the particle is m, and the particle completes one revolution in time τ.

a) Neatly draw the following things:

1. The path of the particle.
2. A dot on the path when the particle is at $\theta = 0°$, $90°$, and $210°$, where θ is measured from the x-axis (positive counter-clockwise).
3. Arrows representing $\hat{e}_R$, $\hat{e}_\theta$, $\vec{v}$, and $\vec{a}$ at each of these points.

b) Calculate all of the quantities in part (3) above at the points defined in part (2), (represent vector quantities in terms of the cartesian base vectors $\hat{\imath}$ and $\hat{\jmath}$). *

c) If this motion was imposed by the tension in a string, what would that tension be? *

d) Is radial tension enough to maintain this motion or is another force needed to keep the motion going (assuming no friction) ? *

e) Again, if this motion was imposed by the tension in a string, what is F_x, the x component of the force in the string, when $\theta = 210°$? Ignore gravity.

15.2.8 The velocity and acceleration of a 1 kg particle, undergoing constant rate circular motion, are known at some instant t:

$$\vec{v} = -10\,\text{m/s}(\hat{\imath} + \hat{\jmath}), \quad \vec{a} = 2\,\text{m/s}^2(\hat{\imath} - \hat{\jmath}).$$

a) Write the position of the particle at time t using $\hat{e}_R$ and $\hat{e}_\theta$ base vectors.

b) Find the net force on the particle at time t.

c) At some later time t^*, the net force on the particle is in the $-\hat{\jmath}$ direction. Find the elapsed time $t - t^*$.

d) After how much time does the force on the particle reverse its direction.

15.2.9 A particle of mass 3 kg moves in the xy-plane so that its position is given by

$$\vec{r}(t) = 4\,\text{m}\left[\cos\left(\frac{2\pi t}{s}\right)\hat{\imath} + \sin\left(\frac{2\pi t}{s}\right)\hat{\jmath}\right]$$

with respect to point O, the origin of a fixed cartesian coordinate system.

a) What is the path of the particle? Show how you know what the path is.

b) What is the angular velocity of the particle? Is it constant? Show how you know if it is constant or not.

c) What is the velocity of the particle in polar coordinates?

d) What is the speed of the particle at $t = 3\,\text{s}$?

e) What net force does it exert on its surroundings at $t = 0\,\text{s}$? Assume the x and y axes are fixed.

f) What is the angular momentum of the particle at $t = 3\,\text{s}$ about point O?

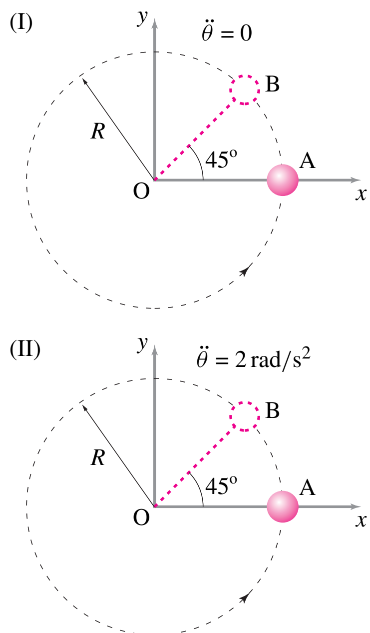
15.2.10 A comparison of constant and nonconstant rate circular motion. A 100 gm mass is going in circles of radius $R = 20\,\text{cm}$ at a constant rate $\dot{\theta} = 3\,\text{rad/s}$. Another identical mass is going in circles of the same radius but at a non-constant rate. The second mass is accelerating at $\ddot{\theta} = 2\,\text{rad/s}^2$ and at position A, it happens to have the same angular speed as the first mass.

a) Find and draw the accelerations of the two masses (call them I and II) at position A.

b) Find $\dot{\vec{H}}_{/O}$ for both masses at position A. *

c) Find $\vec{H}_{/O}$ for both masses at positions A and B. Do the changes in $\vec{H}_{/O}$ between the two positions reflect (qualitatively) the results obtained in (b)? *

d) If the masses are pinned to the center O by massless rigid rods, is tension in the rods enough to keep the two motions going? Explain.



