SAMPLE 15.1 The velocity vector in circular motion. A particle executes circular motion in the xy plane with constant speed $v = 5\,\mathrm{m/s}$. At $t = 0$ the particle is at $\theta = 0$. Given that the radius of the circular orbit is $2.5\,\mathrm{m}$, find the velocity of the particle at $t = 2\,\mathrm{sec}$.

Solution It is given that

$$\begin{aligned} R &= 2.5\,\mathrm{m} \\ v &= \text{constant} = 5\,\mathrm{m/s} \\ \theta(t=0) &= 0. \end{aligned}$$

The velocity of a particle in constant-rate circular motion is:

$$\begin{aligned} \vec{v} &= R\dot{\theta}\hat{e}_\theta \\ \text{where} \quad \hat{e}_\theta &= -\sin\theta\,\hat{\imath} + \cos\theta\,\hat{\jmath}. \end{aligned}$$

Since R is constant and $v = |\vec{v}| = R\dot{\theta}$ is constant,

$$\dot{\theta} = \frac{v}{R} = \frac{5\,\mathrm{m/s}}{2.5\,\mathrm{m}} = 2\,\mathrm{rad/s}$$

is also constant. Thus,

$$\vec{v}(t = 2\,\mathrm{s}) = \underbrace{R\dot{\theta}}_{v}\,\hat{e}_\theta\Big|_{t=2\,\mathrm{s}} = 5\,\mathrm{m/s}\,\hat{e}_\theta(t = 2\,\mathrm{s}).$$

Clearly, we need to find $\hat{e}_\theta$ at $t = 2\,\mathrm{sec}$.

$$\begin{aligned} \text{Now} \quad \dot{\theta} &\equiv \frac{d\theta}{dt} = 2\,\mathrm{rad/s} \\ \Rightarrow \quad \int_0^\theta d\theta &= \int_0^{2\,\mathrm{s}} 2\,\mathrm{rad/s}\,dt \\ \Rightarrow \quad \theta &= (2\,\mathrm{rad/s})\,t\Big|_0^{2\,\mathrm{s}} \\ &= 2\,\mathrm{rad/s}\cdot 2\,\mathrm{s} \\ &= 4\,\mathrm{rad}. \end{aligned}$$

Therefore,

$$\begin{aligned} \hat{e}_\theta &= -\sin 4\hat{\imath} + \cos 4\hat{\jmath} \\ &= 0.76\hat{\imath} - 0.65\hat{\jmath}, \end{aligned}$$

and

$$\begin{aligned} \vec{v}(2\,\mathrm{s}) &= 5\,\mathrm{m/s}(0.76\hat{\imath} - 0.65\hat{\jmath}) \\ &= (3.78\hat{\imath} - 3.27\hat{\jmath})\,\mathrm{m/s}. \end{aligned}$$

$$\boxed{\vec{v} = (3.78\hat{\imath} - 3.27\hat{\jmath})\,\mathrm{m/s}}$$

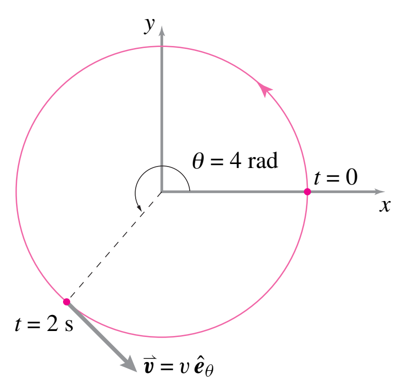

Figure 15.10: The velocity vector $\vec{v}$ at $t = 2\,\mathrm{s}$.

SAMPLE 15.2 Basic kinematics: A point mass executes circular motion with angular acceleration $\ddot{\theta} = 5\,\mathrm{rad/s^2}$. The radius of the circular path is $0.25\ \mathrm{m}$. If the mass starts from rest at $\theta = 0^\circ$, find and draw

1. the velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°,
2. the acceleration of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

Solution We are given, $\ddot{\theta} = 5\,\mathrm{rad/s^2}$, and $R = 0.25\,\mathrm{m}$.

