SAMPLE 14.6 **Force in braking.** A front-wheel-drive car of mass $m = 1200\,\text{kg}$ is cruising at $v = 60\,\text{mph}$ on a straight road when the driver slams on the brake. The car slows down to 20 mph in 4 s while maintaining its straight path.

1. What is the average force (average in time) applied on the car during braking?
2. What is the average power of braking?

Solution

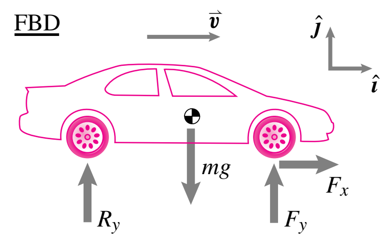

Figure 14.27: Free-body diagram of a front-wheel-drive car during braking. Note that we have (arbitrarily) pointed F_x to the right. The algebra in this problem will tell us that $F_x < 0$.

1. Let us assume that we have an xy coordinate system in which the car is traveling along the x-axis during the entire time under consideration. Then, the velocity of the car before braking, $\vec{v}_1$, and after braking, $\vec{v}_2$, are

$$\vec{v}_1 = v_1\hat{\imath} = 60\,\text{mph}\,\hat{\imath} \quad \text{and} \quad \vec{v}_2 = v_2\hat{\imath} = 20\,\text{mph}\,\hat{\imath}.$$

The linear impulse during braking is $\vec{F}_{\text{ave}}\Delta t$ where $\vec{F} \equiv F_x\hat{\imath}$ (see free-body diagram of the car). Now, from the impulse-momentum relationship,

$$\vec{F}\Delta t = \vec{L}_2 - \vec{L}_1,$$

where $\vec{L}_1$ and $\vec{L}_2$ are linear momenta of the car before and after braking, respectively, and $\vec{F}$ is the average applied force. Therefore,

$$\begin{aligned}
\vec{F} &= \frac{1}{\Delta t}(\vec{L}_2 - \vec{L}_1) = \frac{m}{\Delta t}(\vec{v}_2 - \vec{v}_1) \\
&= \frac{1200\,\text{kg}}{4\,\text{s}}(20 - 60)\,\text{mph}\hat{\imath} \\
&= -12000\frac{\text{kg}}{\text{s}} \cdot \frac{\cancel{\text{mi}}}{\cancel{\text{hr}}} \cdot \frac{1600\ \text{m}}{1\ \cancel{\text{mi}}} \cdot \frac{1\ \cancel{\text{hr}}}{3600\,\text{s}}\hat{\imath} \\
&= -\frac{16,000}{3}\,\text{kg}\cdot\ \text{m/s}^2\hat{\imath} = -5.33\,\text{kN}\hat{\imath}.
\end{aligned}$$

Thus

$$F_x\hat{\imath} = -5.33\,\text{kN}\hat{\imath} \quad \Rightarrow \quad F_x = -5.33\,\text{kN}.$$

$$\boxed{F_x = -5.33\,\text{kN}}$$

2. Let the average power during braking be P_{ave}. Then the work done during braking is $W = \int P_{\text{ave}}dt$. From work-energy principle, we have

$$\begin{aligned}
W &= \Delta E_{\text{K}} \\
\int_{t_1}^{t_2} P_{\text{ave}}dt &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}}(t_2 - t_1) &= \frac{1}{2}m(v_2^2 - v_1^2) \\
P_{\text{ave}} &= \frac{m}{2\Delta t}(v_2^2 - v_1^2)
\end{aligned}$$

Substituting $m = 1200\,\text{kg}$, $\Delta t = 4\,\text{s}$, $v_1 = 60\,\text{mph} = 26.67\,\text{m/s}$ and $v_2 = 20\,\text{mph} = 8.89\,\text{m/s}$, we get

$$P_{\text{ave}} = -94815\,\text{N}{\cdot}\text{m/ s} = -94.815\,\text{kW}.$$

It is easy to check that if we take the average force F_{ave} calculated above and the average speed $v_{\text{ave}} = (v_1 + v_2)/2 = 40\,\text{mph} = 17.77\,\text{m/s}$, then

$$P_{\text{ave}} = F_{\text{ave}}v_{\text{ave}} = -5.33\,\text{kN} \cdot 17.77\,\text{m/s} = -94.815\,\text{kW},$$

as obtained above.