Problem 15.2.10

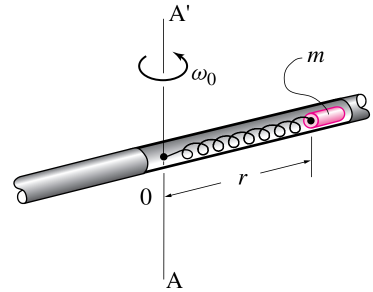
More-Involved Problems

15.2.11 A small mass m is connected to one end of a spring. The other end of the spring is fixed to the center of a circular track. The radius of the track is R, the unstretched length of the spring is ℓ_0 (with $\ell_0 < R$), and the spring constant is k.

a) With what speed should the mass be launched in the track so that it keeps going at a constant speed?

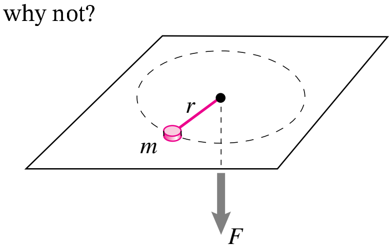
b) If the spring is replaced by another spring of same relaxed length but twice the stiffness, what will be the new required launch speed of the particle?

15.2.12 A bead of mass m is attached to a spring of stiffness k. The bead slides without friction in the tube shown. The tube is driven at a constant angular rate $\dot{\theta}_0$ about axis AA' by a motor (not pictured). There is no gravity. The unstretched spring length is r_0. Find the radial position r of the bead if it is stationary with respect to the rotating tube. *



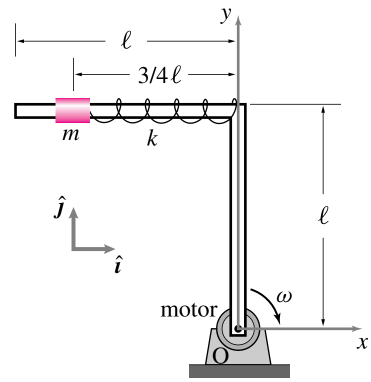
Problem 15.2.12

15.2.13 A particle of mass m is restrained by a string to move with a constant angular speed ω around a circle of radius R on a horizontal frictionless table. If the radius of the circle is reduced slowly to r, by pulling the string with a slowly varying force F through a hole in the table, what will the particle's angular velocity be in the final circular motion? Is kinetic energy changed in moving from circular motion at R to circular motion at r? Why or why not?



Problem 15.2.13

15.2.14 An 'L' shaped rigid, massless, and frictionless bar is made up of two uniform segments of length $\ell = 0.4\,\mathrm{m}$ each. A collar of mass $m = 0.5\,\mathrm{kg}$, attached to a spring at one end, slides frictionlessly on one of the arms of the 'L'. The spring is fixed to the elbow of the 'L' and has a spring constant $k = 6\,\mathrm{N/m}$. The structure rotates clockwise at a constant rate $\omega = 2\,\mathrm{rad/s}$. If the collar is steady at a distance $\frac{3}{4}\ell = 0.3\,\mathrm{m}$ away from the elbow of the 'L', find the relaxed length of the spring, ℓ_0. Neglect gravity. *

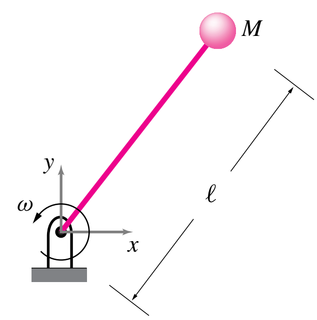


Problem 15.2.14: **Forces in constant rate circular motion.**

15.2.15 A massless rigid rod with length ℓ attached to a ball of mass M spins at a constant angular rate ω which is maintained by a motor (not shown) at the hinge point. The rod can only withstand a tension of T_{cr} before breaking. Find the maximum angular speed of the ball so that the rod does not break assuming

a) there is no gravity, and

b) there is gravity (neglect bending stresses).



Problem 15.2.15

15.2.16 A 1 m long massless string has a particle of 10 grams mass at one end and is tied to a stationary point O at the other end. The particle rotates counter-clockwise in circles on a frictionless horizontal plane. The rotation rate is 2π rev/sec. Assume an xy-coordinate system in the plane with its origin at O.

a) Make a clear sketch of the system.