1. The velocity $\vec{\boldsymbol{v}}$ in circular (constant or non-constant rate) motion is given by:

$$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta.$$

So, to find the velocity at different positions we need $\dot{\theta}$ at those positions. Here the angular acceleration is constant, i.e., $\ddot{\theta} = 5\,\mathrm{rad/s^2}$. Therefore, we can use the formula ①

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\ddot{\theta}\theta$$

to find the angular speed $\dot{\theta}$ at various θ's. But $\dot{\theta}_0 = 0$ (mass starts from rest), therefore $\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$. Now we make a table for computing the velocities at different positions:

Position (θ)	θ in radians	$\dot{\theta} = \sqrt{2\ddot{\theta}\theta}$	$\vec{\boldsymbol{v}} = R\dot{\theta}\hat{\boldsymbol{e}}_\theta$
0°	0	$0\ \mathrm{rad/s}$	$\vec{\boldsymbol{0}}$
30°	$\pi/6$	$\sqrt{10\pi/6} = 2.29\,\mathrm{rad/s}$	$0.57\,\mathrm{m/s}\,\hat{\boldsymbol{e}}_\theta$
90°	$\pi/2$	$\sqrt{10\pi/2} = 3.96\,\mathrm{rad/s}$	$0.99\,\mathrm{m/s}\,\hat{\boldsymbol{e}}_\theta$
210°	$7\pi/6$	$\sqrt{70\pi/6} = 6.05\,\mathrm{rad/s}$	$1.51\,\mathrm{m/s}\,\hat{\boldsymbol{e}}_\theta$

The computed velocities are shown in Fig. 15.11.

① We use this formula because we need $\dot{\theta}$ at different values of θ. In elementary physics books, the same formula is usually written as

$$\dot{\theta}^2 = \dot{\theta}_0^2 + 2\alpha\theta$$

where α is the constant angular acceleration and $\dot{\theta}(=\ \dot{\theta})$ is the angular speed.

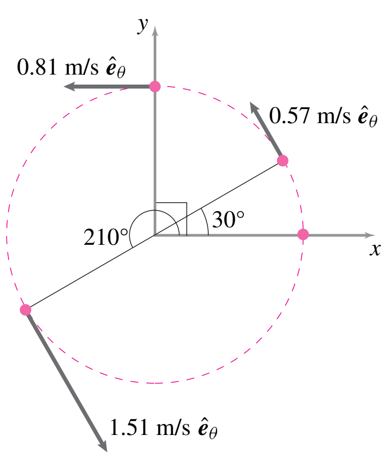

Figure 15.11: Velocity of the mass at $\theta = 0^\circ,\ 30^\circ,\ 90^\circ$, and 210°.

2. The acceleration of the mass is given by

$$\begin{aligned}\vec{\boldsymbol{a}} &= \overbrace{a_R\hat{\boldsymbol{e}}_R}^{\text{radial}} + \overbrace{a_\theta\hat{\boldsymbol{e}}_\theta}^{\text{tangential}} \\ &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Since $\ddot{\theta}$ is constant, the tangential component of the acceleration is constant at all positions. We have already calculated $\dot{\theta}$ at various positions, so we can easily calculate the radial (also called the normal) component of the acceleration. Thus we can find the acceleration. For example, at $\theta = 30^\circ$,

$$\begin{aligned}\vec{\boldsymbol{a}} &= -R\dot{\theta}^2\hat{\boldsymbol{e}}_R + R\ddot{\theta}\hat{\boldsymbol{e}}_\theta \\ &= -0.25\,\mathrm{m}\cdot\frac{10\pi}{6}\,\frac{1}{\mathrm{s}^2}\hat{\boldsymbol{e}}_R + 0.25\,\mathrm{m}\cdot 5\frac{1}{\mathrm{s}^2}\hat{\boldsymbol{e}}_\theta \\ &= -1.31\,\mathrm{m/s^2}\hat{\boldsymbol{e}}_R + 1.25\,\mathrm{m/s^2}\hat{\boldsymbol{e}}_\theta.\end{aligned}$$

Similarly, we find the acceleration of the mass at other positions by substituting the values of R, $\ddot{\theta}$ and $\dot{\theta}$ in the formula and tabulate the results in the table below.

Position (θ)	$a_r = -R\dot{\theta}^2$	$a_\theta = R\ddot{\theta}$	$\vec{a} = a_r\hat{e}_R + a_\theta\hat{e}_\theta$
0°	0	$1.25\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}\hat{e}_\theta$
30°	$-1.31\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-1.31\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
90°	$-3.93\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-3.93\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$
210°	$-9.16\,\mathrm{m/s^2}$	$1.25\,\mathrm{m/s^2}$	$(-9.16\hat{e}_R + 1.25\hat{e}_\theta)\,\mathrm{m/s^2}$

The accelerations computed are shown in Fig. 15.12. The acceleration vector as well as its tangential and radial components are shown in the figure at each position.

