15.1 Kinematics of a particle in planar circular motion

This section concerns the position, velocity and accelerations of one point going in circles. The essence of the content here is this:

> If $\hat{\boldsymbol{e}}_R$ is a unit vector in the plane that is rotating counter-clockwise (CCW) at a rate of $\dot{\theta}$ its rate of change is
> $$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$$
> where $\hat{\boldsymbol{e}}_\theta$ is a unit vector given by rotating $\hat{\boldsymbol{e}}_R$ 90° CCW.

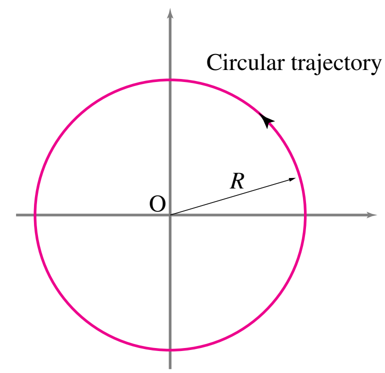


Figure 15.1: Trajectory of particle for circular motion.

If you learn this idea inside and out then either you will have picked up all the other key facts on the way, or you will be able to learn them in a flash. Note, we use $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{r}}$ interchangeably. Likewise for $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_R$.

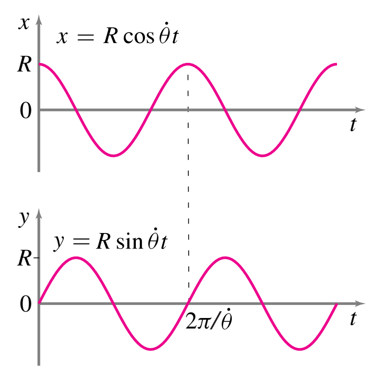


Figure 15.2: Plots of x versus t and y versus t for a particle going in a circle of radius R at constant rate. For simplicity we assumed constant $\dot{\theta}$ with $\theta = \dot{\theta}t$. So both x and y vary as sinusoidal functions of time: $x = R\cos(\dot{\theta}t)$ and $y = R\sin(\dot{\theta}t)$.

Circular motion

The position of a particle going in circles around the origin on the xy plane is

$$\vec{\boldsymbol{r}} = R\cos\theta\,\hat{\boldsymbol{\imath}} + R\sin\theta\,\hat{\boldsymbol{\jmath}},$$

with the radius R a constant. Or, in terms of components,

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta.$$

A natural graphical representation of this motion is a circle (fig. 15.1). Unfortunately, a picture of the circular trajectory doesn't give any information about the speed of the particle on the circle. A plot of a particle moving in circles slowly looks just like a plot of a particle moving quickly.

To get a sense of how position changes in time one can plot the functions $x(t)$ and $y(t)$ (fig. 15.2). Unfortunately this figure only indirectly conveys that the particle is going in circles.

If you want to see both the trajectory and the time history of both variables one can make a 3-D plot of xy position versus time (fig. 15.3). The shadows of this helix on the three coordinate planes are the three graphs just discussed.

Finally, rather than representing time as a spatial coordinate, one can represent time with time itself. How? Make an animated movie showing a particle on the xy plane as it moves. Move your finger around in circles on the table. That's it. Similarly, you can make a dot move in circles on your screen. How do you make all these plots? Using a calculator or computer you can evaluate x and y for a range of values of t. Then, using pencil and paper, a plotting calculator, or a computer, plot x *vs* t, y *vs* t, and y *vs* x. For animations plot x and y over and over again for a sequence of values of t, and show these on your screen at a sequence of times.

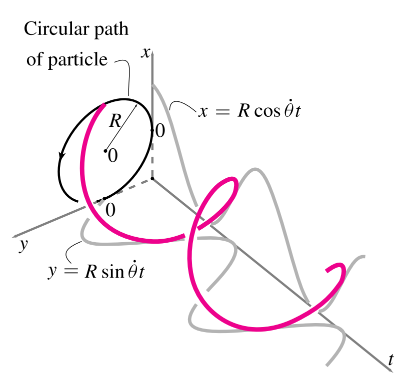


Figure 15.3: Plot of x and y versus time for a particle going in circles at constant rate. x versus t is a cosine curve, y versus t is a sine curve. Together they make up the 3D helix.