$$\boxed{P_{\text{ave}} = -94.815\,\text{kW}}$$

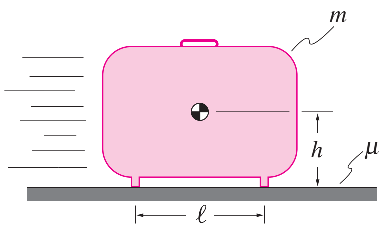

Figure 14.28: A suitcase in motion.

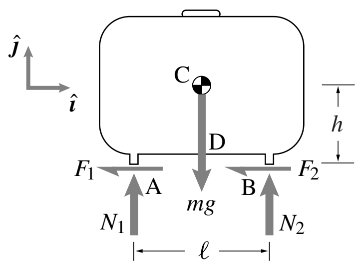

Figure 14.29: FBD of the suitcase.

SAMPLE 14.7 A suitcase skidding on frictional ground. A suitcase of mass m is pushed and sent sliding on a horizontal surface. The suitcase slides without any rotation. A and B are the only contact points of the suitcase with the ground. If the coefficient of friction between the suitcase and the ground is μ, find all the forces applied by the ground on the suitcase. Discuss the results obtained for normal forces.

Solution As usual, we first draw a free-body diagram of the suitcase. The FBD is shown in Fig. 14.29. Assuming Coulomb's law of friction holds, we can write

$$\vec{F}_1 = -\mu N_1 \hat{\imath} \quad \text{and} \quad \vec{F}_2 = -\mu N_2 \hat{\imath}. \tag{14.29}$$

Now we write the balance of linear momentum for the suitcase:

$$\sum \vec{F} = m\vec{a}_{\text{cm}}$$

$$\Rightarrow \qquad -(F_1 + F_2)\hat{\imath} + (N_1 + N_2 - mg)\hat{\jmath} = ma_C\hat{\imath} \tag{14.30}$$

where $\vec{a}_C = a_C\hat{\imath}$ is the unknown acceleration. Dotting eqn. (14.30) with $\hat{\imath}$ and $\hat{\jmath}$ and substituting for F_1 and F_2 from eqn. (14.29) we get

$$-\mu(N_1 + N_2) = ma_C \tag{14.31}$$

$$N_1 + N_2 = mg. \tag{14.32}$$

Equations (14.31) and (14.32) represent 2 scalar equations in three unknowns N_1, N_2 and a. Obviously, we need another equation to solve for these unknowns.

We can write the balance of angular momentum about any point. Points A or B are good choices because they each eliminate some reaction components. Let us write the balance of angular momentum about point A:

$$\sum \vec{M}_{\text{A}} = \dot{\vec{H}}_{\text{A}}$$

$$\begin{aligned}
\sum \vec{M}_{\text{A}} &= \vec{r}_{B/A} \times N_2\hat{\jmath} + \vec{r}_{D/A} \times (-mg)\hat{\jmath} \\
&= \ell\hat{\imath} \times N_2\hat{\jmath} + \frac{\ell}{2}\hat{\imath} \times (-mg)\hat{\jmath} \\
&= (\ell N_2 - mg\frac{\ell}{2})\hat{k}
\end{aligned} \tag{14.33}$$

and

$$\begin{aligned}
\dot{\vec{H}}_{\text{A}} &= \vec{r}_{C/A} \times m\vec{a}_C \\
&= (\frac{\ell}{2}\hat{\imath} + h\hat{\jmath}) \times ma_C\hat{\imath} \\
&= -ma_C h\hat{k}.
\end{aligned} \tag{14.34}$$

