14.2 1D motion with 2D and 3D forces

Even if all the motion is in a single direction, an engineer may still have to consider two- or three-dimensional forces.

Example: Piston in a cylinder.
A piston slides vertically in a cylinder with coefficient of friction μ between the piston and the cylinder wall. Assume the connecting rod has negligible mass so it can be treated as a two-force member as discussed in section 5.2b. The free-body diagram of the piston (with a bit of the connecting rod) is shown in fig. 14.21. The piston is moving up, so the friction force resists the motion and points down. Linear momentum balance for this system is:

$$\begin{aligned} \sum \vec{F}_i &= \dot{\vec{L}} \\ -N\hat{\imath} - \mu N\hat{\jmath} + T\hat{\lambda}_{rod} &= m_{piston}\, a\hat{\jmath}. \end{aligned}$$

If we assume that the acceleration $a\hat{\jmath}$ of the piston is known, as is its mass m_{piston}, the coefficient of friction μ, and the orientation of the connecting rod $\hat{\lambda}_{rod}$, then we can solve for the rod tension T and the normal reaction N.

Note: *even though the piston moves in one direction, the momentum balance equation is a two-dimensional vector equation.*

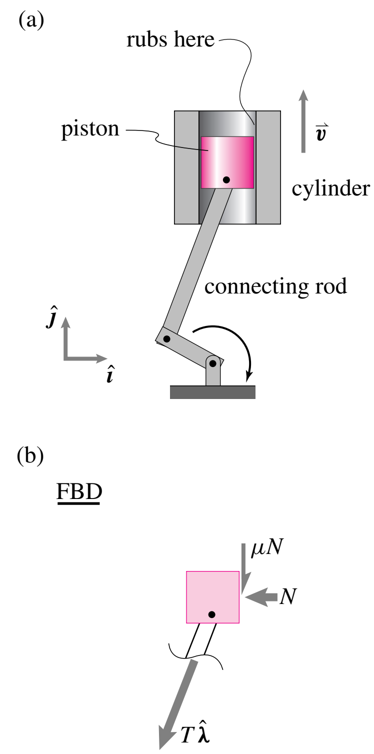

Figure 14.21: (a) shows a piston in a cylinder. (b) shows a free-body diagram of the piston. To draw this FBD, we have assumed: (1) a coefficient of friction μ between the piston and cylinder wall, and (2) negligible mass for the connecting rod, and (3) ignored the spatial extent of the cylinder.

Unlike the 1D mechanics of the previous chapter, in this section on 1D motion, the momentum balance equations are 2D and 3D vector equations. Compared to more general 2D and 3D motion, the 1-D motions we assume in this chapter allow an easy introduction to 2D and 3D dynamics calculations.

Highly constrained bodies

This chapter is about rigid objects that move in straight lines. Most objects will not agree to be the topic of such discussion without being forced into doing so. Without being held in place they would rotate and move in a curvy way. To keep an object that is subject to various forces from rotating or curving takes some constraint by wires, rods, rails, hinges, welds, *etc.*. Of course the presence of constraint is not always associated with the disallowance of rotation — constraints could even cause rotation. But in this chapter, constraints keep a rigid object in straight-line motion.

Constraint forces are of interest

Of common interest is making sure that static and dynamic loads do not cause failure of parts that enforce constraints. For example, suppose a truck hauls a very heavy load that is held down by chains or straps. When the truck accelerates, what is the tension in the chains, and will it exceed their strength limits?

1D mechanics with 1D forces, moving on

This all is in contrast with the situation in 1D "unconstrained" dynamics of the previous chapter. For one-dimensional mechanics we assumed that everything of interest mechanically happened in, say, the $\hat{\imath}$ (x) direction.

That is, we ignored *all* torques and angular momenta, and only considered the $\hat{\imath}$ components of the forces (*i.e.*, $\vec{F}\cdot\hat{\imath}$) and linear momentum ($\vec{L}\cdot\hat{\imath}$), namely F_x and L_x.