Polar coordinates R and θ and unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$

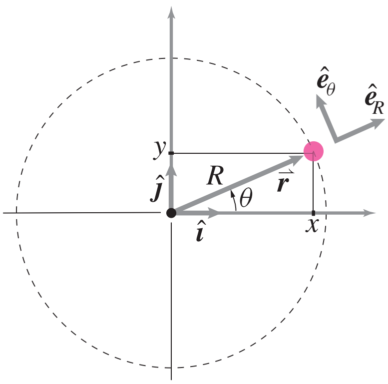


Figure 15.4: The position vector $\vec{\boldsymbol{r}}$ of the particle relative to the center of the circle is $\vec{\boldsymbol{r}}$ (or $\vec{\boldsymbol{R}}$) which is both
$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ and
$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R$ (or $r\hat{\boldsymbol{e}}_r$).
$\vec{\boldsymbol{r}}$ makes an angle θ measured counter-clockwise from the positive x-axis. The unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are in the radial and tangential directions, the directions of increasing R and increasing θ.

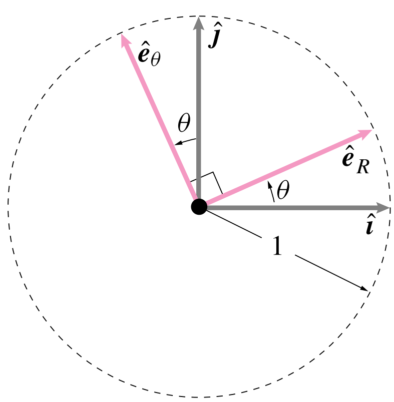


Figure 15.5: You can think of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ as the set $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ rotated counter-clockwise by the angle θ.

Especially for circular motion, it is convenient to represent position, velocity and acceleration with polar, rather than rectangular, coordinates. With polar coordinates we use polar base vectors which, unlike the fixed $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, rotate as the particle goes around. Let's redraw fig. 15.1 and show the unit base vectors

$\hat{\boldsymbol{e}}_R$ ('e R') and $\hat{\boldsymbol{e}}_\theta$ ('e theta').

The radial unit vector $\hat{\boldsymbol{e}}_R$ is directed from the center of the circle towards the point of interest and the transverse vector $\hat{\boldsymbol{e}}_\theta$, perpendicular to $\hat{\boldsymbol{e}}_R$, is tangent to the circle at that point in the direction of increasing θ. As the particle goes around, its $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ unit vectors change accordingly. Two different particles both going in circles with the same center at the same rate each have their own $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ vectors. To be precise we can define $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ as

$$\hat{\boldsymbol{e}}_R \equiv \vec{\boldsymbol{R}}/|\vec{\boldsymbol{R}}| = \vec{\boldsymbol{R}}/R \qquad \text{and} \qquad \hat{\boldsymbol{e}}_\theta \equiv \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_R. \tag{15.1}$$

Note that also $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$.

The velocity and acceleration of a point going in circles, using polar coordinates

In dynamics we are interested in velocity and acceleration so we need to know how to represent these in polar coordinates. First, observe that the position of the particle is (see fig. 15.4)

$$\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R. \tag{15.2}$$

That is, the position vector is the distance from the origin times a unit vector in the direction of the particle's position. Given the position, it is just a matter of *careful* differentiation to find velocity and acceleration. Here is one of many possible ways to derive the polar-coordinate expressions for velocity and acceleration. First, velocity is the time derivative of position, so

$$\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{R}} = \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) = \underbrace{\dot{R}}_{0}\,\hat{\boldsymbol{e}}_R + R\dot{\hat{\boldsymbol{e}}}_R. \tag{15.3}$$

Because a circle has constant radius R, $\dot{R}$ is zero. But how do we calculate the rate of change of $\hat{\boldsymbol{e}}_R$ with respect to time, $\dot{\hat{\boldsymbol{e}}}_R$,?

