15.1 Kinematics of a particle in planar circular motion

This section concerns the position, velocity and accelerations of one point going in circles. The essence of the content here is this:

> If $\hat{\boldsymbol{e}}_R$ is a unit vector in the plane that is rotating counter-clockwise (CCW) at a rate of $\dot{\theta}$ its rate of change is
> $$\dot{\hat{\boldsymbol{e}}}_R = \dot{\theta}\hat{\boldsymbol{e}}_\theta$$
> where $\hat{\boldsymbol{e}}_\theta$ is a unit vector given by rotating $\hat{\boldsymbol{e}}_R$ 90° CCW.

If you learn this idea inside and out then either you will have picked up all the other key facts on the way, or you will be able to learn them in a flash. Note, we use $\vec{\boldsymbol{R}}$ and $\vec{\boldsymbol{r}}$ interchangeably. Likewise for $\hat{\boldsymbol{e}}_r$ and $\hat{\boldsymbol{e}}_R$.

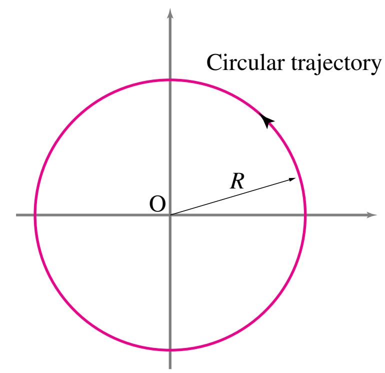


Figure 15.1: Trajectory of particle for circular motion.

Circular motion

The position of a particle going in circles around the origin on the xy plane is

$$\vec{\boldsymbol{r}} = R\cos\theta\,\hat{\boldsymbol{\imath}} + R\sin\theta\,\hat{\boldsymbol{\jmath}},$$

with the radius R a constant. Or, in terms of components,

$$x = R\cos\theta \qquad \text{and} \qquad y = R\sin\theta.$$

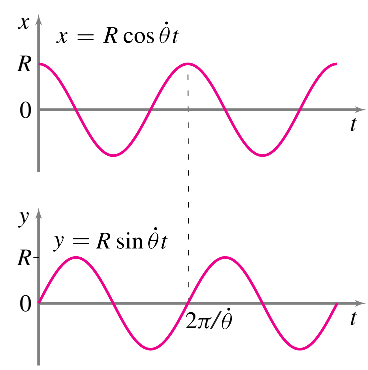


Figure 15.2: Plots of x versus t and y versus t for a particle going in a circle of radius R at constant rate. For simplicity we assumed constant $\dot{\theta}$ with $\theta = \dot{\theta}t$. So both x and y vary as sinusoidal functions of time: $x = R\cos(\dot{\theta}t)$ and $y = R\sin(\dot{\theta}t)$.

A natural graphical representation of this motion is a circle (fig. 15.1). Unfortunately, a picture of the circular trajectory doesn't give any information about the speed of the particle on the circle. A plot of a particle moving in circles slowly looks just like a plot of a particle moving quickly.

To get a sense of how position changes in time one can plot the functions $x(t)$ and $y(t)$ (fig. 15.2). Unfortunately this figure only indirectly conveys that the particle is going in circles.

If you want to see both the trajectory and the time history of both variables one can make a 3-D plot of xy position versus time (fig. 15.3). The shadows of this helix on the three coordinate planes are the three graphs just discussed.

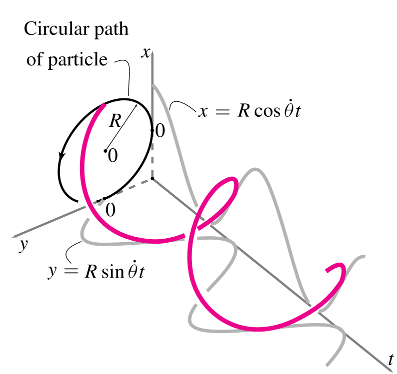


Figure 15.3: Plot of x and y versus time for a particle going in circles at constant rate. x versus t is a cosine curve, y versus t is a sine curve. Together they make up the 3D helix.

Finally, rather than representing time as a spatial coordinate, one can represent time with time itself. How? Make an animated movie showing a particle on the xy plane as it moves. Move your finger around in circles on the table. That's it. Similarly, you can make a dot move in circles on your screen. How do you make all these plots? Using a calculator or computer you can evaluate x and y for a range of values of t. Then, using pencil and paper, a plotting calculator, or a computer, plot x *vs* t, y *vs* t, and y *vs* x. For animations plot x and y over and over again for a sequence of values of t, and show these on your screen at a sequence of times.

Polar coordinates R and θ and unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$

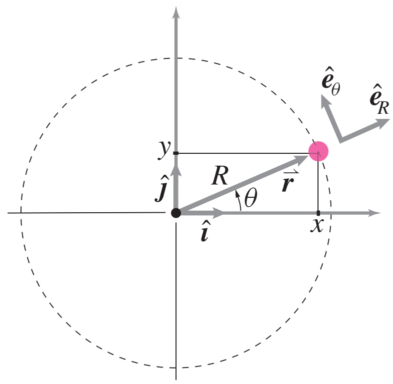


Figure 15.4: The position vector $\vec{\boldsymbol{r}}$ of the particle relative to the center of the circle is $\vec{\boldsymbol{r}}$ (or $\vec{\boldsymbol{R}}$) which is both
$\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$ and
$\vec{\boldsymbol{r}} = R\hat{\boldsymbol{e}}_R$ (or $r\hat{\boldsymbol{e}}_r$).
$\vec{\boldsymbol{r}}$ makes an angle θ measured counterclockwise from the positive x-axis. The unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ are in the radial and tangential directions, the directions of increasing R and increasing θ.

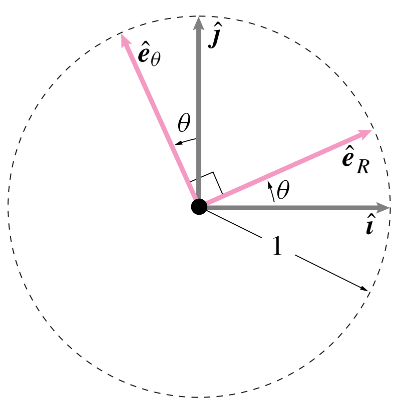


Figure 15.5: You can think of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ as the set $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ rotated counter-clockwise by the angle θ.

Especially for circular motion, it is convenient to represent position, velocity and acceleration with polar, rather than rectangular, coordinates. With polar coordinates we use polar base vectors which, unlike the fixed $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$, rotate as the particle goes around. Let's redraw fig. 15.1 and show the unit base vectors

$\hat{\boldsymbol{e}}_R$ ('e R') and $\hat{\boldsymbol{e}}_\theta$ ('e theta').