Equating (14.33) and (14.34) and dotting both sides with $\hat{k}$ we get the following third scalar equation:

$$\ell N_2 - mg\frac{\ell}{2} = -ma_C h. \tag{14.35}$$

Solving eqns. (14.31) and (14.32) for a we get

$$a_C = -\mu g$$

and substituting this value of a_C in eqn. (14.35) we get

$$\begin{aligned}
N_2 &= \frac{m\mu gh + mg\ell/2}{\ell} \\
&= mg\left(\frac{1}{2} + \frac{h}{\ell}\mu\right).
\end{aligned}$$

Substituting the value of N_2 in either of the equations (14.31) or (14.32) we get

$$N_1 = mg\left(\frac{1}{2} - \frac{h}{\ell}\mu\right).$$

$$N_1 = mg(\tfrac{1}{2} - \tfrac{h}{\ell}\mu),\ N_2 = mg(\tfrac{1}{2} + \tfrac{h}{\ell}\mu),\ f_1 = \mu N_1,\ f_2 = \mu N_2.$$

Discussion: From the expressions for N_1 and N_2 we see that

1. $N_1 = N_2 = \frac{1}{2}mg$ if $\mu = 0$ because without friction there is no deceleration. The problem becomes equivalent to a statics problem.
2. $N_1 = N_2 \approx \frac{1}{2}mg$ if $\ell >> h$. In this case, the moment produced by the friction forces is too small to cause a significant difference in the magnitudes of the normal forces. For example, take $\ell = 20h$ and calculate moment about the center-of-mass to convince yourself.

Graphically, N_1, N_2 and their difference $N_1 - N_2$ are shown in the plot below as a function of h/ℓ for a particular value of μ and mg. As the equations indicate, $N_1 - N_2$ increases steadily as h/ℓ increases, showing how the moment produced by the friction forces makes a bigger and bigger difference between N_1 and N_2 as this moment gets bigger.

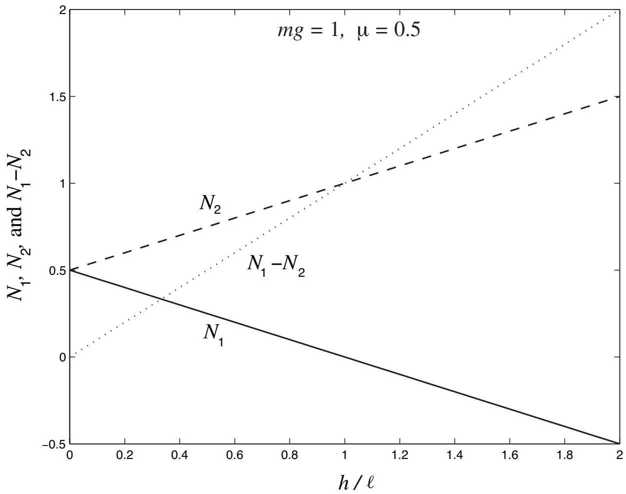

Figure 14.30: The normal forces N_1 and N_2 differ from each other more and more as h/ℓ increases.

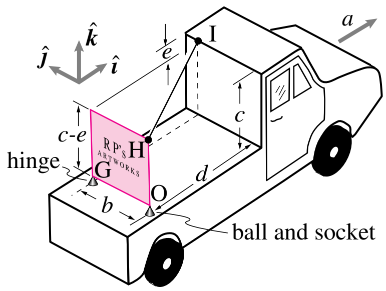

Figure 14.31: An accelerating board in 3-D

SAMPLE 14.8 Uniform acceleration of a board in 3-D. A uniform sign-board of mass $m = 20\,\text{kg}$ sits in the back of an accelerating flatbed truck. The board is supported with a ball-and-socket joint at O and a hinge at G. A light rod from H to I keeps the board from falling over. The truck is on level ground and has forward acceleration $\vec{\boldsymbol{a}} = 0.6\,\text{m/s}^2\hat{\boldsymbol{\imath}}$. The relevant dimensions are $b = 1.5\,\text{m}$, $c = 1.5\,\text{m}$, $d = 3\,\text{m}$, $e = 0.5\,\text{m}$. There is gravity ($g = 10\,\text{m/s}^2$).

1. Draw a free-body diagram of the board.
2. Set up equations to solve for all the unknown forces shown on the FBD.
3. Use the balance of angular momentum about an axis to find the tension in the rod.

