14.1 1-D constrained motion and pulleys

In this section masses are connected together with bars or ropes. These connections are idealized as being inextensible. Consider a car towing another with a strong light chain. We may not want to consider the elasticity of the chain but instead idealize the chain as having a fixed length. Although the idealization of zero deformation is a simplification, it is a simplification that requires special treatment. It is the simplest example of a *kinematic constraint*.

Figure 14.2 shows a schematic of one car pulling another. One-dimensional free-body diagrams are also shown. The force F is the force transmitted from the road to the front car through the tires. The tension T is the tension in the connecting chain. From linear momentum balance for each of the objects (modeled as particles):

$$T = m_1\ddot{x}_1 \quad \text{and} \quad F - T = m_2\ddot{x}_2. \tag{14.1}$$

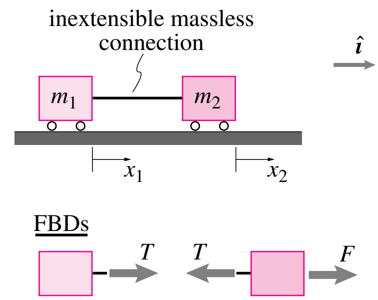


Figure 14.2: A schematic of one car pulling another, or of a boat pulling a barge. Also shown are FBDs of the bodies separately. Because our analysis is only in one spatial dimension, forces with no component in $\hat{\imath}$ direction are not shown.

These equations are exactly the same as for cars connected by a spring, a dashpot, or any idealized-as-massless connector. And all these systems have the same free-body diagrams but different motions. If the connection were with a spring or dashpot the equations above would be supplemented with

$$T = k(x_2 - x_1 - \ell_0) \quad \text{or} \quad T = c(\dot{x}_2 - \dot{x}_1)$$

In this case we need our equations to somehow indicate that the two particles are not allowed to move independently. We need a constraint equation to replace these constitutive laws.

Kinematic constraint: two approaches

There are two basic ways of dealing with kinematic constraints:

1. Use separate free-body diagrams and equations of motion for each particle and then add extra kinematic constraint equations, or
2. do something clever to avoid having to find the constraint forces.

Method 1: Finding the accelerations and the constraint forces together

The geometric (or kinematic) constraint that two masses move together is

$$x_1 = x_2 + \text{Constant}.$$

We can differentiate the kinematic constraint twice to get

$$\ddot{x}_1 = \ddot{x}_2. \tag{14.2}$$

If we take F and the two masses as given, equations 14.1 and 14.2 are three equations for the unknowns $\ddot{x}_1, \ddot{x}_2$, and T. In matrix form, we have:

$$\begin{bmatrix} m_1 & 0 & -1 \\ 0 & m_2 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ T \end{bmatrix} = \begin{bmatrix} 0 \\ F \\ 0 \end{bmatrix}.$$

We can solve these equations to find $\ddot{x}_1, \ddot{x}_2$, and T in terms of F.

Method 2: Finesse having to find the constraint force

On the other hand, if all we are interested in are the accelerations of the cars it would be nice to avoid even having to think about the constraint force. One way to avoid dealing with the constraint force is to draw a free-body diagram of the entire system as in fig. 14.3. If we just call the acceleration of the system $\ddot{x}$ we get, from linear momentum balance, one equation in one unknown:

$$F = (m_1 + m_2)\ddot{x}.$$

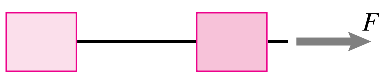


Figure 14.3: A free-body diagram of the whole system. Note that the unknown tension (constraint) force does not show. As usual for 1D mechanics, vertical forces are left off for simplicity (although it would be more correct to include them).

Kinematic constraints

A generalization of the 1D inextensible-cable constraint example above is the rigid-object constraint where not just two, but many particles are assumed to keep constant distance from one another, and in one, two or three dimensions. Another important constraint is an ideal hinge connection between two objects. Much of the theory of mechanics after Newton has been motivated by a desire to deal easily with these and other kinematic constraints. In fact, one way of characterizing the primary difficulty of dynamics as a subject is the difficulty of dealing with kinematic constraints.

Pulleys

Pulleys are used to redirect force to amplify or attenuate force and to amplify or attenuate motion. Like a lever, a pulley system is an example of a mechanical transmission. Objects connected by inextensible ropes around ideal pulleys are also examples of kinematic constraint.