Derivatives of $\hat{\boldsymbol{e}}_R$ and of $\hat{\boldsymbol{e}}_\theta$

To find the velocity in polar coordinates we were just confronted with the problem of finding $\dot{\hat{\boldsymbol{e}}}_R$.

Method 1: One way to find $d\hat{\boldsymbol{e}}_R/dt = \dot{\hat{\boldsymbol{e}}}_R$ uses the geometry of fig. 15.7 and the informal calculus of finite differences (represented by Δ). $\Delta\hat{\boldsymbol{e}}_R$ is evidently (about) in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude $\Delta\theta$ so $\Delta\hat{\boldsymbol{e}}_R \approx (\Delta\theta)\hat{\boldsymbol{e}}_\theta$. Dividing by Δt, we have $\Delta\hat{\boldsymbol{e}}_R/\Delta t \approx (\Delta\theta/\Delta t)\hat{\boldsymbol{e}}_\theta$. So, using this sloppy calculus, we get $\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$. Similarly, and we will need this shortly, we could get $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$.

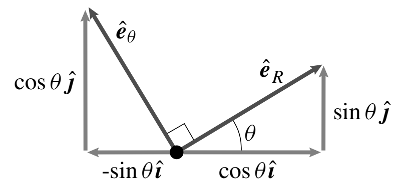


Figure 15.6: Projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y directions. From this picture you can immediately extract that
$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}$ and that
$\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}$.
A similar picture showing the projections of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ d directions would show that
$\hat{\boldsymbol{\imath}} = \cos\theta\hat{\boldsymbol{e}}_R - \sin\theta\hat{\boldsymbol{e}}_\theta$ and that
$\hat{\boldsymbol{\jmath}} = \sin\theta\hat{\boldsymbol{e}}_R + \cos\theta\hat{\boldsymbol{e}}_\theta$.

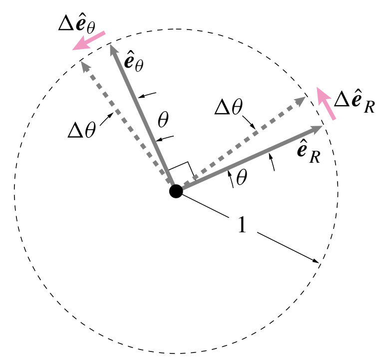


Figure 15.7: A close up view of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$. They make an angle θ with the positive x and y-axis, respectively. As the particle advances an amount $\Delta\theta$ both $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ change. In particular, for small $\Delta\theta$, $\Delta\hat{\boldsymbol{e}}_R$ is approximately in the $\hat{\boldsymbol{e}}_\theta$ direction and $\Delta\hat{\boldsymbol{e}}_\theta$ is approximately in the $-\hat{\boldsymbol{e}}_R$ direction.

Method 2 This method is a little less geometric and a little more algebraic. We start with the decomposition of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ into cartesian coordinates. These decompositions are found by looking at the projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y-directions (see fig. 15.6).

$$\begin{aligned} \hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}} \end{aligned} \tag{15.4}$$

We can find $\dot{\hat{\boldsymbol{e}}}_R$ by differentiating, taking into account that θ is changing with time but that the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are fixed (so they don't change with time).

$$\begin{aligned} \dot{\hat{\boldsymbol{e}}}_R &= \frac{d}{dt}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\sin\theta\hat{\boldsymbol{\imath}} + \dot{\theta}\cos\theta\hat{\boldsymbol{\jmath}} = \dot{\theta}\hat{\boldsymbol{e}}_\theta \\ \dot{\hat{\boldsymbol{e}}}_\theta &= \frac{d}{dt}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\hat{\boldsymbol{\jmath}}) = -\dot{\theta}\hat{\boldsymbol{e}}_R \end{aligned}$$

We had to use the chain rule, that is

$$\frac{d\sin\theta(t)}{dt} = \frac{d\sin\theta}{d\theta}\frac{d\theta(t)}{dt} = \dot{\theta}\cos\theta.$$