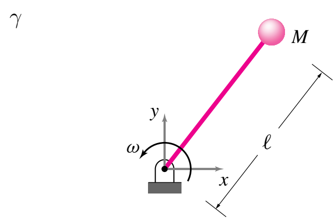
b) What is the tension in the string (in Newtons)? *

c) What is the angular momentum of the mass about O? *

d) When the string makes a 45° angle with the positive x and y axis on the plane, the string is quickly and cleanly cut. What is the position of the mass 1 sec later? Make a sketch of the particle's trajectory. *

15.2.17 A ball of mass M fixed to an inextensible rod of length ℓ and negligible mass rotates about a frictionless hinge as shown in the figure. A motor (not shown) at the hinge point accelerates the mass-rod system from rest by applying a constant torque M_O. The rod is initially lined up with the positive x-axis. The rod can only withstand a tension of T_{cr} before breaking. At what time will the rod break and after how many revolutions? Neglect bending stresses.

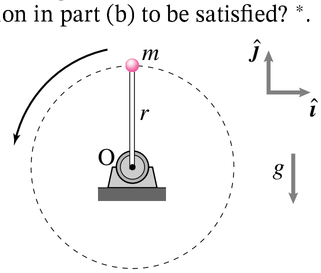
a) Neglect gravity.
b) Include gravity.



Problem 15.2.17

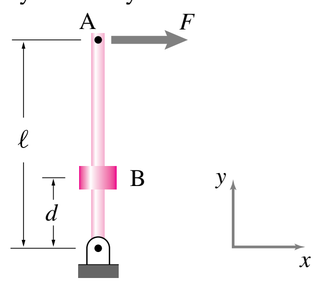
15.2.18 A particle of mass m, tied to one end of a rod whose other end is fixed at point O to a motor, moves in a circular path in the vertical plane at a constant rate. Gravity acts in the $-\hat{\jmath}$ direction.

a) Find the difference between the maximum and minimum tension in the rod. *
b) Find the ratio $\frac{\Delta T}{T_{max}}$ where $\Delta T = T_{max} - T_{min}$. A criterion for ignoring gravity might be if the variation in tension is less than 2% of the maximum tension; i.e., when $\frac{\Delta T}{T_{max}} < 0.02$. For a given length r of the rod, find the rotation rate ω for which this condition is met. *.
c) For $\omega = 300\,\text{rpm}$, what would be the length of the rod for the condition in part (b) to be satisfied? *.



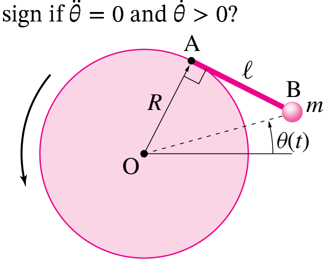
Problem 15.2.18

15.2.19 A massless rigid bar of length L is hinged at the bottom. A force F is applied at point A at the end of the bar. A mass m is glued to the bar at point B, a distance d from the hinge. There is no gravity. What is the acceleration of point A at the instant shown? Assume the angular velocity is initially zero.



Problem 15.2.19

15.2.20 The mass m is attached rigidly to the rotating disk by the light rod AB of length ℓ. Neglect gravity. Find M_A (the moment on the rod AB from its support point at A) in terms of $\dot{\theta}$ and $\ddot{\theta}$. What is the sign of M_A if $\dot{\theta} = 0$ and $\ddot{\theta} > 0$? What is the sign if $\ddot{\theta} = 0$ and $\dot{\theta} > 0$?



Problem 15.2.20

Pendulum problems

15.2.21 Simple pendulum, comprehensive version. This problem covers many aspects of a simple pendulum. A point mass M hangs on a massless string or rod of length L. The gravitational force is Mg. The pendulum is in a vertical plane. At any time t, the angle between the straight down line and the pendulum, measured counter clockwise, is $\theta(t)$. Neglect air friction. When numbers are called for use $M = 1\,\text{kg}$, $L = 1\,\text{m}$ and $g = 10\,\text{m/s}^2$.