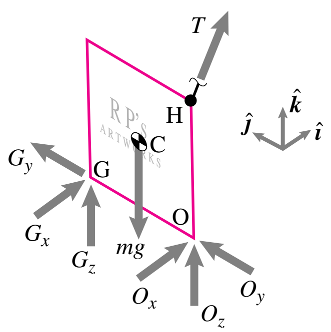

Figure 14.32: FBD of the board

Solution

1. The free-body diagram of the board is shown in Fig. 14.32.
2. Linear momentum balance for the board:

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}, \quad \text{or}$$

$$(G_x + O_x)\hat{\boldsymbol{\imath}} + (G_y + O_y)\hat{\boldsymbol{\jmath}} + (G_z + O_z - mg)\hat{\boldsymbol{k}} + T\hat{\boldsymbol{\lambda}}_{HI} = ma\hat{\boldsymbol{\imath}} \tag{14.36}$$

where

$$\hat{\boldsymbol{\lambda}}_{HI} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\sqrt{d^2 + b^2 + e^2}} = \frac{d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}}{\ell},$$

and ℓ is the length of the rod HI.

Dotting eqn. (14.36) with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three scalar equations:

$$G_x + O_x + T\frac{d}{\ell} = ma \tag{14.37}$$

$$G_y + O_y + T\frac{b}{\ell} = 0 \tag{14.38}$$

$$G_z + O_z + T\frac{e}{\ell} = mg \tag{14.39}$$

Angular momentum balance about point G:

$$\sum \vec{\boldsymbol{M}}_G = \dot{\vec{\boldsymbol{H}}}_G$$

$$\begin{aligned}
\sum \vec{\boldsymbol{M}}_G &= \vec{\boldsymbol{r}}_{C/G} \times (-mg\hat{\boldsymbol{k}}) + \vec{\boldsymbol{r}}_{O/G} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) + \vec{\boldsymbol{r}}_{H/G} \times T\hat{\boldsymbol{\lambda}}_{HI} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times (-mg\hat{\boldsymbol{k}}) - b\hat{\boldsymbol{\jmath}} \times (O_x\hat{\boldsymbol{\imath}} + O_z\hat{\boldsymbol{k}}) \\
&\quad +[-b\hat{\boldsymbol{\jmath}} + (c-e)\hat{\boldsymbol{k}}] \times \frac{T}{\ell}(d\hat{\boldsymbol{\imath}} + b\hat{\boldsymbol{\jmath}} + e\hat{\boldsymbol{k}}) \\
&= \left(\frac{b}{2}mg - bO_z - be\frac{T}{\ell} - (c-e)b\frac{T}{\ell}\right)\hat{\boldsymbol{\imath}} \\
&\quad +(c-e)d\frac{T}{\ell}\hat{\boldsymbol{\jmath}} + \left(bO_x + bd\frac{T}{\ell}\right)\hat{\boldsymbol{k}}
\end{aligned} \tag{14.40}$$

and

$$\begin{aligned}
\dot{\vec{\boldsymbol{H}}}_G &= \vec{\boldsymbol{r}}_{C/G} \times ma\hat{\boldsymbol{\imath}} \\
&= (-\frac{b}{2}\hat{\boldsymbol{\jmath}} + \frac{c-e}{2}\hat{\boldsymbol{k}}) \times ma\hat{\boldsymbol{\imath}} \\
&= \frac{b}{2}ma\hat{\boldsymbol{k}} + \frac{c-e}{2}ma\hat{\boldsymbol{\jmath}}.
\end{aligned} \tag{14.41}$$

Equating (14.40) and (14.41) and dotting both sides with $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$ we get the following three additional scalar equations:

$$O_z + \frac{c}{\ell}T = \frac{1}{2}mg \tag{14.42}$$

$$\frac{d}{\ell}T = \frac{1}{2}ma \tag{14.43}$$

$$O_x + \frac{d}{\ell}T = \frac{1}{2}ma \tag{14.44}$$

Now we have six scalar equations in seven unknowns — O_x, O_y, O_z, G_x, G_y, G_z, and T. From basic linear algebra, we know that we cannot find unique solutions for all these unknowns from the given equations. A closer inspection of eqns. (14.37–14.39) and (14.42–14.44) shows that we can easily solve for O_x, O_z, G_x, G_z, and T, but O_y and G_y cannot be determined uniquely because they appear together as the sum $G_y + O_y$. ① Fortunately, we can find the tension in the wire HI without worrying about the values of O_y and G_y as we show below.