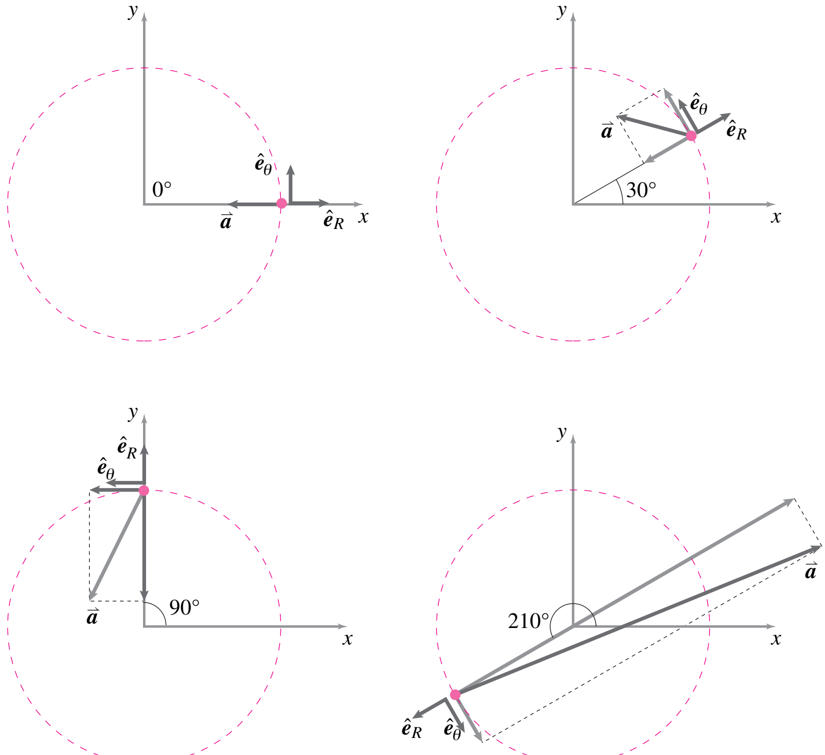

Figure 15.12: Acceleration of the mass at $\theta = 0^\circ$, 30°, 90°, and 210°. The radial and tangential components are shown with grey arrows. As the angular velocity increases, the radial component of the acceleration increases; therefore, the total acceleration vector leans more and more towards the radial direction.

SAMPLE 15.3 In an experiment, the magnitude of angular deceleration of a rotating ball is found to be proportional to its angular speed $\dot\theta$ (*i.e.*, $\ddot\theta \propto -\dot\theta$). Assume that the proportionality constant is k.

1. Find $\dot\theta$ as a function of t, given that $\dot\theta(t=0) = \dot\theta_0$.
2. Given that $k = 0.1/\,\mathrm{s}$, how much time does it take for $\dot\theta$ to reduce to half the initial value?

Solution The equation given is:

$$\ddot\theta = \frac{d\dot\theta}{dt} = -k\dot\theta. \tag{15.9}$$

1. We can solve this equation in a couple of ways.

   Method-1: Let us guess a solution of the exponential form with arbitrary constants and plug it into eqn. (15.9) to check if our solution works. Let $\dot\theta(t) = C_1 e^{C_2 t}$. Substituting in eqn. (15.9), we get

$$\begin{aligned} C_1 C_2 e^{C_2 t} &= -kC_1 e^{C_2 t} \\ \Rightarrow \quad C_2 &= -k, \\ \text{also,} \quad \dot\theta(0) &= \dot\theta_0 = C_1 e^{C_2 \cdot 0} \\ \Rightarrow \quad C_1 &= \dot\theta_0. \end{aligned}$$

   Therefore,

$$\dot\theta(t) = \dot\theta_0 e^{-kt}. \tag{15.10}$$

$$\boxed{\dot\theta(t) = \dot\theta_0\, e^{-kt}}$$

   Method-2: Equation (15.9) can also be solved by direct integration as follows.

$$\begin{aligned} \frac{d\dot\theta}{\dot\theta} &= -k\,dt \\ \Rightarrow \quad \int_{\dot\theta_0}^{\dot\theta(t)} \frac{d\dot\theta}{\dot\theta} &= -\int_0^t k\,dt \\ \Rightarrow \quad \ln\dot\theta\Big|_{\dot\theta_0}^{\dot\theta(t)} &= -kt \\ \Rightarrow \quad \ln\left(\frac{\dot\theta(t)}{\dot\theta_0}\right) &= -kt \end{aligned}$$

   Therefore,

$$\dot\theta(t) = \dot\theta_0\, e^{-kt},$$

   which is the same solution as equation (15.10).

