14.1 1-D constrained motion and pulleys

In this section masses are connected together with bars or ropes. These connections are idealized as being inextensible. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as having a fixed length. Although the idealization of zero deformation is a simplification, it is a simplification that requires special treatment. It is the simplest example of a *kinematic constraint*.

Figure 14.2 shows a schematic of one car pulling another. One-dimensional free-body diagrams are also shown. The force F is the force transmitted from the road to the front car through the tires. The tension T is the tension in the connecting chain. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1\ddot{x}_1 \quad \text{and} \quad F - T = m_2\ddot{x}_2. \tag{14.1}$$

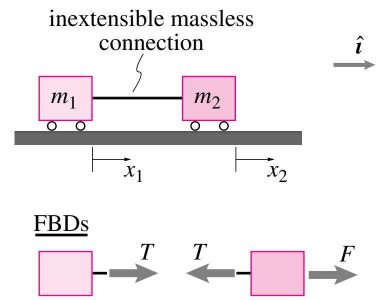

Figure 14.2: A schematic of one car pulling another, or of a boat pulling a barge. Also shown are FBDs of the bodies separately. Because our analysis is only in one spatial dimension, forces with no component in $\hat{\imath}$ direction are not shown.

These equations are exactly the same as for cars connected by a spring, a dashpot, or any idealized-as-massless connector. And all these systems have the same free-body diagrams but different motions. If the connection were with a spring or dashpot the equations above would be supplemented with

$$T = k(x_2 - x_1 - \ell_0) \quad \text{or} \quad T = c(\dot{x}_2 - \dot{x}_1)$$

In this case we need our equations to somehow indicate that the two particles are not allowed to move independently. We need a constraint equation to replace these constitutive laws.

Kinematic constraint: two approaches

There are two basic ways of dealing with kinematic constraints:

1. Use separate free-body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or
2. do something clever to avoid having to find the constraint forces.

Method 1: Finding the accelerations and the constraint forces together

The geometric (or kinematic) constraint that two masses move together is

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \tag{14.2}$$

If we take F and the two masses as given, equations 14.1 and 14.2 are three equations for the unknowns $\ddot{x}_1, \ddot{x}_2$, and T. In matrix form, we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

We can solve these equations to find $\ddot{x}_1, \ddot{x}_2$, and T in terms of F.

Method 2: Finesse having to find the constraint force

On the other hand, if all we are interested in are the accelerations of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free-body diagram of the entire system as in fig. 14.3. If we just call the acceleration of the system $\ddot{x}$ we get, from linear momentum balance, one equation in one unknown:

$$F = (m_1 + m_2)\ddot{x}.$$

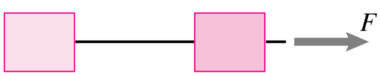

Figure 14.3: A free-body diagram of the whole system. Note that the unknown tension (constraint) force does not show. As usual for 1D mechanics, vertical forces are left off for simplicity (although it would be more correct to include them).

Kinematic constraints

A generalization of the 1D inextensible-cable constraint example above is the rigid-object constraint where not just two, but many particles are assumed to keep constant distance from one another, and in one, two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics as a subject is the difficulty of dealing with kinematic constraints.

Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Objects connected by inextensible ropes around ideal pulleys are also examples of kinematic constraint.

Constant length and constant tension

Problems with pulleys are solved by using two facts about idealized strings. First, an ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant (not varying in time):

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ldots = \text{constant}. \qquad (14.3)$$

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string①.The tension on one side of a pulley is the same as the tension on the other side:

①See fig. 5.27 on page 262 and the related text which shows why $T_1 = T_2$ for one round pulley idealized as frictionless and massless.

$$T_1 = T_2 = T_3 \ldots. \tag{14.4}$$

Example: Length of string calculation

We use the trivial pulley example in fig. 14.4. Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv constant. \tag{14.5}$$

One portion of the string touches half of the pulley circumference, πr, even if x_A and x_B change in time and different portions of string wrap around the pulley at different times.