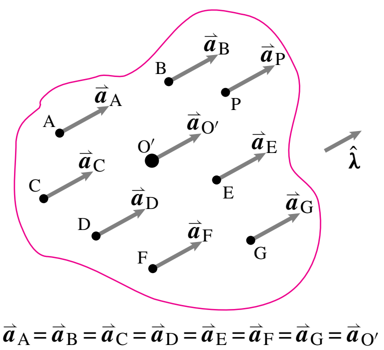

Figure 14.22: Parallel motion: all points on the body have the same acceleration $\vec{a} = a\hat{\lambda}$. For straight-line motion: $\hat{\lambda}(t)$=constant in time and $\vec{v} = v\hat{\lambda}$.

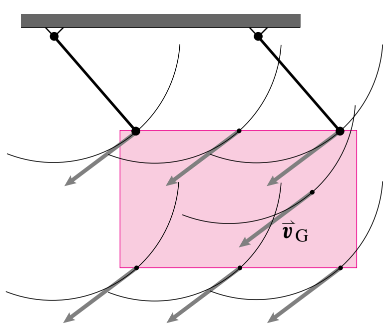

Figure 14.23: A swing showing instantaneous parallel motion which is *curvilinear*. At every instant, each point has the same velocity as the others, but the motion is not in a straight line.

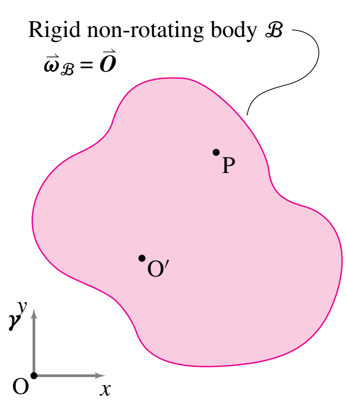

Figure 14.24: A non-rotating body $\mathcal{B}$ with points O' and P.

Kinematics of parallel motion and straight line motion

Let's consider a set of points in the system of interest. Let's call them A to G, or generically, P. For convenience we distinguish a reference point O'. O' may be the center-of-mass, the origin of a local coordinate system, or a fleck of dirt that serves as a marker. By *parallel motion*, we mean that the system happens to move in such a way that $\vec{a}_P = \vec{a}_{O'}$, and $\vec{v}_P = \vec{v}_{O'}$ (fig. 14.22). That is,

$$\vec{a}_A = \vec{a}_B = \vec{a}_C = \vec{a}_D = \vec{a}_E = \vec{a}_F = \vec{a}_G = \vec{a}_P = \vec{a}_{O'}$$

at every instant in time. We also assume that $\vec{v}_A = \ldots = \vec{v}_P = \vec{v}_{O'}$.

A special case of parallel motion is straight-line motion.

a system moves with straight-line motion if it moves like a non-rotating rigid body, in a straight line.

For straight-line motion, the velocity of the body is in a fixed unchanging direction. If we call a unit vector in that direction $\hat{\lambda}$, then we have

$$\vec{v}(t) = v(t)\hat{\lambda}, \qquad \vec{a}(t) = a(t)\hat{\lambda} \qquad \text{and} \qquad \vec{r}(t) = \vec{r}_0 + s(t)\hat{\lambda}$$

for every point in the system. $\vec{r}_0$ is the position of a point at time 0 and s is the distance the point moves in the $\hat{\lambda}$ direction. Every point in the system has the same s, v, a, and $\hat{\lambda}$ as the other points. There are a variety of problems of practical interest that can be idealized as fitting into this class, notably, the motions of things constrained to move on belts, roads, and rails, like the train on a straight track.

Example: Parallel swing is not straight-line motion
The swing shown does not rotate — all points on the swing have the same velocity. The velocities of all particles are parallel but, since paths are curved, this motion is not straight-line motion. Such curvilinear parallel motion will be discussed later in the book.

Velocity of a point

The velocity of any point P on a non-rotating rigid body (such as for straight-line motion) is the same as that of any reference point on the body (see Fig. 14.24).

$$\vec{v}_P = \vec{v}_{O'}$$

A more general case, which you will learn in later chapters, is shown as 5b in Table II at the back of the book. This formula concerns rotational rate which we will measure with the vector $\vec{\omega}$. For now all you need to know is that $\vec{\omega} = \vec{0}$ when something is not rotating. In 5b in Table II, if you set $\vec{\omega}_{\mathcal{B}} = 0$ and $\vec{v}_{P/\mathcal{B}} = \vec{0}$ it says that $\vec{v}_P = \dot{\vec{r}}_{O'/O}$ or in shorthand, $\vec{v}_P = \vec{v}_{O'}$, as we have written above.