Constant length and constant tension

Problems with pulleys are solved by using two facts about idealized strings. First, an ideal string is inextensible so the sum of the string lengths, over the different inter-pulley sections, adds to a constant (not varying in time):

$$\ell_1 + \ell_2 + \ell_3 + \ell_4 + \ldots = \text{constant}. \qquad (14.3)$$

Second, for round pulleys of negligible mass and no bearing friction, tension is constant along the length of the string[1].The tension on one side of a pulley is the same as the tension on the other side:

$$T_1 = T_2 = T_3 \dots . \tag{14.4}$$

[1]See fig. 5.27 on page 262 and the related text which shows why $T_1 = T_2$ for one round pulley idealized as frictionless and massless.

Example: Length of string calculation

We use the trivial pulley example in fig. 14.4. Starting from point A, we add up the lengths of string

$$\ell_{tot} = x_A + \pi r + x_B \equiv constant. \tag{14.5}$$

One portion of the string touches half of the pulley circumference, πr, even if x_A and x_B change in time and different portions of string wrap around the pulley at different times.

We now formally deduce the relations between the velocities and accelerations of points A and B. Differentiating equation 14.5 with respect to time once and then again, we get

$$\begin{aligned} \dot{\ell}_{tot} = 0 \quad &= \quad \dot{x}_A + 0 + \dot{x}_B \\ \Rightarrow \dot{x}_A \quad &= \quad -\dot{x}_B \\ \Rightarrow \ddot{x}_A \quad &= \quad -\ddot{x}_B \end{aligned} \tag{14.6}$$

When point A is displaced to the right by an amount Δx_A, point B is displaced exactly the same amount but to the left; that is, $\Delta x_A = -\Delta x_B$. Note, to find the kinematic relations 14.6 for the pulley system, we never need to know the total length of the string, only that it is constant in time. The constant-in-time quantities (the pulley half-circumference and the string length) get 'killed' in the process of differentiation.

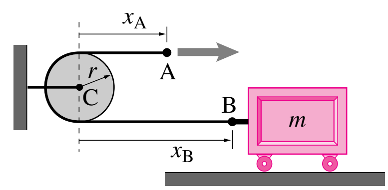


Figure 14.4: One mass, one pulley, and one string

Commonly we think of pulleys as small and thus never account for the pulley-contacting string length. Luckily this approximation generally leads to no error because we most often are interested in displacements, velocities, and accelerations. And in these cases, as in the example above, the pulley contact length drops out of the equations anyway.

The classic simple uses of pulleys

First imagine trying to move a load with no pulley as in fig. 14.5a. The force you apply goes right to the mass. This is like direct drive with no transmission.

Now you would like to use pulleys to help you move the mass. In the cases we consider here the mass is on a frictionless support and we are trying to accelerate it. But the concepts are the same if there are also resisting forces on the mass. What can we do with one pulley? Three possibilities are shown in fig. 14.5b-d which might, at a blinking glance, look roughly the same. But they are quite different. Here we discuss each design qualitatively.

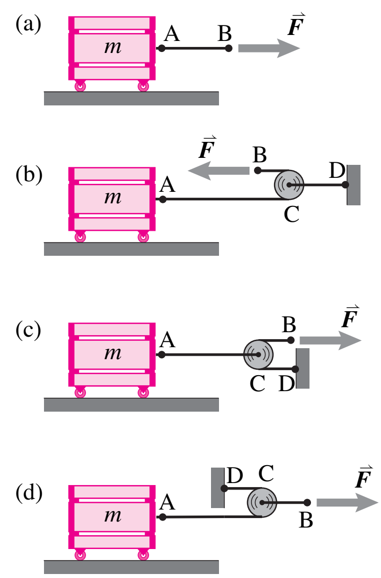


Figure 14.5: The four classic cases: (a) no pulley, (b) a pulley system with no mechanical advantage, (c) a pulley system that multiplies force and attenuates motion, and (d) a pulley system that attenuates force and amplifies motion.

In fig. 14.5b we pull one direction and the mass accelerates the other way. This illustrates

> the simplest use of a pulley, to redirect an applied force.

The force on the mass has magnitude $|\vec{F}|$ and there is no mechanical advantage.

Figure 14.5c shows

> the classic use of a pulley, to multiply a force.