The radial unit vector $\hat{\boldsymbol{e}}_R$ is directed from the center of the circle towards the point of interest and the transverse vector $\hat{\boldsymbol{e}}_\theta$, perpendicular to $\hat{\boldsymbol{e}}_R$, is tangent to the circle at that point in the direction of increasing θ. As the particle goes around, its $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ unit vectors change accordingly. Two different particles both going in circles with the same center at the same rate each have their own $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ vectors. To be precise we can define $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ as

$$\hat{\boldsymbol{e}}_R \equiv \vec{\boldsymbol{R}}/|\vec{\boldsymbol{R}}| = \vec{\boldsymbol{R}}/R \qquad \text{and} \qquad \hat{\boldsymbol{e}}_\theta \equiv \hat{\boldsymbol{k}} \times \hat{\boldsymbol{e}}_R. \tag{15.1}$$

Note that also $\hat{\boldsymbol{e}}_R \times \hat{\boldsymbol{e}}_\theta = \hat{\boldsymbol{k}}$.

The velocity and acceleration of a point going in circles, using polar coordinates

In dynamics we are interested in velocity and acceleration so we need to know how to represent these in polar coordinates. First, observe that the position of the particle is (see fig. 15.4)

$$\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R. \tag{15.2}$$

That is, the position vector is the distance from the origin times a unit vector in the direction of the particle's position. Given the position, it is just a matter of *careful* differentiation to find velocity and acceleration. Here is one of many possible ways to derive the polar-coordinate expressions for velocity and acceleration. First, velocity is the time derivative of position, so

$$\vec{\boldsymbol{v}} = \frac{d}{dt}\vec{\boldsymbol{R}} = \frac{d}{dt}(R\hat{\boldsymbol{e}}_R) = \underbrace{\dot{R}}_{0}\,\hat{\boldsymbol{e}}_R + R\dot{\hat{\boldsymbol{e}}}_R. \tag{15.3}$$

Because a circle has constant radius R, $\dot{R}$ is zero. But how do we calculate the rate of change of $\hat{\boldsymbol{e}}_R$ with respect to time, $\dot{\hat{\boldsymbol{e}}}_R$,?

Derivatives of $\hat{\boldsymbol{e}}_R$ and of $\hat{\boldsymbol{e}}_\theta$

To find the velocity in polar coordinates we were just confronted with the problem of finding $\dot{\hat{\boldsymbol{e}}}_R$.

Method 1: One way to find $d\hat{\boldsymbol{e}}_R/dt = \dot{\hat{\boldsymbol{e}}}_R$ uses the geometry of fig. 15.7 and the informal calculus of finite differences (represented by Δ). $\Delta\hat{\boldsymbol{e}}_R$ is evidently (about) in the direction $\hat{\boldsymbol{e}}_\theta$ and has magnitude $\Delta\theta$ so $\Delta\hat{\boldsymbol{e}}_R \approx (\Delta\theta)\hat{\boldsymbol{e}}_\theta$. Dividing by Δt, we have $\Delta\hat{\boldsymbol{e}}_R/\Delta t \approx (\Delta\theta/\Delta t)\hat{\boldsymbol{e}}_\theta$. So, using this sloppy calculus, we get $\dot{\hat{\boldsymbol{e}}}_R = \dot\theta\hat{\boldsymbol{e}}_\theta$. Similarly, and we will need this shortly, we could get $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot\theta\hat{\boldsymbol{e}}_R$.

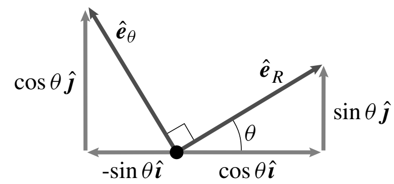


Figure 15.6: Projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y directions. From this picture you can immediately extract that
$\hat{\boldsymbol{e}}_R = \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}$ and that
$\hat{\boldsymbol{e}}_\theta = -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}$.
A similar picture showing the projections of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ in the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ d directions would show that
$\hat{\boldsymbol{\imath}} = \cos\theta\hat{\boldsymbol{e}}_R - \sin\theta\hat{\boldsymbol{e}}_\theta$ and that
$\hat{\boldsymbol{\jmath}} = \sin\theta\hat{\boldsymbol{e}}_R + \cos\theta\hat{\boldsymbol{e}}_\theta$.

Method 2 This method is a little less geometric and a little more algebraic. We start with the decomposition of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ into cartesian coordinates. These decompositions are found by looking at the projections of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ in the x and y-directions (see fig. 15.6).

$$\begin{aligned} \hat{\boldsymbol{e}}_R &= \cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}} \\ \hat{\boldsymbol{e}}_\theta &= -\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}} \end{aligned} \tag{15.4}$$

We can find $\dot{\hat{\boldsymbol{e}}}_R$ by differentiating, taking into account that θ is changing with time but that the unit vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ are fixed (so they don't change with time).

$$\begin{aligned} \dot{\hat{\boldsymbol{e}}}_R &= \frac{d}{dt}(\cos\theta\hat{\boldsymbol{\imath}} + \sin\theta\,\hat{\boldsymbol{\jmath}}) = -\dot\theta\sin\theta\hat{\boldsymbol{\imath}} + \dot\theta\cos\theta\,\hat{\boldsymbol{\jmath}} = \dot\theta\hat{\boldsymbol{e}}_\theta \\ \dot{\hat{\boldsymbol{e}}}_\theta &= \frac{d}{dt}(-\sin\theta\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}) = -\dot\theta\hat{\boldsymbol{e}}_R \end{aligned}$$

We had to use the chain rule, that is

$$\frac{d\sin\theta(t)}{dt} = \frac{d\sin\theta}{d\theta}\frac{d\theta(t)}{dt} = \dot\theta\cos\theta.$$

Now, two different ways, we know

$$\dot{\hat{\boldsymbol{e}}}_R = \dot\theta\hat{\boldsymbol{e}}_\theta \quad \text{and} \quad \dot{\hat{\boldsymbol{e}}}_\theta = -\dot\theta\hat{\boldsymbol{e}}_R. \tag{15.5}$$

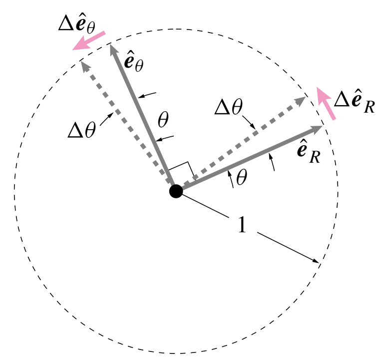


Figure 15.7: A close up view of the unit vectors $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$. They make an angle θ with the positive x and y-axis, respectively. As the particle advances an amount $\Delta\theta$ both $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ change. In particular, for small $\Delta\theta$, $\Delta\hat{\boldsymbol{e}}_R$ is approximately in the $\hat{\boldsymbol{e}}_\theta$ direction and $\Delta\hat{\boldsymbol{e}}_\theta$ is approximately in the $-\hat{\boldsymbol{e}}_R$ direction.