2. We need to find t for $\dot\theta = \dot\theta_0/2$, given that $k = 0.1$. From eqn. (15.10), we get

$$\begin{aligned} \frac{\dot\theta}{\dot\theta_0} &= e^{-kt} \\ \Rightarrow \quad t &= \frac{1}{-k}\ln\left(\frac{\dot\theta}{\dot\theta_0}\right) \\ &= \frac{1}{-0.1}\ln\left(\frac{1}{2}\right) = \frac{-0.693}{-0.1/\,\mathrm{s}} = 6.93\,\mathrm{s}. \end{aligned}$$

$$\boxed{t = 6.93\,\mathrm{s} \text{ for } \dot\theta(t) = \dot\theta_0/2}$$

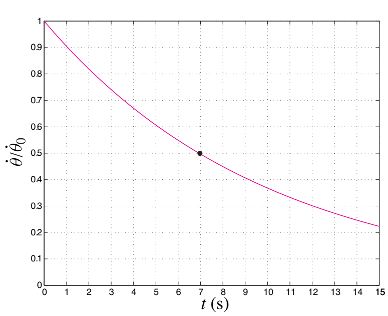

Figure 15.13: Plot of $\dot\theta(t)/\dot\theta_0 = e^{-kt}$ for $k = 0.1/\,\mathrm{s}$. The angular speed $\dot\theta$ reduces to half of its initial value in 6.93 s. Note that this time is the same for $\dot\theta$ to reduce to half its value at any given time t (not just at $t = 0$).

SAMPLE 15.4 Using kinematic formulae: The spinning wheel of a stationary exercise bike is brought to rest from 100 rpm by applying brakes over a period of 5 seconds.

1. Find the average angular deceleration of the wheel.
2. Find the number of revolutions it makes during the braking.

Solution We are given,

$$\dot{\theta}_0 = 100\,\text{rpm}, \quad \dot{\theta}_{\text{final}} = 0, \quad \text{and} \quad t = 5\,\text{s}.$$

1. Let α be the average (constant) deceleration. Then

$$\dot{\theta}_{\text{final}} = \dot{\theta}_0 - \alpha t.$$

Therefore,

$$\begin{aligned}
\alpha &= \frac{\dot{\theta}_0 - \dot{\theta}_{\text{final}}}{t} \\
&= \frac{100\,\text{rpm} - 0\,\text{rpm}}{5\,\text{s}} \\
&= \frac{100\,\text{rev}}{60\,\text{s}} \cdot \frac{1}{5\,\text{s}} \\
&= 0.33\frac{\text{rev}}{\text{s}^2}.
\end{aligned}$$

$$\boxed{\alpha = 0.33\frac{\text{rev}}{\text{s}^2}}$$

2. To find the number of revolutions made during the braking period, we use the formula

$$\theta(t) = \underbrace{\theta_0}_{0} + \dot{\theta}_0 t + \frac{1}{2}(-\alpha)t^2 = \dot{\theta}_0 t - \frac{1}{2}\alpha t^2.$$

Substituting the known values, we get

$$\begin{aligned}
\theta &= \frac{100\,\text{rev}}{60\,\text{s}} \cdot 5\,\text{s} - \frac{1}{2}0.33\frac{\text{rev}}{\text{s}^2} \cdot 25\,\text{s}^2 \\
&= 8.33\,\text{rev} - 4.12\,\text{rev} \\
&= 4.21\,\text{rev}.
\end{aligned}$$

$$\boxed{\theta = 4.21\,\text{rev}}$$

Comments:

- Note the negative sign used in both the formulae above. Since α is deceleration, that is, a negative acceleration, we have used negative sign with α in the formulae.
- Note that it is not always necessary to convert rpm in rad/s. Here we changed rpm to rev/ s because time was given in seconds.

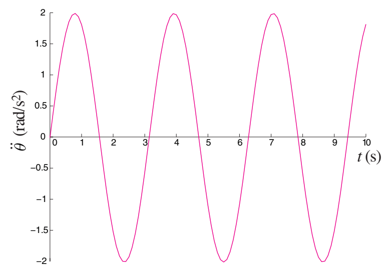

Figure 15.14: Time varying angular acceleration, $\ddot{\theta}(t) = c\sin\beta t$.