a) **Find the equations of motion.** That is, assume that you know both θ and $\dot{\theta}$, find $\ddot{\theta}$. There are several ways to do this problem.* Find the equations using
   1. Linear momentum balance
   2. Angular momentum balance
   3. Conservation of energy
b) **Tension.** Assuming that you know θ and $\dot{\theta}$, find the tension T in the string.
c) **Reaction components.** Assuming you know θ and $\dot{\theta}$, find the x and y components of the force that the hinge support causes on the pendulum. Clearly define the directions of positive x and y with a sketch.
d) **Reduction to first order equations.** The equation that you found in (a) is a nonlinear second order ordinary differential equation. It can be changed to a pair of first order equations by defining a new variable $\omega \equiv \dot{\theta}$. Write the equation from (a) as a pair of first order equations. *
e) **Numerical solution.** Given the initial conditions $\theta(t=0) = \pi/2$ and $\omega(t=0) = \dot{\theta}(t=0) = 0$. Using numerical integration, find: $\theta(t)$, $\dot{\theta}(t)$ & $T(t)$. Make a single plot, or three vertically aligned plots, of these variables for one full oscillation of the pendulum.
f) **Maximum tension.** Using your numer-
ical solutions, find the maximum value of the tension in the rod as the mass swings. *
g) Plot the x and y reaction components as a function of time.
h) **Period of oscillation.** How long does it take to make one oscillation?
i) **Other observations.** Some questions:
   1. Does the solution to (f) depend on the length of the string?
   2. Is the solution to (f) exactly 30 or just a number near 30? If it is exact can you find the result analytically?
   3. Is the period found in (h) longer or shorter than the period found by solving the linear equation $\ddot{\theta} + (g/l)\theta = 0$, based on the (inappropriate-to-use in this case) small angle approximation $\sin\theta = \theta$? Explain intuitively why you expect the period to be longer or shorter?

15.2.22 Tension in a simple pendulum string. A simple pendulum of length 2 m with mass 3 kg is released from rest at an initial angle of 60° from the vertically down position.

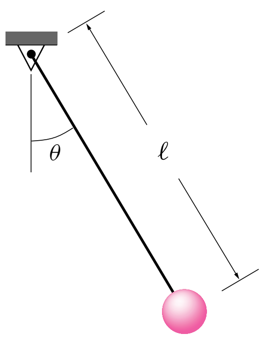
a) What is the tension in the string just after the pendulum is released?

b) What is the tension in the string when the pendulum has reached 30° from the vertical?

15.2.23 Cartesian coordinates Find the nonlinear governing differential equation for a simple pendulum

$$\ddot{\theta} = -\frac{g}{\ell}\sin\theta$$

using linear momentum in Cartesian coordinates and without using the polar coordinate formulas for velocity and acceleration. Of course you can use that $x = \ell\cos\theta$ and $y = \ell\sin\theta$ for x pointing down.

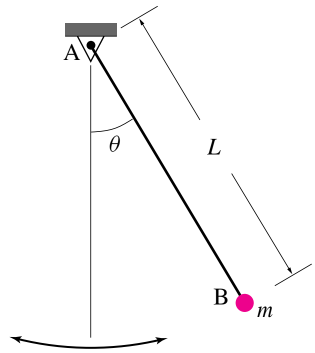


Problem 15.2.23

15.2.24 Tension in a rope-swing rope. Model a swinging person as a point mass. The swing starts from rest at an angle $\theta = 90°$. When the rope passes through vertical the tension in the rope is higher (it is hard to hang on). A person wants to know ahead of time if she is strong enough to hold on. How hard does she have to hang on compared, say, to her own weight? You are to find the solution two ways. Use the same m, g, and L for both solutions.

a) Find $\ddot{\theta}$ as a function of g, L, θ, and m. This equation is the governing differential equation. Write it as a system of first order equations. Solve them numerically. Once you know $\dot{\theta}$ at the time the rope is vertical you can use other mechanics relations to find the tension. If you like, you can plot the tension as a function of time as the mass falls.

b) Use conservation of energy to find $\dot{\theta}$ at $\theta = 0$. Then use other mechanics relations to find the tension. *



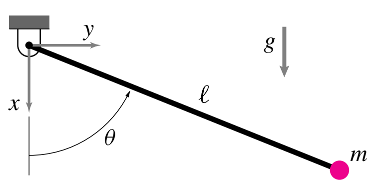
Problem 15.2.24

15.2.25 Pendulum. A pendulum with a negligible-mass rod and point mass m is released from rest at the horizontal position $\theta = \pi/2$.

a) Find the acceleration (a vector) of the mass just after it is released at $\theta = \pi/2$ in terms of ℓ, m, g and any base vectors you define clearly.

b) Find the acceleration (a vector) of the mass when the pendulum passes through the vertical at $\theta = 0$ in terms of ℓ, m, g and any base vectors you define clearly

c) Find the string tension when the pendulum passes through the vertical at $\theta = 0$ (in terms of ℓ, m and g).