Continuing the quest for velocity and acceleration

Now that we know how $\hat{\boldsymbol{e}}_R$ changes in time we can continue our quest for $\vec{\boldsymbol{v}}$. Continuing from eqn. (15.3) we now have

$$\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{R}}} = R\dot{\hat{\boldsymbol{e}}}_R = R\dot\theta\hat{\boldsymbol{e}}_\theta. \tag{15.6}$$

Similarly we can find the acceleration $\ddot{\vec{\boldsymbol{R}}}$ by differentiating once again,

$$\vec{\boldsymbol{a}} = \ddot{\vec{\boldsymbol{R}}} = \dot{\vec{\boldsymbol{v}}} = \frac{d}{dt}(R\dot\theta\hat{\boldsymbol{e}}_\theta) = \underbrace{\dot R\dot\theta\hat{\boldsymbol{e}}_\theta}_{\vec{\boldsymbol{0}}} + R\ddot\theta\hat{\boldsymbol{e}}_\theta + R\dot\theta\dot{\hat{\boldsymbol{e}}}_\theta \tag{15.7}$$

The first term on the right hand side is zero because $\dot{R}$ is 0 for circular motion. The third term is evaluated using the formula we just found for the rate of change of $\hat{\boldsymbol{e}}_\theta$: $\dot{\hat{\boldsymbol{e}}}_\theta = -\dot{\theta}\hat{\boldsymbol{e}}_R$. So, using that $\vec{\boldsymbol{R}} = R\hat{\boldsymbol{e}}_R$,

$$\vec{\boldsymbol{a}} = -\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta \qquad (15.8)$$

The velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ for a particle going in circles at constant rate are shown in fig. 15.8.

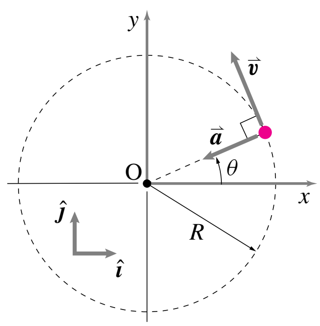


Figure 15.8: The directions of velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ are shown for a particle going in circles at *constant* rate. The velocity is tangent to the circle and the acceleration is directed towards the center of the circle.

Example: A person standing on the earth's equator
A person standing on the equator has velocity

$$\begin{aligned}\vec{\boldsymbol{v}} = \dot{\theta}R\hat{\boldsymbol{e}}_\theta \quad &\approx \quad \left(\frac{2\pi\,\text{rad}}{24\,\text{hr}}\right)4000\,\text{mi}\hat{\boldsymbol{e}}_\theta \\ &\approx \quad 1050\,\text{mph}\hat{\boldsymbol{e}}_\theta \approx 1535\,\text{ft/s}\hat{\boldsymbol{e}}_\theta\end{aligned}$$

and acceleration

$$\begin{aligned}\vec{\boldsymbol{a}} = -\dot{\theta}^2R\hat{\boldsymbol{e}}_R \quad &\approx \quad -\left(\frac{2\pi\,\text{rad}}{24\,\text{hr}}\right)^2 4000\,\text{mi}\hat{\boldsymbol{e}}_R \\ &\approx \quad -274\,\text{mi/hr}^2\hat{\boldsymbol{e}}_R \approx -0.11\,\text{ft/s}^2\hat{\boldsymbol{e}}_R.\end{aligned}$$

The velocity of a person standing on the equator, due to the earth's rotation, is about 1000 mph tangent to the earth. Her acceleration is about $0.11\,\text{ft/s}^2 \approx 0.03\,\text{m/s}^2$ towards the center of the earth, about 1/300 of g, about 1/300 the acceleration of an object in near-earth-surface frictionless free-fall.

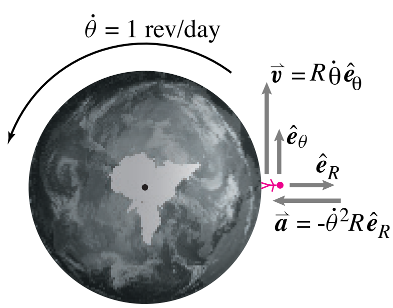


Figure 15.9

Alternate expressions for the velocity and acceleration formulas

Note that we can define a scalar velocity $v = R\dot{\theta}$. We informally call this scalar the speed even though it can be positive or negative. So

$$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta = v\hat{\boldsymbol{e}}_\theta.$$

Similarly the acceleration is

$$\vec{\boldsymbol{a}} = -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta = -\frac{v^2}{R}\hat{\boldsymbol{e}}_R + \dot{v}\hat{\boldsymbol{e}}_\theta.$$

where $\dot{v}$ is the rate of change of tangential speed(1). Thus the acceleration is made of two terms. One proportional to the speed squared and directed towards the center of the circle, and one proportional to the rate of change of speed and directed tangent to the circle.

(1)**Caution:** Note that the rate of change of speed is not the magnitude of the acceleration: $\dot{v} \neq |\vec{\boldsymbol{a}}|$ or in other words: $\frac{d}{dt}|\vec{\boldsymbol{v}}| \neq |\frac{d}{dt}\vec{\boldsymbol{v}}|$. Consider the case of a car driving in circles at constant rate. Its rate of change of speed is zero, yet it has an acceleration.

Centripetal acceleration

The term

$$-R\dot{\theta}^2\hat{\boldsymbol{e}}_R = -\dot{\theta}^2\vec{\boldsymbol{R}}$$

is called the *centripetal* acceleration. Why, intuitively, is the centripetal acceleration proportional to the speed *squared*? Well, the acceleration is the change in the velocity vector per unit time. There are two effects

1. If the speed is twice as big then the velocity is twice as big.
2. And, for a given radius, the angle it rotates per unit time is twice as big.

These two proportionalities with speed, the size of the velocity vector which rotates and the rate at which it rotates both apply. So if the speed is twice as big the acceleration is 4 times as big. Hence the v^2.

15.1 Summary: the motion quantities

We can use our results for velocity and acceleration to better evaluate the momenta and energy quantities. These results will allow us to do mechanics problems associated with circular motion. For one particle in circular motion we have:

Linear momentum:

$$\vec{\boldsymbol{L}} = \quad \vec{\boldsymbol{v}}m = \quad R\dot{\theta}\hat{\boldsymbol{e}}_\theta m,$$

Rate of change of linear momentum:

$$\dot{\vec{\boldsymbol{L}}} = \quad \vec{\boldsymbol{a}}m = \quad (-\dot{\theta}^2\vec{\boldsymbol{R}} + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta)m,$$

Angular momentum:

$$\vec{\boldsymbol{H}}_{/\mathrm{O}} = \quad \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{v}}m = \quad R^2\dot{\theta}m\hat{\boldsymbol{k}},$$

Rate of change of angular momentum:

$$\dot{\vec{\boldsymbol{H}}}_{/\mathrm{O}} = \quad \vec{\boldsymbol{r}}_{/0} \times \vec{\boldsymbol{a}}m = \quad R^2\ddot{\theta}m\hat{\boldsymbol{k}},$$

Kinetic energy:

$$E_\mathrm{K} = \quad \frac{1}{2}v^2 m = \quad \frac{1}{2}R^2\dot{\theta}^2 m, \text{and}$$

Rate of change of kinetic energy:

$$\dot{E}_\mathrm{K} = \quad \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}}\, m = \quad mR^2\dot{\theta}\ddot{\theta}$$

15.2 d'Alembert's mechanics: beginners beware

This box *is* an aside. It will not help you do dynamics problems. As warned on page 167,it's worse than that. The material in this box usually harms more than it helps. The d'Alembert approach to mechanics, described here, cannot be well-absorbed by beginners. Students attempting to use d'Alembert methods make frequent mistakes. We advise against the use of d'Alembert mechanics for beginners. We don't allow its use in homework and exams.