Now, two different ways, we know

$$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta \quad \text{and} \quad \dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R. \tag{15.5}$$

Continuing the quest for velocity and acceleration

Now that we know how $\hat{\boldsymbol{e}}_R$ changes in time we can continue our quest for $\vec{\boldsymbol{v}}$. Continuing from eqn. (15.3) we now have

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{R}}} = R\dot{\hat{\boldsymbol{e}}}_R = R\dot{\theta}\hat{\boldsymbol{e}}_\theta. \tag{15.6}$$

Similarly we can find the acceleration $\ddot{\vec{\boldsymbol{R}}}$ by differentiating once again,

$$\vec{\boldsymbol{a}} = \ddot{\vec{\boldsymbol{R}}} = \dot{\vec{\boldsymbol{v}}} = \frac{d}{dt}(R\dot{\theta}\hat{\boldsymbol{e}}_\theta) = \underbrace{\dot{R}\dot{\theta}\hat{\boldsymbol{e}}_\theta}_{\vec{\boldsymbol{0}}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta + R\dot{\theta}\dot{\hat{\boldsymbol{e}}}_\theta \tag{15.7}$$

The first term on the right hand side is zero because $\dot{R}$ is 0 for circular motion. The third term is evaluated using the formula we just found for the rate of change of $\hat{\boldsymbol{e}}_\theta$: $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. So, using that $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$,

$$\vec{\boldsymbol{a}} = -\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta \tag{15.8}$$

The velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ for a particle going in circles at constant rate are shown in fig. 15.8.

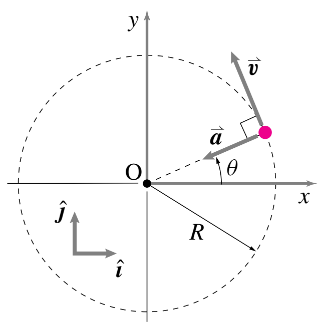


Figure 15.8: The directions of velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ are shown for a particle going in circles at *constant* rate. The velocity is tangent to the circle and the acceleration is directed towards the center of the circle.

Example: A person standing on the earth's equator
A person standing on the equator has velocity

$$\vec{\boldsymbol{v}} = \dot{\theta}R\hat{\boldsymbol{e}}_\theta \approx \left(\frac{2\pi\,\text{rad}}{24\,\text{hr}}\right) 4000\,\text{mi}\hat{\boldsymbol{e}}_\theta \approx 1050\,\text{mph}\hat{\boldsymbol{e}}_\theta \approx 1535\,\text{ft/s}\hat{\boldsymbol{e}}_\theta$$

and acceleration

$$\vec{\boldsymbol{a}} = -\dot{\theta}^2R\hat{\boldsymbol{e}}_R \approx -\left(\frac{2\pi\,\text{rad}}{24\,\text{hr}}\right)^2 4000\,\text{mi}\hat{\boldsymbol{e}}_R \approx -274\,\text{mi/hr}^2\hat{\boldsymbol{e}}_R \approx -0.11\,\text{ft/s}^2\hat{\boldsymbol{e}}_R.$$

The velocity of a person standing on the equator, due to the earth's rotation, is about 1000 mph tangent to the earth. Her acceleration is about $0.11\,\text{ft/s}^2 \approx 0.03\,\text{m/s}^2$ towards the center of the earth, about 1/300 of g, about 1/300 the acceleration of an object in near-earth-surface frictionless free-fall.

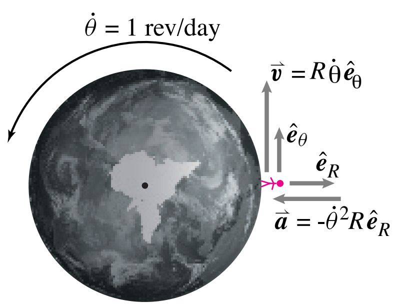


Figure 15.9

Alternate expressions for the velocity and acceleration formulas

Note that we can define a scalar velocity $v = R\dot{\theta}$. We informally call this scalar the speed even though it can be positive or negative. So

$$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta = v\hat{\boldsymbol{e}}_\theta.$$

Similarly the acceleration is

$$\vec{\boldsymbol{a}} = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -\frac{v^2}{R}\hat{\boldsymbol{e}}_R + \dot{v}\hat{\boldsymbol{e}}_\theta.$$

where $\dot{v}$ is the rate of change of tangential speed①. Thus the acceleration is made of two terms. One proportional to the speed squared and directed towards the center of the circle, and one proportional to the rate of change of speed and directed tangent to the circle.