Problem 15.2.25

15.2.26 Write a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} = -\frac{g}{\ell}\sin\theta$, over a given time interval $(0, t)$, and initial conditions $\theta(0)$, and $\dot{\theta}(0)$. The output should be a vector of time instants, t_i, in the given time interval and the corresponding θ_i and $\dot{\theta}_i$.

Now use your computer program to find the solution of

$$\ddot{\theta} = -\sin\theta, \qquad \theta(0) = \pi/4,\ \dot{\theta}(0) = 0.$$

Compare the solution obtained with the analytical solution of the corresponding simple pendulum equation, $\ddot{\theta} = -\theta$ with the same initial conditions. In particular,

a) Find the difference in the time period of oscillations of the two systems.

b) Plot $\dot{\theta}$ obtained from the two solutions against time and comment on the differences.

c) Plot $\dot{\theta}$ against θ from the two solutions on the same plot and compare the two phase portraits. Comment on the differences.

15.2.27 Solve the nonlinear pendulum equation numerically taking 20 different initial angular positions between $\theta = 0$ and $\theta = \pi$, each time releasing the pendulum gently from rest. Find the time period of oscillation, T, from each solution and plot it against the amplitude of motion, *i.e.,* $\theta(0)$.

a) How does the period of oscillation depend on the amplitude for small amplitudes?

b) What is the limiting value of the time period for large amplitudes, *i.e.,* $\theta(0) \to \pi$?

c) How does T depend on the amplitude over the entire range?

15.2.28 A pendulum of mass m and length ℓ is released from rest at $\theta(0) = 60°$. It executes oscillatory motion. If the pendulum were to be released from two different positions, $\theta(0) = 45°$ and $\theta(0) = 0°$, with some corresponding initial angular speed such that the ensuing motion were exactly the same as that with $\theta(0) = 60°$ and $\dot{\theta}(0) = 0$, find the required initial angular speeds.

a) First, find the corresponding $\dot{\theta}(0)$ without any computer simulation.

b) Verify your answer by plotting computer generated solutions for the three different initial conditions.

c) What is the general relationship between $\theta(0)$ and $\dot{\theta}(0)$ that produces a predefined motion generated by, say, a given set of $\theta(0) = \theta_0$ and $\dot{\theta}(0) = \dot{\theta}_0$.

15.2.29 Use a computer program to solve the nonlinear pendulum equation, $\ddot{\theta} + \lambda^2 \sin\theta = 0$, where $\lambda^2 = 1.56/\mathrm{s}^2$, with the following 11 initial conditions: $[\theta(0), \dot{\theta}(0)]$ = [1, 0], [2, 0], [3, 0], [4, -1], [-4, 1], [4, -1.02], [-4, 1.02], [4, -1.1], [-4, 1.1], [4, 1.4], and [-4, 1.4], where θ is in radians and $\dot{\theta}$ in rad/s. Obtain each solution over the time interval $t = 0$ to $t = 20\,\mathrm{s}$.

a) Plot all solutions in the phase space (*i.e.,* θ vs $\dot{\theta}$) in a single graph.

b) What does the extension of the plot beyond $\theta = \pm\pi$ mean?

c) Which initial conditions give solutions outside the separatrix? What do these solutions mean? Are these solutions periodic?

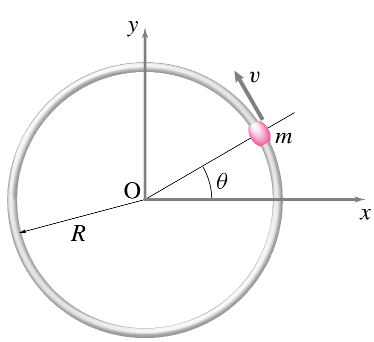
d) If you added a little bit of viscous damping to the pendulum motion, can you guess what will happen to the solutions inside the separatrix? [Hint: think about energy associated with these solutions.]