Here's a detailed solution to this problem.

Example: Pulley in figure 14.5c

The methods here will solve every pulley problem②. First draw the FBDs and rope geometry sketch (fig. 14.6). First linear momentum balance (and force balance for the negligible-mass pulley):

$$\text{LMB} \quad \Rightarrow \quad m\ddot{x}_A = T_A \quad \text{and} \quad T_A = 2T_B. \tag{14.7}$$

Next the rope length kinematics:

$$\ell_1 = x_C - x_A \quad \text{and} \quad \ell_2 = (x_B - x_C) + (x_D - x_C) + \pi r$$

Differentiate the rope-length equations twice, remembering that ℓ_1, ℓ_2, r and x_D are all constants, to get

$$\ddot{x}_C = \ddot{x}_A \quad \text{and} \quad \ddot{x}_B = 2\ddot{x}_C. \tag{14.8}$$

The second equation above ($\ddot{x}_B = 2\ddot{x}_C$) is the only one that you couldn't write down at a glance. Solve these equations in terms of $F = T_B$ to get:

$$T_A = 2F \quad \text{and} \quad \ddot{x}_A = 2F/m \quad \text{and} \quad \ddot{x}_B = 4F/m.$$

② **How to solve pulley problems**
Follow these rules, and you can solve all pulley problems (see the example from fig. 14.5c in the text).

1. Draw a free-body diagram of each pulley and each mass, taking account that the tension in any given rope is constant along its length.
2. Write linear momentum balance for each pulley and mass (this might just be force balance if you neglect the mass of, say, a pulley).
3. Do length accounting for each rope, taking account that its length is constant. To avoid sign errors, *use a reference point that is off to the side of the whole mechanism*, even if there is no such point in the original problem (the wall at 0 in the solution to fig. 14.5c above).
4. Differentiate the string length equations (from the step above), twice.
5. Solve the equations from parts 2 and 4 (above) for desired unknowns.

Figure 14.5d shows

> a less common use of a pulley, to multiply motion.

In this case an analysis similar to that above (that you are asked to do in problem 14.1.6 on page 711) shows that:

$$T_A = F/2 \quad \text{and} \quad \ddot{x}_A = F/(2m) \quad \text{and} \quad \ddot{x}_B = F/(4m)$$

Despite the superficial similarity, this setup is the opposite to the use in fig. 14.5c.

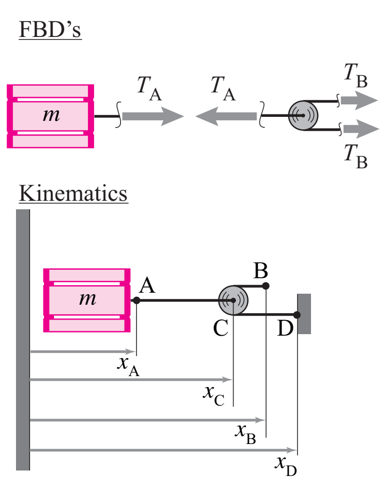


Figure 14.6: FBDs and kinematics for the mechanism in fig. 14.5c

Force amplification is motion attenuation

In fig. 14.5c a force F is applied at B. The resulting force on the mass at A is $2F$ and point A moves with half the acceleration of point B. However, in fig. 14.5d the resulting force on the mass at A is $F/2$, and point A moves with twice the acceleration of point B.

These illustrate a general duality rule

If force is amplified then motion is equally attenuated. If motion is amplified the force is equally attenuated ③.

③Caveats: Negligible dissipation and negligible mass of the transmission parts.

High gear and low gear. This rule applies to all frictionless passive transmissions (*e.g.,* levers, gear trains, hydraulic systems) with negligible inertia: In a low gear in a car (or bicycle) the force at the wheel is large for a given force of the engine (leg muscle) but the wheel doesn't turn much for a given displacement of the engine (foot). In a high gear the force at the wheel is small but the wheel turns a lot for a given amount of engine rotation (or foot displacement). These are, of course, special cases of the much more general rule: you can't win!

Power balance

In every special case we can derive the duality rule (above) using momentum balance and kinematics. But the general result is best understood with energy balance:

the power of the applied force is the power applied to the mass.