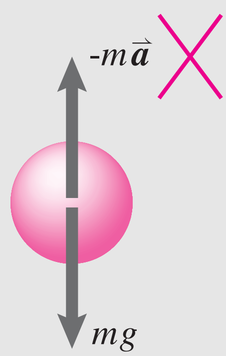
But you might be curious about this forbidden fruit. To demystify the taboo, we briefly describe the approach. You might as well learn it here instead of somewhere else.

The d'Alembert approach has an intuitive appeal to experts. And the d'Alembert equations are the first step in deriving the more advanced (*e.g.,* Lagrangian, Hamiltonian, 'method of virtual speed', and 'Kane') approaches to dynamics.

How does it go?

First, label the free-body diagram: *'free-body diagram including inertial forces.'* Then, in addition to the applied forces draw pseudo-forces equal to $-m\vec{a}$ for every mass particle m. These pseudo-forces are shown in the 'FBD' of a falling ball. The pseudo-forces are sometimes called 'inertial' forces or 'd'Alembert forces.

Free body diagram *including* inertial forces



d'Alembert FBD
(NOT RECOMMENDED!!!)

For the d'Alembert approach, instead of momentum balance equations you write 'pseudo-statics' equations of 'force' balance and 'moment' balance

$$\overbrace{\underbrace{\sum \vec{F}}_{\text{including inertial forces}} = \vec{0}}^{\text{pseudo-statics force balance}}$$

$$\overbrace{\underbrace{\sum \vec{M}_C}_{\text{including torques from inertial forces}} = \vec{0}}^{\text{pseudo-statics moment balance}}$$

These equations include the actual forces as well as the 'inertial' forces shown on the 'd'Alembert free-body diagram'.

By these means, the dynamics equations have been reduced to statics equations. Linear momentum balance is replaced by pseudo-statics force balance. Angular momentum balance is replaced by pseudo-statics moment balance.

The moving of the inertial terms from the right side of the equation to the left leads to both conceptual simplicity *and* puts the equations of dynamics in a form that is closer to most people's intuitions. The simplification is not so great as it may seem at first sight. Accelerations still need to be calculated and the sums involved in calculation of rate of change of linear and angular momentum still need to be calculated, only now they are sums of pseudo *inertial* forces.

Consider the example of sitting in a car as the car rounds a corner to the left. In the momentum balance approach, we write

$$\vec{F} = \underbrace{m\vec{a}}_{\dot{\vec{L}}}$$

and say the force from the car on you to the left is equal to the rate of change of your linear momentum as you accelerate to the left. In the d'Alembert approach, we write

$$\vec{F} - \underbrace{m\vec{a}}_{\text{inertia force}} = \vec{0}$$

and think the inertia force to the right is balanced by the interaction force of the car on your body to the left.

It is a puzzle of human consciousness why such a trivial algebraic manipulation, namely,

$$\vec{F} = \mathrm{m}\vec{a} \;\Rightarrow\; \vec{F} - m\vec{a} = \vec{0}$$

should lead to such a great conceptual confusion. But, it is an empirical fact that most of us are susceptible to this confusion.

That is, if you follow your likely first intuition and think of $m\vec{a}$ as a force you will probably join the ranks of many other talented students who consequently make many sign errors.

Every teacher of mechanics has encountered the confusion in their students about whether $-m\vec{a}$ is or is not a force (and most likely in themselves as well.) To avoid such confusion, many teachers or texts take a firm stand and say

- '$m\vec{a}$ *is not* a force!'; but, as if believing in a different god, others will say with equal conviction

- '-$m\vec{a}$ *is* a force!'.

In this book, we take the former approach. We take the equation

$$\vec{F} = \mathrm{m}\vec{a}$$

to mean:

forces from interactions $= m\cdot$ (acceleration of mass).

If you insist on working with the d'Alembert approach instead, you must do so confidently and clearly. To repeat,

- instead of labeling your free-body diagram '*FBD*', label it '*FBD including inertial forces*',
- instead of using '*Linear Momentum Balance*', use '*Pseudo-Force Balance*', and
- instead of using '*Angular Momentum Balance*' use '*Pseudo-Moment Balance*'

We do *not* recommend d'Alembert mechanics to beginners, but if you insist, good luck to you and don't blame us for your (almost inevitable) sign errors!

SAMPLE 15.1 The velocity vector in circular motion. A particle executes circular motion in the xy plane with constant speed $v = 5\,\mathrm{m/s}$. At $t = 0$ the particle is at $\theta = 0$. Given that the radius of the circular orbit is 2.5 m, find the velocity of the particle at $t = 2$ sec.

Solution It is given that

$$\begin{aligned} R &= 2.5\,\mathrm{m} \\ v &= \text{constant} = 5\,\mathrm{m/s} \\ \theta(t=0) &= 0. \end{aligned}$$

The velocity of a particle in constant-rate circular motion is:

$$\begin{aligned} \vec{\boldsymbol{v}} &= R\dot{\theta}\hat{\boldsymbol{e}}_\theta \\ \text{where} \quad \hat{\boldsymbol{e}}_\theta &= -\sin\theta\,\hat{\boldsymbol{\imath}} + \cos\theta\,\hat{\boldsymbol{\jmath}}. \end{aligned}$$

Since R is constant and $v = |\vec{\boldsymbol{v}}| = R\dot{\theta}$ is constant,

$$\dot{\theta} = \frac{v}{R} = \frac{5\,\mathrm{m/s}}{2.5\,\mathrm{m}} = 2\,\mathrm{rad/s}$$

is also constant. Thus,

$$\vec{\boldsymbol{v}}(t = 2\,\mathrm{s}) = \underbrace{R\dot{\theta}}_{v}\,\hat{\boldsymbol{e}}_\theta\Big|_{t=2\,\mathrm{s}} = 5\,\mathrm{m/s}\,\hat{\boldsymbol{e}}_\theta(t = 2\,\mathrm{s}).$$

Clearly, we need to find $\hat{\boldsymbol{e}}_\theta$ at $t = 2$ sec.