①**Caution:** Note that the rate of change of speed is not the magnitude of the acceleration: $\dot{v} \neq |\vec{\boldsymbol{a}}|$ or in other words: $\frac{d}{dt}|\vec{\boldsymbol{v}}| \neq |\frac{d}{dt}\vec{\boldsymbol{v}}|$. Consider the case of a car driving in circles at constant rate. Its rate of change of speed is zero, yet it has an acceleration.

Centripetal acceleration

The term

$$-R\dot{\theta}^2\hat{\boldsymbol{e}}_R = -\dot{\theta}^2\vec{\boldsymbol{R}}$$

is called the *centripetal* acceleration. Why, intuitively, is the centripetal acceleration proportional to the speed *squared*? Well, the acceleration is the change in the velocity vector per unit time. There are two effects

1. If the speed is twice as big then the velocity is twice as big.
2. And, for a given radius, the angle it rotates per unit time is twice as big.

These two proportionalities with speed, the size of the velocity vector which rotates and the rate at which it rotates both apply. So if the speed is twice as big the acceleration is 4 times as big. Hence the v^2.

15.1 Summary: the motion quantities

We can use our results for velocity and acceleration to better evaluate the momenta and energy quantities. These results will allow us to do mechanics problems associated with circular motion. For one particle in circular motion we have:

Linear momentum:

$$\vec{\boldsymbol{L}} = \vec{\boldsymbol{v}}m = R\dot{\theta}\hat{\boldsymbol{e}}_\theta m,$$

Rate of change of linear momentum:

$$\dot{\vec{\boldsymbol{L}}} = \vec{\boldsymbol{a}}m = (-\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta)m,$$

Angular momentum:

$$\vec{\boldsymbol{H}}_{/O} = \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{v}}m = R^2\dot{\theta}m\hat{\boldsymbol{k}},$$

Rate of change of angular momentum:

$$\dot{\vec{\boldsymbol{H}}}_{/O} = \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}m = R^2\ddot{\theta}m\hat{\boldsymbol{k}},$$

Kinetic energy:

$$E_K = \frac{1}{2}v^2m = \frac{1}{2}R^2\dot{\theta}^2m,\text{and}$$

Rate of change of kinetic energy:

$$\dot{E_K} = \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}\,m = mR^2\dot{\theta}\ddot{\theta}$$

15.2 d'Alembert's mechanics: beginners beware

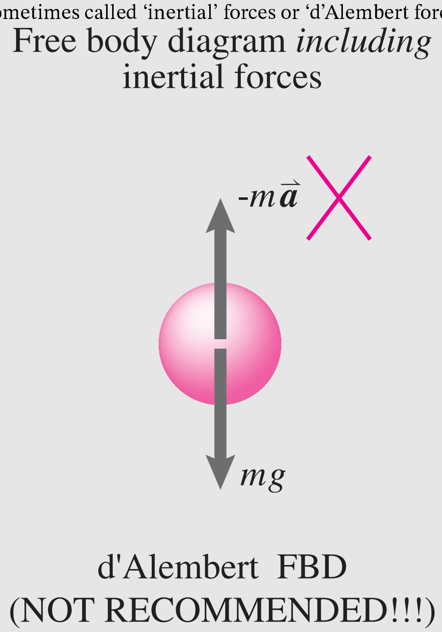
This box *is* an aside. It will not help you do dynamics problems. As warned on page 167,it's worse than that. The material in this box usually harms more than it helps. The d'Alembert approach to mechanics, described here, cannot be well-absorbed by beginners. Students attempting to use d'Alembert methods make frequent mistakes. We advise against the use of d'Alembert mechanics for beginners. We don't allow its use in homework and exams.