We now formally deduce the relations between the velocities and accelerations of points A and B. Differentiating equation 14.5 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 &= \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A &= -\dot{x}_B \\ \Rightarrow \ddot{x}_A &= -\ddot{x}_B \end{aligned} \tag{14.6}$$

When point A is displaced to the right by an amount Δx_A, point B is displaced exactly the same amount but to the left; that is, $\Delta x_A = -\Delta x_B$. Note, to find the kinematic relations 14.6 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant-in-time quantities (the pulley half-circumference and the string length) get 'killed' in the process of differentiation.

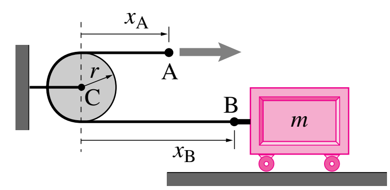

Figure 14.4: One mass, one pulley, and one string

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations. And in these cases, as in the example above, the pulley contact length drops out of the equations anyway.

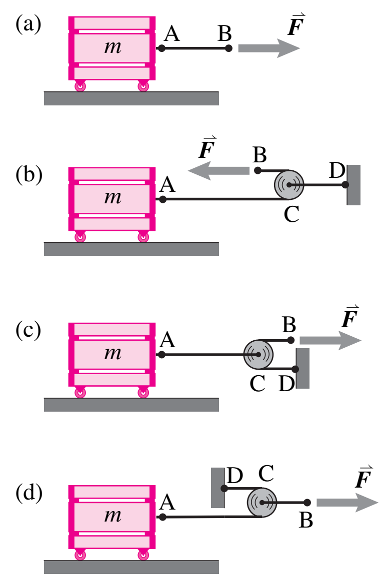

Figure 14.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that multiplies force and attenuates motion, and (d) a pulley system that attenuates force and amplifies motion.

The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in fig. 14.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in fig. 14.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively.

In fig. 14.5b we pull one direction and the mass accelerates the other way. This illustrates

> the simplest use of a pulley, to redirect an applied force.

The force on the mass has magnitude $|\vec{\boldsymbol{F}}|$ and there is no mechanical advantage.

Figure 14.5c shows

> the classic use of a pulley, to multiply a force.

Here's a detailed solution to this problem.

Example: Pulley in figure 14.5c

The methods here will solve every pulley problem 2. First draw the FBDs and rope geometry sketch (fig. 14.6). First linear momentum balance (and force balance for the negligible-mass pulley):

$$\text{LMB} \quad \Rightarrow \quad m\ddot{x}_A = T_A \qquad \text{and} \qquad T_A = 2T_B. \tag{14.7}$$

Next the rope length kinematics:

$$\ell_1 = x_C - x_A \qquad \text{and} \qquad \ell_2 = (x_B - x_C) + (x_D - x_C) + \pi r$$

Differentiate the rope-length equations twice, remembering that ℓ_1, ℓ_2, r and x_D are all constants, to get

$$\ddot{x}_C = \ddot{x}_A \qquad \text{and} \qquad \ddot{x}_B = 2\ddot{x}_C. \tag{14.8}$$

The second equation above ($\ddot{x}_B = 2\ddot{x}_C$) is the only one that you couldn't write down at a glance. Solve these equations in terms of $F = T_B$ to get:

$$T_A = 2F \qquad \text{and} \qquad \ddot{x}_A = 2F/m \qquad \text{and} \qquad \ddot{x}_B = 4F/m.$$

Figure 14.5d shows

> a less common use of a pulley, to multiply motion.

In this case an analysis similar to that above (that you are asked to do in problem 14.1.6 on page 711) shows that:

$$T_A = F/2 \qquad \text{and} \qquad \ddot{x}_A = F/(2m) \qquad \text{and} \qquad \ddot{x}_B = F/(4m)$$

Despite the superficial similarity, this setup is the opposite to the use in fig. 14.5c.

Force amplification is motion attenuation

In fig. 14.5c a force F is applied at B. The resulting force on the mass at A is $2F$ and point A moves with half the acceleration of point B. However, in fig. 14.5d the resulting force on the mass at A is $F/2$, and point A moves with twice the acceleration of point B.

These illustrate a general duality rule

2 **How to solve pulley problems**
Follow these rules, and you can solve all pulley problems (see the example from fig. 14.5c in the text).