More circular motion problems

15.2.30 Bead on a hoop with friction. A bead slides on a rigid, stationary, circular wire. The coefficient of friction between the bead and the wire is μ. The bead is loose on the wire (not a tight fit but not so loose that you have to worry about rattling). Assume gravity is negligible.

a) Given v, m, R, & μ; what is $\dot{v}$? *

b) If $v(\theta = 0) = v_0$, how does v depend on θ, μ, v_0 and m? *

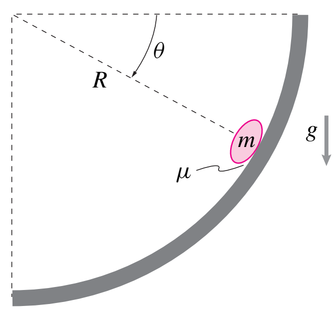


Problem 15.2.30

15.2.31 Particle in a chute. One of a million non-interacting rice grains is sliding in a circular chute with radius R. Its mass is m and it slides with coefficient of friction μ (Actually it slides, rolls and tumbles — μ is just the effective coefficient of friction from all of these interactions.) Gravity g acts downwards.

a) Find a differential equation that is satisfied by θ that governs the speed of the rice as it slides down the hoop. Parameters in this equation can be m, g, R and μ [Hint: Draw FBD, write eqs of mechanics, express as ODE.]

b) Find the particle speed at the bottom of the chute if $R = 0.5m$, $m = 0.1$ grams, $g = 10\,\mathrm{m/s^2}$, and $\mu = 0.2$ as well as the initial values of $\theta_0 = 0$ and its initial downward speed is $v_0 = 10\,\mathrm{m/s}$. [Hint: you are probably best off using a numerical solution.]

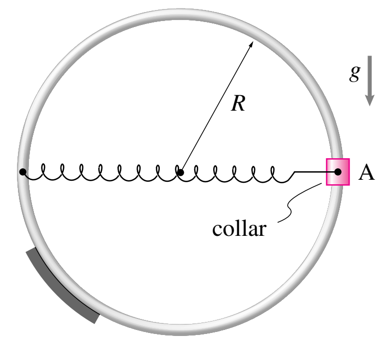


Problem 15.2.31

15.2.32 Due to a push which happened in the past, the collar with mass m is sliding up at speed v_0 on the circular ring when it passes through the point A. The ring is frictionless. A spring of constant k and unstretched length R is also pulling on the collar.

a) What is the acceleration of the collar at A. Solve in terms of R, v_0, m, k, g and any base vectors you define.

b) What is the force on the collar from the ring when it passes point A? Solve in terms of R, v_0, m, k, g and any base vectors you define.

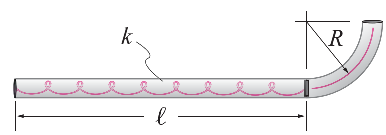


Problem 15.2.32

15.2.33 A toy used to shoot pellets is made out of a thin tube which has a spring of spring constant k on one end. The spring is placed in a straight section of length ℓ; it is unstretched when its length is ℓ. The straight part is attached to a (quarter) circular tube of radius R, which points up in the air.

a) A pellet of mass m is placed in the device and the spring is pulled to the left by an amount $\Delta\ell$. Ignoring friction along the travel path, what is the pellet's velocity $\vec{v}$ as it leaves the tube? *

b) What force acts on the pellet just prior to its departure from the tube? What about just after? *

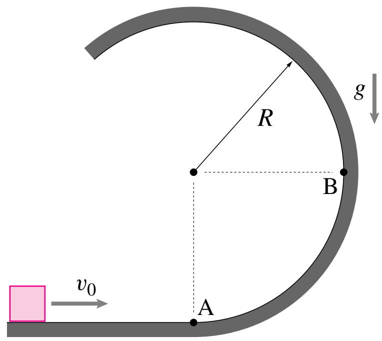


Problem 15.2.33

15.2.34 A block with mass m is moving to the right at speed v_0 when it reaches a circular frictionless portion of the ramp.

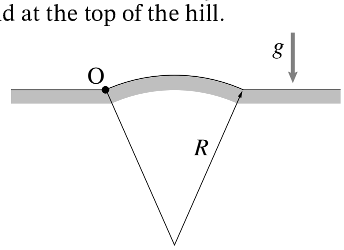
a) What is the speed of the block when it reaches point B? Solve in terms of R, v_0, m and g.

b) What is the force on the block from the ramp just after it gets onto the ramp at point A? Solve in terms of R, v_0, m and g. Remember, force is a vector.



Problem 15.2.34

15.2.35 A car moves with speed v along the surface of the hill shown which can be approximated as a circle of radius R. The car starts at a point on the hill at point O. Compute the magnitude of the speed v such that the car just leaves the ground at the top of the hill.



Problem 15.2.35