$$\begin{aligned} \text{Now} \quad \dot{\theta} &\equiv \frac{d\theta}{dt} = 2\,\mathrm{rad/s} \\ \Rightarrow \quad \int_0^\theta d\theta &= \int_0^{2\,\mathrm{s}} 2\,\mathrm{rad/s}\,dt \\ \Rightarrow \quad \theta &= (2\,\mathrm{rad/s})\,t\Big|_0^{2\,\mathrm{s}} \\ &= 2\,\mathrm{rad/s}\cdot 2\,\mathrm{s} \\ &= 4\,\mathrm{rad}. \end{aligned}$$

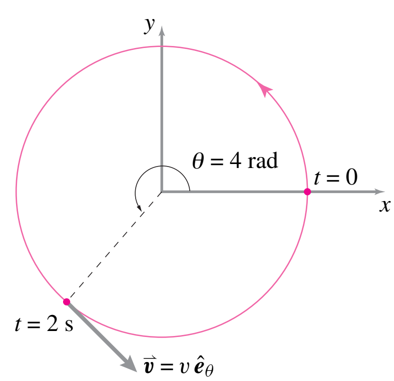


Figure 15.10: The velocity vector $\vec{\boldsymbol{v}}$ at $t = 2\,\mathrm{s}$.

Therefore,

$$\begin{aligned} \hat{\boldsymbol{e}}_\theta &= -\sin 4\hat{\boldsymbol{\imath}} + \cos 4\hat{\boldsymbol{\jmath}} \\ &= 0.76\hat{\boldsymbol{\imath}} - 0.65\hat{\boldsymbol{\jmath}}, \end{aligned}$$

and

$$\begin{aligned} \vec{\boldsymbol{v}}(2\,\mathrm{s}) &= 5\,\mathrm{m/s}(0.76\hat{\boldsymbol{\imath}} - 0.65\hat{\boldsymbol{\jmath}}) \\ &= (3.78\hat{\boldsymbol{\imath}} - 3.27\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = (3.78\hat{\boldsymbol{\imath}} - 3.27\hat{\boldsymbol{\jmath}})\,\mathrm{m/s}}$$

SAMPLE 15.2 Basic kinematics: A point mass executes circular motion with angular acceleration $\ddot{\theta} = 5\,\text{rad/s}^2$. The radius of the circular path is 0.25 m. If the mass starts from rest at $\theta = 0^\circ$, find and draw

1. the velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°,
2. the acceleration of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

Solution We are given, $\ddot{\theta} = 5\,\text{rad/s}^2$, and $R = 0.25\,\text{m}$.

1. The velocity $\vec{\boldsymbol{v}}$ in circular (constant or non-constant rate) motion is given by:

$$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta.$$

So, to find the velocity at different positions we need $\dot{\theta}$ at those positions. Here the angular acceleration is constant, i.e., $\ddot{\theta} = 5\,\text{rad/s}^2$. Therefore, we can use the formula ①

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\ddot{\theta}\theta$$

to find the angular speed $\dot{\theta}$ at various θ's. But $\dot{\theta}_0 = 0$ (mass starts from rest), therefore $\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$. Now we make a table for computing the velocities at different positions:

① We use this formula because we need $\dot{\theta}$ at different values of θ. In elementary physics books, the same formula is usually written as

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\alpha\theta$$

where α is the constant angular acceleration and $\dot{\theta}(=\ \dot{\theta})$ is the angular speed.

Position (θ)	θ in radians	$\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$	$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$
0°	0	0 rad/s	$\vec{\mathbf{0}}$
30°	$\pi/6$	$\sqrt{10\pi/6} = 2.29\,\text{rad/s}$	$0.57\,\text{m/s}\,\hat{\boldsymbol{e}}_\theta$
90°	$\pi/2$	$\sqrt{10\pi/2} = 3.96\,\text{rad/s}$	$0.99\,\text{m/s}\,\hat{\boldsymbol{e}}_\theta$
210°	$7\pi/6$	$\sqrt{70\pi/6} = 6.05\,\text{rad/s}$	$1.51\,\text{m/s}\,\hat{\boldsymbol{e}}_\theta$

The computed velocities are shown in Fig. 15.11.

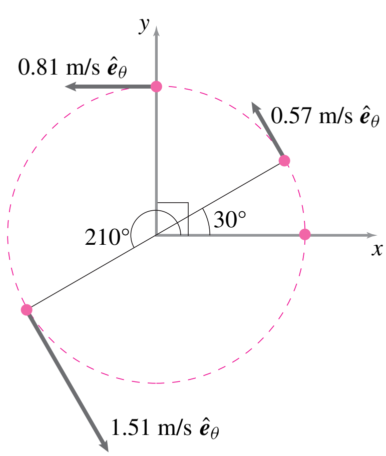


Figure 15.11: Velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

2. The acceleration of the mass is given by

$$\begin{aligned}\vec{\boldsymbol{a}} &= \overbrace{a_R\hat{\boldsymbol{e}}_R}^{\text{radial}} + \overbrace{a_\theta\hat{\boldsymbol{e}}_\theta}^{\text{tangential}} \\ &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Since $\ddot{\theta}$ is constant, the tangential component of the acceleration is constant at all positions. We have already calculated $\dot{\theta}$ at various positions, so we can easily calculate the radial (also called the normal) component of the acceleration. Thus we can find the acceleration. For example, at $\theta = 30^\circ$,

$$\begin{aligned}\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta \\ &= -0.25\,\text{m}\cdot\frac{10\pi}{6}\frac{1}{\text{s}^2}\hat{\boldsymbol{e}}_R + 0.25\,\text{m}\cdot 5\frac{1}{\text{s}^2}\hat{\boldsymbol{e}}_\theta \\ &= -1.31\,\text{m/s}^2\hat{\boldsymbol{e}}_R + 1.25\,\text{m/s}^2\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Similarly, we find the acceleration of the mass at other positions by substituting the values of R, $\ddot{\theta}$ and $\dot{\theta}$ in the formula and tabulate the results in the table below.

Position (θ)	$a_r = -R\dot{\theta}^2$	$a_\theta = R\ddot{\theta}$	$\vec{a} = a_r\hat{e}_R + a_\theta\hat{e}_\theta$
0°	0	$1.25\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}\hat{e}_\theta$
30°	$-1.31\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-1.31\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
90°	$-3.93\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-3.93\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
210°	$-9.16\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-9.16\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$

The accelerations computed are shown in Fig. 15.12. The acceleration vector as well as its tangential and radial components are shown in the figure at each position.