1. Draw a free-body diagram of each pulley and each mass, taking account that the tension in any given rope is constant along its length.
2. Write linear momentum balance for each pulley and mass (this might just be force balance if you neglect the mass of, say, a pulley).
3. Do length accounting for each rope, taking account that its length is constant. To avoid sign errors, *use a reference point that is off to the side of the whole mechanism,* even if there is no such point in the original problem (the wall at 0 in the solution to fig. 14.5c above).
4. Differentiate the string length equations (from the step above), twice.
5. Solve the equations from parts 2 and 4 (above) for desired unknowns.

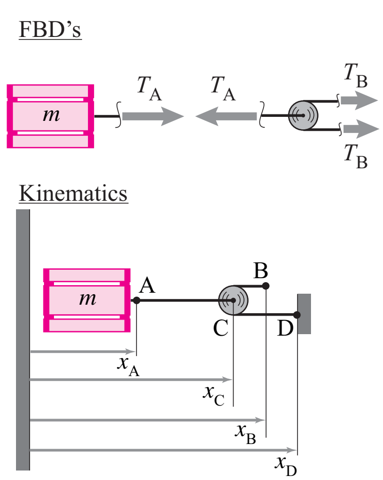

Figure 14.6: FBDs and kinematics for the mechanism in fig. 14.5c

If force is amplified then motion is equally attenuated. If motion is amplified the force is equally attenuated ③.

③Caveats: Negligible dissipation and negligible mass of the transmission parts.

High gear and low gear. This rule applies to all frictionless passive transmissions (*e.g.,* levers, gear trains, hydraulic systems) with negligible inertia: In a low gear in a car (or bicycle) the force at the wheel is large for a given force of the engine (leg muscle) but the wheel doesn't turn much for a given displacement of the engine (foot). In a high gear the force at the wheel is small but the wheel turns a lot for a given amount of engine rotation (or foot displacement). These are, of course, special cases of the much more general rule: you can't win!

Power balance

In every special case we can derive the duality rule (above) using momentum balance and kinematics. But the general result is best understood with energy balance:

the power of the applied force is the power applied to the mass.

Example: Power balance and the pulley in figure 14.5c
There are various ways of setting up the energy (or power) balance equations. Here is one:

$$\text{Power into the transmission} = \text{Power out of the transmission}$$

The power in is $T_B \cdot \dot{x}_B$. The power out, delivered to A, is $T_A \cdot \dot{x}_A$. So,

$$T_B \cdot \dot{x}_B = T_A \cdot \dot{x}_A \quad \Rightarrow \quad \frac{T_A}{T_B} = \frac{\dot{x}_B}{\dot{x}_A}.$$

But because $\dot{x}_a/\dot{x}_B$ is a constant, differentiating we get $\ddot{x}_A/\ddot{x}_B = \dot{x}_A/\dot{x}_B$. So we also have

$$\frac{T_A}{T_B} = \frac{\ddot{x}_B}{\ddot{x}_A}.$$

The 'effective mass' of a point of force application The feel of the machine is of concern for machines that people handle. One aspect of feel is the *effective mass* (sometimes called 'reflected inertia') is defined by the response of a point to an applied force.

$$m_{\text{eff}} = \frac{|\vec{\boldsymbol{F}}_{\text{B}}|}{|\vec{\boldsymbol{a}}_{\text{B}}|}.$$

For the case of fig. 14.5a and fig. 14.5b the effective mass of point B is just m. For the case of fig. 14.5c the block at A has $2|\vec{\boldsymbol{F}}|$ acting on it and point B has twice the acceleration of point A. So the acceleration of point B is $4F/m = F/(m/4)$ and the effective mass of point B is $m/4$. For the case of fig. 14.5d, the mass only has $|\vec{\boldsymbol{F}}|/2$ acting on it and point B only has half the acceleration of point A, so the effective mass is $4m$.

These special cases exemplify the general rule:

> The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems shown in fig. 14.5c and fig. 14.5d which look so similar to a novice, actually differ by a factor of $2^2 \cdot 2^2 = 16$. With a given F and m point B in fig. 14.5c has 16 times the acceleration of point B in fig. 14.5d.