Acceleration of a point

Similarly, the acceleration of every point on a non-rotating rigid body is the same as every other point. The more general case, not needed in this chapter, is shown as entry 5c in Table II at the back of the book.

General results

Before we proceed with discussion of the details of the mechanics of straight-line motion we present some ideas that are also more generally applicable. That is, the concept of the center-of-mass allows some useful simplifications of the general expressions for $\vec{\boldsymbol{L}}$, $\dot{\vec{\boldsymbol{L}}}$, $\vec{\boldsymbol{H}}_{/\mathrm{C}}$, $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}}$ and E_{K}.

Linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$

Although we are dealing with zillions of atoms in a given object, the linear momentum and angular momentum are simple to evaluate:

$$\vec{\boldsymbol{L}} = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{\mathrm{cm}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{L}}} = m_{\mathrm{tot}}\vec{\boldsymbol{a}}_{\mathrm{cm}}.$$

Actually, as the front inside cover states, these formulas are good for any motion of any system. The nice simplification for the straight-line motion of this chapter is that all points on a given object have the same velocity and acceleration. So we don't need to find or track the center of mass, but can track the motion of any point on the object.

Angular momentum $\vec{H}_{/C}$ and its rate of change, $\dot{\vec{H}}_C$ for straight-line motion

For the motions in this chapter, where $\vec{\boldsymbol{a}}_i = \vec{\boldsymbol{a}}_{cm}$ and thus $\vec{\boldsymbol{a}}_{i/cm} = \vec{\boldsymbol{0}}$, angular momentum considerations are simplified, as explained in Box 14.1 on page 3(1).

$$\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \vec{\boldsymbol{v}}_{\mathrm{cm}} m_{\mathrm{tot}} \qquad \text{and} \qquad \dot{\vec{\boldsymbol{H}}}_{\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times \vec{\boldsymbol{a}}_{\mathrm{cm}} m_{\mathrm{tot}}$$

(1) Calculating rate of change of angular momentum will get more difficult as the book progresses. For a rigid body $\mathcal{B}$ in more general motion, the calculation of rate of change of angular momentum involves the angular velocity $\vec{\boldsymbol{\omega}}_{\mathcal{B}}$, its rate of change $\dot{\vec{\boldsymbol{\omega}}}_{\mathcal{B}}$, and the moment of inertia matrix $[\boldsymbol{I}^{\mathrm{cm}}]$. If you look in the back of the book at Table I, entries 6c and 6d, you will see formulas that reduce to the formulas below if you assume no rotation and thus use $\vec{\omega} = \vec{\boldsymbol{0}}$ and $\dot{\vec{\omega}} = \vec{\boldsymbol{0}}$. On the other hand, rate of change of linear momentum is simple, at least in concept, in this chapter, as well as in the rest of this book. For all motions of all systems we have

$$\dot{\vec{\boldsymbol{L}}} = m_{tot}\vec{\boldsymbol{a}}_{cm}.$$

14.1 Angular momentum for straight-line motion

For straight-line motion, and parallel motion in general, we can derive the simplification in the calculation of $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ as follows:

$$\begin{aligned}
\vec{\boldsymbol{H}}_{/\mathrm{C}} &\equiv \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_i (\text{ definition}) \\
&= \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{v}}_{\mathrm{cm}} \quad (\text{since, } \vec{\boldsymbol{v}}_i = \vec{\boldsymbol{v}}_{\mathrm{cm}}) \\
&= \left(\sum \vec{\boldsymbol{r}}_{i/\mathrm{C}}\, \mathrm{m}_i\right) \times \vec{\boldsymbol{v}}_{\mathrm{cm}}, \\
&= \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times (m_{\mathrm{tot}} \vec{\boldsymbol{v}}_{\mathrm{cm}}), \\
&\quad (\text{ since, } \textstyle\sum \vec{\boldsymbol{r}}_{i/\mathrm{C}}\, \mathrm{m}_i \equiv m_{\mathrm{tot}} \vec{\boldsymbol{r}}_{\mathrm{cm/C}}).
\end{aligned}$$

The derivation that $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{\mathrm{cm/C}} \times (m\vec{\boldsymbol{a}}_{\mathrm{cm}})$ follows from $\dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \equiv \sum \vec{\boldsymbol{r}}_{i/\mathrm{C}} \times \mathrm{m}_i \vec{\boldsymbol{a}}_i$ by the same reasoning.