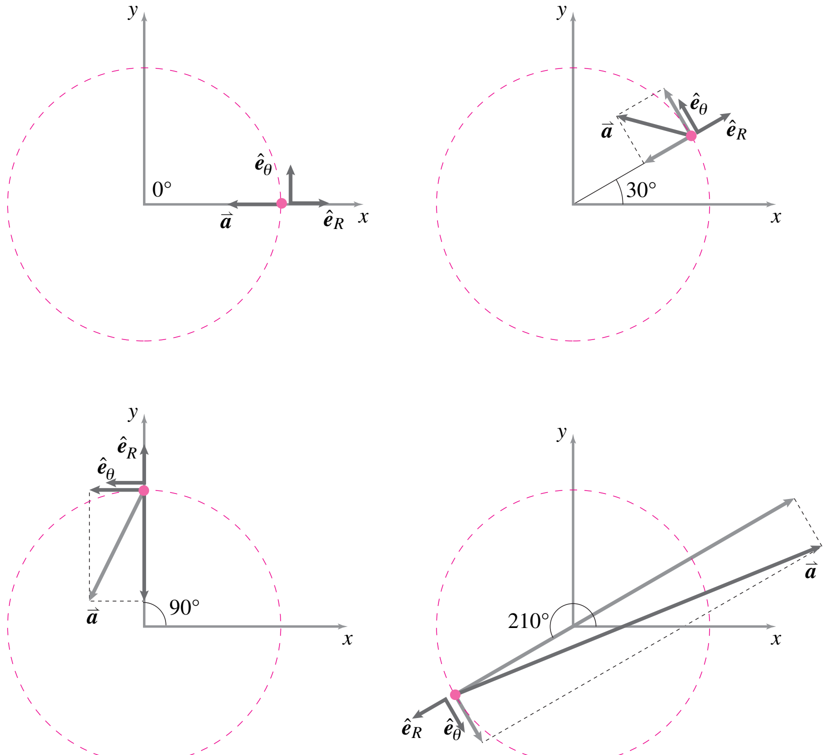


Figure 15.12: Acceleration of the mass at $\theta = 0^\circ$, 30°, 90°, and 210°. The radial and tangential components are shown with grey arrows. As the angular velocity increases, the radial component of the acceleration increases; therefore, the total acceleration vector leans more and more towards the radial direction.

SAMPLE 15.3 In an experiment, the magnitude of angular deceleration of a rotating ball is found to be proportional to its angular speed $\dot{\theta}$ (*i.e.*, $\ddot{\theta} \propto -\dot{\theta}$). Assume that the proportionality constant is k.

1. Find $\dot{\theta}$ as a function of t, given that $\dot{\theta}(t=0) = \dot{\theta}_0$.
2. Given that $k = 0.1/\,\text{s}$, how much time does it take for $\dot{\theta}$ to reduce to half the initial value?

Solution The equation given is:

$$\ddot{\theta} = \frac{d\dot{\theta}}{dt} = -k\dot{\theta}. \tag{15.9}$$

1. We can solve this equation in a couple of ways.

   Method-1: Let us guess a solution of the exponential form with arbitrary constants and plug it into eqn. (15.9) to check if our solution works. Let $\dot{\theta}(t) = C_1 e^{C_2 t}$. Substituting in eqn. (15.9), we get

$$\begin{aligned} C_1 C_2 e^{C_2 t} &= -kC_1 e^{C_2 t} \\ \Rightarrow \quad C_2 &= -k, \\ \text{also,} \quad \dot{\theta}(0) &= \dot{\theta}_0 = C_1 e^{C_2 \cdot 0} \\ \Rightarrow \quad C_1 &= \dot{\theta}_0. \end{aligned}$$

   Therefore,

$$\dot{\theta}(t) = \dot{\theta}_0 e^{-kt}. \tag{15.10}$$

$$\boxed{\dot{\theta}(t) = \dot{\theta}_0\, e^{-kt}}$$

   Method-2: Equation (15.9) can also be solved by direct integration as follows.

$$\begin{aligned} \frac{d\dot{\theta}}{\dot{\theta}} &= -k\,dt \\ \Rightarrow \quad \int_{\dot{\theta}_0}^{\dot{\theta}(t)} \frac{d\dot{\theta}}{\dot{\theta}} &= -\int_0^t k\,dt \\ \Rightarrow \quad \ln \dot{\theta}\Big|_{\dot{\theta}_0}^{\dot{\theta}(t)} &= -kt \\ \Rightarrow \quad \ln\left(\frac{\dot{\theta}(t)}{\dot{\theta}_0}\right) &= -kt \end{aligned}$$

   Therefore,

$$\dot{\theta}(t) = \dot{\theta}_0\, e^{-kt},$$

   which is the same solution as equation (15.10).

2. We need to find t for $\dot{\theta} = \dot{\theta}_0/2$, given that $k = 0.1$. From eqn. (15.10), we get

$$\begin{aligned} \frac{\dot{\theta}}{\dot{\theta}_0} &= e^{-kt} \\ \Rightarrow \quad t &= \frac{1}{-k}\ln\left(\frac{\dot{\theta}}{\dot{\theta}_0}\right) \\ &= \frac{1}{-0.1}\ln\left(\frac{1}{2}\right) = \frac{-0.693}{-0.1/\,\text{s}} = 6.93\,\text{s}. \end{aligned}$$

$$\boxed{t = 6.93\,\text{s for } \dot{\theta}(t) = \dot{\theta}_0/2}$$

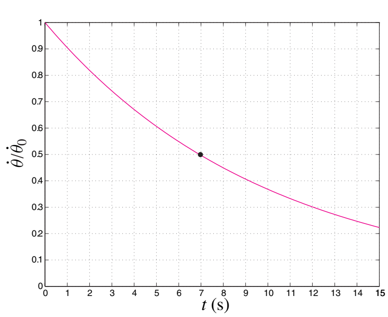


Figure 15.13: Plot of $\dot{\theta}(t)/\dot{\theta}_0 = e^{-kt}$ for $k = 0.1/\,\text{s}$. The angular speed $\dot{\theta}$ reduces to half of its initial value in 6.93 s. Note that this time is the same for $\dot{\theta}$ to reduce to half its value at any given time t (not just at $t = 0$).

SAMPLE 15.4 **Using kinematic formulae:** The spinning wheel of a stationary exercise bike is brought to rest from 100 rpm by applying brakes over a period of 5 seconds.