① Note that G_y and O_y will always appear together as the sum $G_y + O_y$ even if you took the angular momentum balance about some other point. This is because they have the same line of action. Thus, they cannot be found independently. This mathematical problem corresponds to the physical reality that the supports at points O and G could be squeezing the plate along the line OG with, say, $O_y = 1000\,\text{N}$ and $G_y = -1000\,\text{N}$ even if there were no gravity, and the truck was not accelerating. To make prestress problems like this tractable, people often make assumptions like, 'Assume $G_y = 0$', that is, they try to get rid of the redundancy in supports to make the problem statically determinate.

3. Balance of angular momentum about axis OG gives:

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG}\cdot\sum\vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\lambda}}_{OG}\cdot\dot{\vec{\boldsymbol{H}}}_G\\ &= \hat{\boldsymbol{\lambda}}_{OG}\cdot(\vec{\boldsymbol{r}}_{C/G}\times ma\hat{\boldsymbol{\imath}}).\end{aligned} \tag{14.45}$$

Since all reaction forces and the weight go through axis OG, they do not produce any moment about this axis (convince yourself that the forces from the reactions have no torque about the axis by calculation or geometry). Therefore,

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG}\cdot\sum\vec{\boldsymbol{M}}_G &= \hat{\boldsymbol{\jmath}}\cdot(\vec{\boldsymbol{r}}_{H/G}\times T\hat{\boldsymbol{\lambda}}_{HI})\\ &= T\frac{d(c-e)}{\ell}.\end{aligned} \tag{14.46}$$

$$\begin{aligned}\hat{\boldsymbol{\lambda}}_{OG}\cdot(\vec{\boldsymbol{r}}_{C/G}\times ma\hat{\boldsymbol{\imath}}) &= \hat{\boldsymbol{\jmath}}\cdot\left[\left(\frac{b}{2}\hat{\boldsymbol{\jmath}}+\frac{c-e}{2}\hat{\boldsymbol{k}}\right)\times ma\hat{\boldsymbol{\imath}}\right]\\ &= ma\frac{(c-e)}{2}.\end{aligned} \tag{14.47}$$

Equating (14.46) and (14.47), as required by eqn. (14.45), we get

$$\begin{aligned}T &= \frac{ma\ell}{2d}\\ &= \frac{20\,\text{kg}\cdot 0.6\,\text{m/s}^2\cdot 3.39\,\text{m}}{2\cdot 3\,\text{m}}\\ &= 6.78\,\text{N}.\end{aligned}$$

$$\boxed{T_{HI} = 6.78\,\text{N}}$$

SAMPLE 14.9 Computer solution of algebraic equations. In the previous sample problem (Sample 14.8), six equations were obtained to solve for the six unknown forces (assuming $G_y = 0$). (i) Set up the six equations in matrix form and (ii) solve the matrix equation on a computer. Check the solution by substituting the values obtained in one or two equations.

Solution

1. The six scalar equations — (14.37), (14.38), (14.39), (14.42), (14.43), and (14.44) are amenable to hand calculations. We, however, set up these equations in matrix form and solve the matrix equation on the computer. The matrix form of the equations is:

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & \frac{d}{\ell} \\ 0 & 1 & 0 & 0 & 0 & \frac{b}{\ell} \\ 0 & 0 & 1 & 0 & 1 & \frac{e}{\ell} \\ 0 & 0 & 1 & 0 & 0 & \frac{c}{\ell} \\ 0 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \\ 1 & 0 & 0 & 0 & 0 & \frac{d}{\ell} \end{bmatrix} \begin{Bmatrix} O_x \\ O_y \\ O_z \\ G_x \\ G_z \\ T \end{Bmatrix} = \begin{Bmatrix} ma \\ 0 \\ mg \\ mg/2 \\ ma/2 \\ ma/2 \end{Bmatrix}. \tag{14.48}$$

The above equation can be written, in matrix notation, as

$$\mathbf{A}\,\mathbf{x} = \mathbf{b}$$

where **A** is the coefficient matrix, **x** is the vector of the unknown forces, and **b** is the vector on the right hand side of the equation. Now we are ready to solve the system of equations on the computer.