But for straight-line motion (and, slightly more generally, for any parallel motion), the calculations turn out to be the same as we would get if we put a single point mass at the center-of-mass②:

②**Caution:** The special motions in this chapter are almost the only cases where the angular momentum and its rate of change are so easy to calculate.

$$\begin{aligned} \vec{\boldsymbol{H}}_{/C} &\equiv \sum(\vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{v}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\text{total}} \vec{\boldsymbol{v}}_{\text{cm}}), \\ \dot{\vec{\boldsymbol{H}}}_{/C} &\equiv \sum(\vec{\boldsymbol{r}}_{i/C} \times m_i \vec{\boldsymbol{a}}_i) &= \vec{\boldsymbol{r}}_{cm/C} \times (m_{\text{total}} \vec{\boldsymbol{a}}_{cm}). \end{aligned}$$

Note, there is some subtlety in the definition of $\vec{\boldsymbol{H}}_{/C}$, as explained in section B.

Kinetic energy

Generally things will not be so simple, but for straight-line motion, or any parallel motion where all points on an object have the same velocity and acceleration, kinetic energy and its rate of change are also easy to calculate:

$$E_K = m_{\text{tot}} v_{\text{cm}}^2/2 \qquad \text{and} \qquad \dot{E}_K = m_{\text{tot}} v_{\text{cm}} a_{\text{cm}}.$$

The kinetic energy works the same as if all the mass was concentrated at the center of mass. This result does not generalize to more complex motions.

Approach

To study systems in straight-line motion (as always) we:

- draw a free-body diagram, showing the appropriate forces and couples at places where connections are ‘cut’,
- state reasonable kinematic assumptions based on the motions that the constraints allow,
- write linear and/or angular momentum balance equations and/or energy balance, and
- solve for quantities of interest.

Angular momentum balance about a judiciously chosen axis is a particularly useful tool for reducing the number of equations that need to be solved.

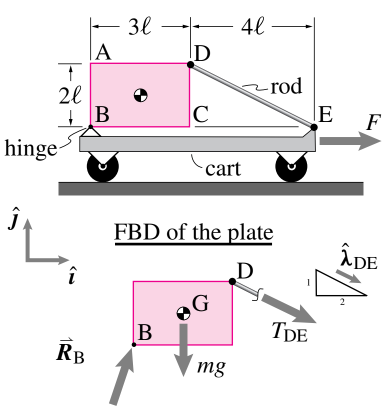

Figure 14.25: Uniform plate supported by a hinge and a rod on an accelerating cart.

Example: Plate on a cart
A uniform rectangular plate $ABCD$ of mass m is supported by a light rigid rod DE and a hinge joint at point B. The dimensions are as shown. The cart has acceleration $a_x \hat{\boldsymbol{\imath}}$ due to a force $F\hat{\boldsymbol{\imath}}$ and the constraints of the wheels. Referring to the free-body diagram in fig. 14.25 and writing angular momentum balance for the plate about point B, we can get an equation for the tension in the rod T_{DE} in terms of m and a_x:

$$
\begin{aligned}
\sum \vec{\boldsymbol{M}}_{/B} &= \dot{\vec{\boldsymbol{H}}}_{/B} \\
\left\{\vec{\boldsymbol{r}}_{D/B} \times (T_{DE}\hat{\boldsymbol{\lambda}}_{DE}) + \vec{\boldsymbol{r}}_{G/B} \times (-mg\hat{\boldsymbol{\jmath}}) \right. &= \left. \vec{\boldsymbol{r}}_{G/B} \times (ma_x\hat{\boldsymbol{\imath}})\right\} \\
\{\} \cdot \hat{\boldsymbol{k}} \Rightarrow \qquad T_{DE} &= \frac{\sqrt{5}}{7} m(a_x - \frac{3}{2}g).
\end{aligned}
$$

Summarizing note:

> angular momentum balance is important even when there is no rotation.