SAMPLE 15.5 Non-constant acceleration: A particle of mass 500 grams executes circular motion with radius $R = 100\,\text{cm}$ and angular acceleration $\ddot{\theta}(t) = c\sin\beta t$, where $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

1. Find the position of the particle after 10 seconds if the particle starts from rest, that is, $\theta(0) = 0$.
2. How much kinetic energy does the particle have at the position found above?

Solution

1. We are given $\ddot{\theta}(t) = c\sin\beta t$, $\dot{\theta}(0) = 0$ and $\theta(0) = 0$. We have to find $\theta(10\,\text{s})$. Basically, we have to solve a second order differential equation with given initial conditions.

$$\begin{aligned}
\ddot{\theta} \equiv \frac{d}{dt}(\dot{\theta}) &= c\sin\beta t \\
\Rightarrow \int_{\dot{\theta}_0=0}^{\dot{\theta}(t)} d\dot{\theta} &= \int_0^t c\sin\beta\tau\, d\tau \\
\dot{\theta}(t) &= -\frac{c}{\beta}\cos\beta\tau\Big|_0^t = \frac{c}{\beta}(1-\cos\beta t).
\end{aligned}$$

Thus, we get the expression for the angular speed $\dot{\theta}(t)$. We can solve for the position $\theta(t)$ by integrating once more:

$$\begin{aligned}
\dot{\theta} \equiv \frac{d}{dt}(\theta) &= \frac{c}{\beta}(1-\cos\beta t) \\
\Rightarrow \int_{\theta_0=0}^{\theta(t)} d\theta &= \int_0^t \frac{c}{\beta}(1-\cos\beta\tau) \\
\theta(t) &= \frac{c}{\beta}\left[\tau - \frac{\sin\beta\tau}{\beta}\right]_0^t \\
&= \frac{c}{\beta^2}(\beta t - \sin\beta t).
\end{aligned}$$

Now substituting $t = 10\,\text{s}$ in the last expression along with the values of other constants, we get

$$\begin{aligned}
\theta(10\,\text{s}) &= \frac{2\,\text{rad/s}^2}{(2\,\text{rad/s})^2}[2\,\text{rad/s}\cdot 10\,\text{s} - \sin(2\,\text{rad/s}\cdot 10\,\text{s})] \\
&= 9.54\,\text{rad}.
\end{aligned}$$

$\boxed{\theta = 9.54\,\text{rad}}$

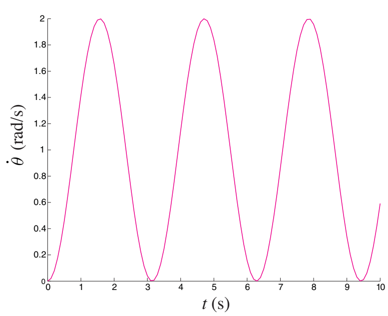

Figure 15.15: Angular speed, $\dot{\theta}(t) = \frac{c}{\beta}(1-\cos\beta t)$, plotted against time for $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

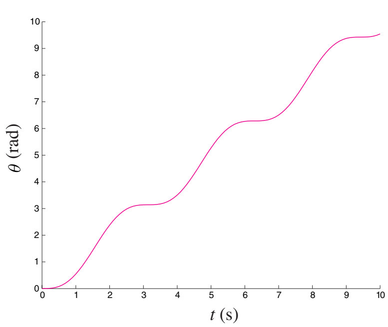

Figure 15.16: Angular position, $\theta(t) = \frac{c}{\beta^2}(\beta t - \sin\beta t)$, plotted against time for $c = 2\,\text{rad/s}^2$ and $\beta = 2\,\text{rad/s}$.

2. The kinetic energy of the particle is given by

$$\begin{aligned}
E_K &= \frac{1}{2}mv^2 = \frac{1}{2}m(R\dot{\theta})^2 \\
&= \frac{1}{2}mR^2[\underbrace{\frac{c}{\beta}(1-\cos\beta t)}_{\dot{\theta}(t)}]^2 \\
&= \frac{1}{2}\,0.5\,\text{kg}\cdot 1\,\text{m}^2\cdot\left[\frac{2\,\text{rad/s}^2}{2\,\text{rad/s}}\cdot(1-\cos(20))\right]^2 \\
&= 0.086\,\text{kg}\cdot\text{m}^2\cdot\text{s}^2 = 0.086\text{Joule}.
\end{aligned}$$

$\boxed{E_K = 0.086\text{J}}$