2. We use the following pseudo-code to solve the above matrix equation. ②

```
m = 20, a = 0.6,
b = 1.5, c = 1.5, d = 3, e = 0.5, g = 10,
l = sqrt(b^2 + d^2 + e^2),

A = [1 0 0 1 0 d/l
     0 1 0 0 0 b/l
     0 0 1 0 1 e/l
     0 0 1 0 0 c/l
     0 0 0 0 0 d/l
     1 0 0 0 0 d/l]
b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =                 % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
  102.0000
    6.7823
```

The solution obtained from the computer means:

$$O_x = 0,\ O_y = -3\,\mathrm{N},\ O_z = 97\,\mathrm{N},\ G_x = 6\,\mathrm{N},\ G_z = 102\,\mathrm{N},\ T = 6.78\,\mathrm{N}.$$

We now hand-check the solution by substituting the values obtained in, say, Eqns. (14.38) and (14.43). Before we substitute the values of forces, we need to calculate the length ℓ.

$$\begin{aligned} \ell &= \sqrt{d^2 + b^2 + e^2} \\ &= 3.3912\,\mathrm{m}. \end{aligned}$$

② Be careful with units. Most computer programs will not take care of your units. They only deal with numerical input and output. You should, therefore, make sure that your variables have proper units for the required calculations. Either do dimensionless calculations or use consistent units for all quantities.

Therefore,

$$\text{Eqn. (14.38):} \quad O_y + T\frac{b}{\ell} = -3\,\mathrm{N} + 6.78\,\mathrm{N}\cdot\frac{1.5\,\mathrm{m}}{3.3912\,\mathrm{m}} \stackrel{\surd}{=} 0,$$

$$\text{Eqn. (14.43):} \quad \frac{d}{\ell}T - \frac{1}{2}ma = \frac{3\,\mathrm{m}}{3.3912\,\mathrm{m}}6.78\,\mathrm{N} - \frac{1}{2}20\,\mathrm{kg}\,0.6\,\mathrm{m/s^2} \stackrel{\surd}{=} 0.$$

Thus, the computer solution agrees with our equations.

Comments: We could have solved the six equations for seven unknowns without assuming $G_y = 0$ if our computer program or package allows us to do so. We will, of course, not get a unique solution. For example, by taking the following $\mathbf{A}$, a 6×7 matrix, and solving $\mathbf{A\,x = b}$ for $\mathbf{x} = [O_x\ O_y\ O_z; G_x\ G_y\ G_z\ T]^T$ with the same $\mathbf{b}$ as input above, we get the solution as shown below.

```
 A = [1 0 0 1 0 0 d/l
      0 1 0 0 1 0 b/l
      0 0 1 0 0 1 e/l
      0 0 1 0 0 0 c/l
      0 0 0 0 0 0 d/l
      1 0 0 0 0 0 d/l]
 b = [m*a, 0, m*g, m*g/2, m*a/2, m*a/2]'

{Solve A x = b for x}
x =               % this is the computer output
         0
   -3.0000
   97.0000
    6.0000
         0
  102.0000
    6.7823
```

This is the same solution as we got before except that it includes $G_y = 0$ in the solution. Now, if we add a vector $\Delta\,\mathbf{x} = [0\ \alpha\ 0\ 0\ -\alpha\ 0\ 0]^T$ to $\mathbf{x}$ where α is any number, and compute $\mathbf{A\,(x{+}\Delta x)}$, we get back $\mathbf{b}$. That is, the six equilibrium conditions are satisfied irrespective of the actual values of O_y and G_y as long as the value of $O_y + G_y$ remains the same.