Sliding and pseudo-sliding objects

A car coming to a stop can be roughly modeled as a rigid body that translates and does not rotate. That is, at least for a first approximation, the rotation of the car due to the suspension and tire deformation, can be neglected. The free-body diagram will show various forces with lines of action that do not all act through a single point so that angular momentum balance must be used to analyze the system. Similarly, a bicycle which is braking or a box that is skidding (if not tipping) may be analyzed by assuming straight-line motion.

Example: Car skidding
Consider the accelerating four-wheel drive car in fig. 14.26. The motion quantities for the car are $\dot{\vec{\boldsymbol{L}}} = m_{car}\vec{\boldsymbol{a}}_{car}$ and $\dot{\vec{\boldsymbol{H}}}_{/C} = \vec{\boldsymbol{r}}_{cm/C} \times \vec{\boldsymbol{a}}_{car} m_{car}$. We could calculate angular momentum balance relative to the car's center of mass in which case $\sum \vec{\boldsymbol{M}}_{\text{cm}} = \dot{\vec{\boldsymbol{H}}}_{\text{cm}} = \vec{\boldsymbol{0}}$ (because the position of the center-of-mass relative to the center-of-mass is $\vec{\boldsymbol{0}}$).

As mentioned, it is often useful to calculate angular momentum balance of sliding objects about points of contact (such as where tires contact the road) or about points that lie on lines of action of applied forces when writing angular momentum balance to solve for forces or accelerations. To do so usually eliminates some unknown reactions from the equations to be solved. For example, the angular momentum balance equation about the rear-wheel contact of a car does not contain the rear-wheel contact forces.

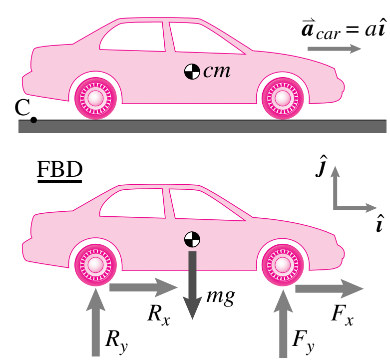

Figure 14.26: A four-wheel drive car accelerating but not tipping. See fig. 2.72 on page 192 for more about FBDs involving wheel contact.

Wheels

The function of wheels is to allow easy sliding-like (pseudo-sliding) motion between objects, at least in the direction they are pointed. On the other hand, wheels do sometimes slip due to:

- being overpowered (as in a screeching accelerating car),
- being braked hard, or
- having very bad bearings (like a rusty toy car).

How wheels are treated when analyzing cars, bikes, and the like depends on both the application and on the level of detail one requires. In *this chapter*, we will always assume that wheels have negligible mass. Thus, when we treat the special case of un-driven and un-braked wheels our free-body diagrams will be as in fig. 2.73a on page 192 and *not* like the one in fig. 2.73b. With the ideal wheel approximation, all of the various cases for a car traveling to the right are shown with partial free-body diagrams of a wheel in fig. 2.72. For the purposes of actually solving problems, we have accepted Coulomb's law of friction as a model for contacting interaction (see page 188 in sec. 2.4).

3-D forces in straight-line motion

The ideas we have discussed apply as well in three dimensions as in two. As you learned from doing statics problems, working out the details in 3D, where vector methods must be used carefully, is more involved than in 2D. As for statics, three-dimensional problems often yield simple results and simple intuitions by considering angular momentum balance about an axis.

Angular momentum balance about an axis

The simplest way to think of angular momentum balance about an axis is to look at angular momentum balance about a point and then take a dot product with a unit vector along an axis:

$$\hat{\boldsymbol{\lambda}} \cdot \left\{ \sum \vec{\boldsymbol{M}}_{/\mathrm{C}} = \dot{\vec{\boldsymbol{H}}}_{/\mathrm{C}} \right\}.$$

Note that the axis need not correspond to any mechanical device in any way resembling an axle. The equation above applies for any point C and any vector $\hat{\boldsymbol{\lambda}}$. If you choose C and $\hat{\boldsymbol{\lambda}}$ judiciously many terms in your equations may drop out.