1. Find the average angular deceleration of the wheel.
2. Find the number of revolutions it makes during the braking.

Solution We are given,

$$\dot{\theta}_0 = 100\,\text{rpm}, \quad \dot{\theta}_{\text{final}} = 0, \quad \text{and} \quad t = 5\,\text{s}.$$

1. Let α be the average (constant) deceleration. Then

$$\dot{\theta}_{\text{final}} = \dot{\theta}_0 - \alpha t.$$

Therefore,

$$\begin{aligned}\alpha &= \frac{\dot{\theta}_0 - \dot{\theta}_{\text{final}}}{t}\\ &= \frac{100\,\text{rpm} - 0\,\text{rpm}}{5\,\text{s}}\\ &= \frac{100\,\text{rev}}{60\,\text{s}}\cdot\frac{1}{5\,\text{s}}\\ &= 0.33\frac{\text{rev}}{\text{s}^2}.\end{aligned}$$

$$\alpha = 0.33\frac{\text{rev}}{\text{s}^2}$$

2. To find the number of revolutions made during the braking period, we use the formula

$$\theta(t) = \underbrace{\theta_0}_{0} + \dot{\theta}_0 t + \frac{1}{2}(-\alpha)t^2 = \dot{\theta}_0 t - \frac{1}{2}\alpha t^2.$$

Substituting the known values, we get

$$\begin{aligned}\theta &= \frac{100\,\text{rev}}{60\,\text{s}}\cdot 5\,\text{s} - \frac{1}{2}0.33\frac{\text{rev}}{\text{s}^2}\cdot 25\,\text{s}^2\\ &= 8.33\,\text{rev} - 4.12\,\text{rev}\\ &= 4.21\,\text{rev}.\end{aligned}$$

$$\theta = 4.21\,\text{rev}$$

Comments:

- Note the negative sign used in both the formulae above. Since α is deceleration, that is, a negative acceleration, we have used negative sign with α in the formulae.
- Note that it is not always necessary to convert rpm in rad/s. Here we changed rpm to rev/s because time was given in seconds.

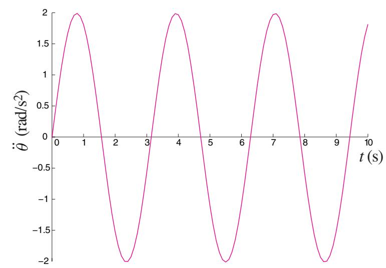


Figure 15.14: Time varying angular acceleration, $\ddot{\theta}(t) = c\sin\beta t$.

SAMPLE 15.5 Non-constant acceleration: A particle of mass 500 grams executes circular motion with radius $R = 100\,\text{cm}$ and angular acceleration $\ddot{\theta}(t) = c\sin\beta t$, where $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

1. Find the position of the particle after 10 seconds if the particle starts from rest, that is, $\theta(0) = 0$.
2. How much kinetic energy does the particle have at the position found above?

Solution

1. We are given $\ddot{\theta}(t) = c\sin\beta t$, $\dot{\theta}(0) = 0$ and $\theta(0) = 0$. We have to find $\theta(10\,\text{s})$. Basically, we have to solve a second order differential equation with given initial conditions.

$$\begin{aligned} \ddot{\theta} \equiv \frac{d}{dt}(\dot{\theta}) &= c\sin\beta t \\ \Rightarrow \int_{\dot{\theta}_0=0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t c\sin\beta\tau\, d\tau \\ \dot{\theta}(t) &= -\frac{c}{\beta}\cos\beta\tau\Big|_0^t = \frac{c}{\beta}(1-\cos\beta t). \end{aligned}$$

Thus, we get the expression for the angular speed $\dot{\theta}(t)$. We can solve for the position $\theta(t)$ by integrating once more:

$$\begin{aligned} \dot{\theta} \equiv \frac{d}{dt}(\theta) &= \frac{c}{\beta}(1-\cos\beta t) \\ \Rightarrow \int_{\theta_0=0}^{\theta(t)} d\theta &= \int_0^t \frac{c}{\beta}(1-\cos\beta\tau) \\ \theta(t) &= \frac{c}{\beta}\left[\tau - \frac{\sin\beta\tau}{\beta}\right]_0^t \\ &= \frac{c}{\beta^2}(\beta t - \sin\beta t). \end{aligned}$$

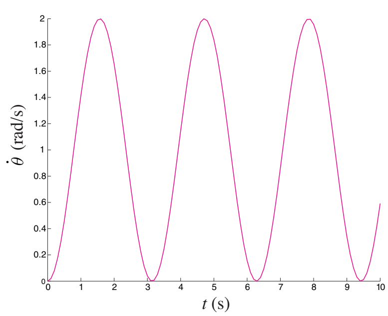


Figure 15.15: Angular speed, $\dot{\theta}(t) = \frac{c}{\beta}(1-\cos\beta t)$, plotted against time for $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

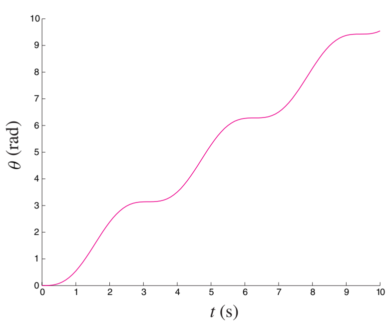


Figure 15.16: Angular position, $\theta(t) = \frac{c}{\beta^2}(\beta t - \sin\beta t)$, plotted against time for $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

Now substituting $t = 10\,\text{s}$ in the last expression along with the values of other constants, we get

$$\begin{aligned} \theta(10\,\text{s}) &= \frac{2\,\text{rad/s}^2}{(2\,\text{rad/s})^2}[2\,\text{rad/s}\cdot 10\,\text{s} - \sin(2\,\text{rad/s}\cdot 10\,\text{s})] \\ &= 9.54\,\text{rad}. \end{aligned}$$

$\theta = 9.54\,\text{rad}$

2. The kinetic energy of the particle is given by

$$\begin{aligned} E_K &= \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\theta})^2 \\ &= \frac{1}{2}mR^2[\underbrace{\frac{c}{\beta}(1-\cos\beta t)}_{\dot{\theta}(t)}]^2 \\ &= \frac{1}{2}\,0.5\,\text{kg}\cdot 1\,\text{m}^2\cdot\left[\frac{2\,\text{rad/s}^2}{2\,\text{rad/s}}\cdot(1-\cos(20))\right]^2 \\ &= 0.086\,\text{kg}\cdot\text{m}^2\cdot\text{s}^2 = 0.086\text{Joule}. \end{aligned}$$

$E_K = 0.086\text{J}$

15.1 Kinematics of a particle in circular motion

Preparatory Problems

15.1.1 A particle goes on a circular path with radius R making the angle $\theta = ct$ measured counter clockwise from the positive x axis. Assume $R = 5\,\text{cm}$ and $c = 2\pi\,\text{s}^{-1}$.

a) Plot the path.

b) What is the angular rate in revolutions per second?

c) Put a dot on the path for the location of the particle at $t = t^* = 1/6$ s.

d) What are the x and y coordinates of the particle position at $t = t^*$? Mark them on your plot.

e) Draw the vectors $\hat{\boldsymbol{e}}_\theta$ and $\hat{\boldsymbol{e}}_R$ at $t = t^*$.

f) What are the x and y components of $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ at $t = t^*$?

g) What are the R and θ components of $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ at $t = t^*$?

h) Draw an arrow representing both the velocity and the acceleration at $t = t^*$.

i) Find the $\hat{\boldsymbol{e}}_R$ and $\hat{\boldsymbol{e}}_\theta$ components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$.

j) Find the x and y components of position $\vec{\boldsymbol{r}}$, velocity $\vec{\boldsymbol{v}}$ and acceleration $\vec{\boldsymbol{a}}$ at $t = t^*$. Find the velocity and acceleration two ways:

   1. Differentiate the position given as $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$.
   2. Differentiate the position give as $\vec{\boldsymbol{r}} = r\hat{\boldsymbol{e}}_r$ and then convert the results to Cartesian coordinates.