But you might be curious about this forbidden fruit. To demystify the taboo, we briefly describe the approach. You might as well learn it here instead of somewhere else.

The d'Alembert approach has an intuitive appeal to experts. And the d'Alembert equations are the first step in deriving the more advanced (*e.g.,* Lagrangian, Hamiltonian, 'method of virtual speed', and 'Kane') approaches to dynamics.

How does it go?

First, label the free-body diagram: *'free-body diagram including inertial forces.'* Then, in addition to the applied forces draw pseudo-forces equal to $-m\vec{a}$ for every mass particle m. These pseudo-forces are shown in the 'FBD' of a falling ball. The pseudo-forces are sometimes called 'inertial' forces or 'd'Alembert forces.



For the d'Alembert approach, instead of momentum balance equations you write 'pseudo-statics' equations of 'force' balance and 'moment' balance

$$\overbrace{\underbrace{\sum \vec{F}}_{\text{including inertial forces}} = \vec{0}}^{\text{pseudo-statics force balance}}$$

$$\overbrace{\underbrace{\sum \vec{M}_C}_{\text{including torques from inertial forces}} = \vec{0}}^{\text{pseudo-statics moment balance}}$$

These equations include the actual forces as well as the 'inertial' forces shown on the 'd'Alembert free-body diagram'.

By these means, the dynamics equations have been reduced to statics equations. Linear momentum balance is replaced by pseudo-statics force balance. Angular momentum balance is replaced by pseudo-statics moment balance.

The moving of the inertial terms from the right side of the equation to the left leads to both conceptual simplicity *and* puts the equations of dynamics in a form that is closer to most people's intuitions. The simplification is not so great as it may seem at first sight. Accelerations still need to be calculated and the sums involved in calculation of rate of change of linear and angular momentum still need to be calculated, only now they are sums of pseudo *inertial* forces.

Consider the example of sitting in a car as the car rounds a corner to the left. In the momentum balance approach, we write

$$\vec{F} = \underbrace{m\vec{a}}_{\dot{\vec{L}}}$$

and say the force from the car on you to the left is equal to the rate of change of your linear momentum as you accelerate to the left. In the d'Alembert approach, we write

$$\vec{F} - \underbrace{m\vec{a}}_{\text{inertia force}} = \vec{0}$$

and think the inertia force to the right is balanced by the interaction force of the car on your body to the left.

It is a puzzle of human consciousness why such a trivial algebraic manipulation, namely,

$$\vec{F} = \mathrm{m}\vec{a} \;\Rightarrow\; \vec{F} - m\vec{a} = \vec{0}$$

should lead to such a great conceptual confusion. But, it is an empirical fact that most of us are susceptible to this confusion.

That is, if you follow your likely first intuition and think of $m\vec{a}$ as a force you will probably join the ranks of many other talented students who consequently make many sign errors.

Every teacher of mechanics has encountered the confusion in their students about whether $-m\vec{a}$ is or is not a force (and most likely in themselves as well.) To avoid such confusion, many teachers or texts take a firm stand and say

- '$m\vec{a}$ *is not* a force!'; but, as if believing in a different god, others will say with equal conviction

- '-$m\vec{a}$ *is* a force!'.

In this book, we take the former approach. We take the equation

$$\vec{F} = \mathrm{m}\vec{a}$$

to mean:

forces from interactions $= m\cdot$ (acceleration of mass).

If you insist on working with the d'Alembert approach instead, you must do so confidently and clearly. To repeat,

- instead of labeling your free-body diagram '*FBD*', label it '*FBD including inertial forces*',
- instead of using '*Linear Momentum Balance*', use '*Pseudo-Force Balance*', and
- instead of using '*Angular Momentum Balance*' use '*Pseudo-Moment Balance*'

We do *not* recommend d'Alembert mechanics to beginners, but if you insist, good luck to you and don't blame us for your (almost inevitable) sign errors!