15.1.2 A bead goes around a circular track of radius 1 ft at a constant speed. It makes it around the track in exactly 1 s.

a) Find the speed of the bead. Does this vary in time?

b) Find the magnitude of acceleration of the bead. Does this vary in time?

c) Is the magnitude of the acceleration the derivative of the speed (*i.e.*, $|\vec{\boldsymbol{a}}| \stackrel{?}{=} \frac{d}{dt}|\vec{\boldsymbol{v}}|$)

15.1.3 If a particle moves along a circle at constant rate (constant $\dot{\theta}$) following the equation

$$\vec{\boldsymbol{r}}(t) = R\cos(\dot{\theta}t)\hat{\boldsymbol{\imath}} + R\sin(\dot{\theta}t)\hat{\boldsymbol{\jmath}}$$

which of these things are true and why? If not true, explain why.

a) $\vec{\boldsymbol{v}} = \vec{\boldsymbol{0}}$

b) $\vec{\boldsymbol{v}} =$ constant

c) $|\vec{\boldsymbol{v}}| =$ constant

d) $\vec{\boldsymbol{a}} = \vec{\boldsymbol{0}}$

e) $\vec{\boldsymbol{a}} =$ constant

f) $|\vec{\boldsymbol{a}}| =$ constant

g) $\vec{\boldsymbol{v}} \perp \vec{\boldsymbol{a}}$

15.1.4 A particle moves according to:

$$\begin{aligned} x(t) &= R\cos(ct), \\ y(t) &= R\sin(ct). \end{aligned}$$

where $R = 1\,\text{m}$ and $c = 5\,\text{rad/s}$.

a) Show that the speed of the particle is constant.

b) How much time does the particle take to go from P at $(0, 1\,\text{m})$ to Q at $(1\,\text{m}, 0)$?

c) What is the acceleration of the particle at point Q?

More-Involved Problems

15.1.5 A 200 mm diameter gear rotates at a constant speed of 100 rpm.

a) What is the speed of a peripheral point on the gear?

b) If no point on the gear is to exceed the centripetal acceleration of $25\,\text{m/s}^2$, find the maximum allowable angular speed (in rpm) of the gear.

15.1.6 A particle is in circular motion in the xy-plane at the constant angular speed of $\dot{\theta} = 2\,\text{rad/s}$ at radius 0.5 m. At $t = 0$ the particle is at $\theta = 0$.

a) Draw the path and mark the position of the particle at $t = 0.5\,\text{s}$ and $t = 15\,\text{s}$.

b) Find the velocity and acceleration of the particle at $t = 0.5\,\text{s}$ and $t = 15\,\text{s}$.

.

15.1.7 A particle undergoes constant rate circular motion in the xy-plane. At some instant t_0, its velocity is $\vec{\boldsymbol{v}}(t_0) = -3\,\text{m/s}\hat{\boldsymbol{\imath}} + 4\,\text{m/s}\hat{\boldsymbol{\jmath}}$ and after 5 s the velocity is $v(t_0 + 5\,\text{s}) = (5/\sqrt{2})\,\text{m/s}(\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})$. If the particle has not yet completed one revolution between the two instants, find

a) the angular speed of the particle,

b) the distance traveled by the particle in 5 s, and

c) the acceleration of the particle at the two instants.

15.1.8 A bead on a circular path of radius R in the xy-plane has rate of change of angular speed $\alpha = bt^2$. The bead starts from rest at $\theta = 0$.

a) What is the bead's angular position θ (measured from the positive x-axis) and angular speed $\dot{\theta}$ as a function of time ?

b) What is the angular speed as function of angular position?

15.1.9 A bead on a circular wire has an angular speed given by $\dot{\theta} = c\theta^{1/2}$. The bead starts from rest at $\theta = 0$. What is the angular position and speed of the bead as a function of time? [This problem is subtle because it has multiple solutions. One answer you can find with a quick guess. Another you can find by separation of variables. The full general solution is an appropriate mixture of these two.)] *

15.1.10 Solve $\ddot{\theta} = C$, given $\dot{\theta}(0) = \dot{\theta}_0$, $\theta(0) = \theta_0$ and that C is a constant. That is, find θ in terms of some or all of C, $\dot{\theta}_0$, θ_0 and t.

15.1.11 Given that $\ddot{\theta} - \lambda^2\theta = 0$, $\theta(0) = \pi/2$, $\dot{\theta}(0) = 0$, and $\lambda = 3/\,\text{s}$ find the value of θ at $t = 1\,\text{s}$.

15.1.12 Two runners run on a circular track side-by-side at the same constant angular rate $\dot{\theta} = 0.25\,\mathrm{rad/s}$ about the center of the track. The inside runner is in a lane of radius $r_i = 35\,\mathrm{m}$ and the outside runner is in a lane of radius $r_o = 37\,\mathrm{m}$. What is the velocity of the outside runner relative to the inside runner?

15.1.13 A particle oscillates on the arc of a circle with radius R according to the equation $\theta = \theta_0 \cos(\lambda t)$. What are the conditions on R, θ_0, and λ so that the maximum acceleration in this motion occurs at $\theta = 0$. "Acceleration" here means the magnitude of the acceleration vector.

15.1.14 A particle moves on a circular arc starting from rest at $\theta_0 = 0$. As θ increases, the magnitude of the acceleration is constant. Assume, all in consistent units, that $R = 1$ and $|\vec{\boldsymbol{a}}| = 1$.

a) Write the statement 'the magnitude of acceleration is constant' as an equation in terms of $\dot{\theta}$ and $\ddot{\theta}$.

b) Find a solution to the equation with the given initial conditions (analytically or numerically).

c) Find and plot $\dot{\theta}$ vs t and θ vs t.

d) In circular motion does $|\vec{\boldsymbol{a}}|$=constant necessarily mean that the motion is at or is gradually approaching constant rate circular motion? Is so, why? If not show a counter-example.

15.1.15 A particle moves in a circle so that its acceleration $\vec{\boldsymbol{a}}$ always makes a fixed angle ϕ with the position vector $-\vec{\boldsymbol{r}}$, with $0 \leq \phi \leq \pi/2$. For example, $\phi = 0$ would be constant rate circular motion. Assume $\phi = \pi/4$, $R = 1\,\mathrm{m}$ and $\dot{\theta}_0 = 1\,\mathrm{rad/s}$. How long does it take the particle to reach

a) the speed of sound ($\approx 300\,\mathrm{m/s}$)?

b) the speed of light ($\approx 3 \cdot 10^8\,\mathrm{m/s}$)?

c) ∞?