Example: Power balance and the pulley in figure 14.5c
There are various ways of setting up the energy (or power) balance equations. Here is one:

$$\text{Power into the transmission} = \text{Power out of the transmission}$$

The power in is $T_B \cdot \dot{x}_B$. The power out, delivered to A, is $T_A \cdot \dot{x}_A$. So,

$$T_B \cdot \dot{x}_B = T_A \cdot \dot{x}_A \quad \Rightarrow \quad \frac{T_A}{T_B} = \frac{\dot{x}_B}{\dot{x}_A}.$$

But because $\dot{x}_a/\dot{x}_B$ is a constant, differentiating we get $\ddot{x}_A/\ddot{x}_B = \dot{x}_A/\dot{x}_B$. So we also have

$$\frac{T_A}{T_B} = \frac{\ddot{x}_B}{\ddot{x}_A}.$$

The 'effective mass' of a point of force application The feel of the machine is of concern for machines that people handle. One aspect of feel is the *effective mass* (sometimes called 'reflected inertia') is defined by the response of a point to an applied force.

$$m_{\text{eff}} = \frac{|\vec{\boldsymbol{F}}_{\text{B}}|}{|\vec{\boldsymbol{a}}_{\text{B}}|}.$$

For the case of fig. 14.5a and fig. 14.5b the effective mass of point B is just m. For the case of fig. 14.5c the block at A has $2|\vec{\boldsymbol{F}}|$ acting on it and point B has twice the acceleration of point A. So the acceleration of point B is $4F/m = F/(m/4)$ and the effective mass of point B is $m/4$. For the case of fig. 14.5d, the mass only has $|\vec{\boldsymbol{F}}|/2$ acting on it and point B only has half the acceleration of point A, so the effective mass is $4m$.

These special cases exemplify the general rule:

> The effective mass of one end of a transmission is the mass of the other end multiplied by the square of the motion amplification ratio.

In terms of the effective mass, the systems shown in fig. 14.5c and fig. 14.5d which look so similar to a novice, actually differ by a factor of $2^2 \cdot 2^2 = 16$. With a given F and m point B in fig. 14.5c has 16 times the acceleration of point B in fig. 14.5d.

SAMPLE 14.1 **Find the motion of two cars.** One car is towing another of equal mass on level ground. The thrust of the wheels of the first car is F. The second car rolls frictionlessly. Find the acceleration of the system two ways:

1. using separate free-body diagrams,
2. using a system free-body diagram.

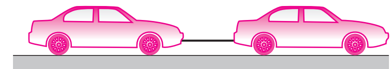


Figure 14.7

Solution

1. The free-body diagram of each car is shown below, in fig. 14.8.

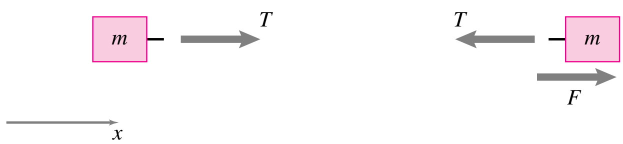


Figure 14.8: Partial free-body diagrams of the two cars (the vertical ground reactions are not shown as they are of no interest to us for the horizontal motion.

From the linear momentum balance of each car, we get

$$
\begin{aligned}
m\ddot{x}_1 &= T && (14.9)\\
F - T &= m\ddot{x}_2 && (14.10)
\end{aligned}
$$

The kinematic constraint of towing (the cars move together, *i.e.*, no relative displacement between the cars) gives

$$\ddot{x}_1 - \ddot{x}_2 = 0 \qquad (14.11)$$

Solving eqns. (14.9), (14.10), and (14.11) simultaneously, we get

$$\ddot{x}_1 = \ddot{x}_2 = \frac{F}{2m} \quad (T = \frac{F}{2})$$

2. The free-body diagram of the two cars together is shown below, in fig. 14.9.

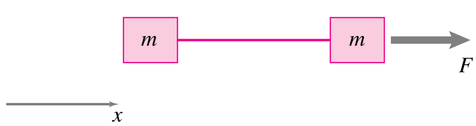


Figure 14.9:

From the linear momentum balance of the two cars as one system, we get

$$
\begin{aligned}
m\ddot{x} + m\ddot{x} &= F\\
\ddot{x} &= F/2m
\end{aligned}
$$

$$\boxed{\ddot{x} = \ddot{x}_1 = \ddot{x}_2 = F/2m}$$

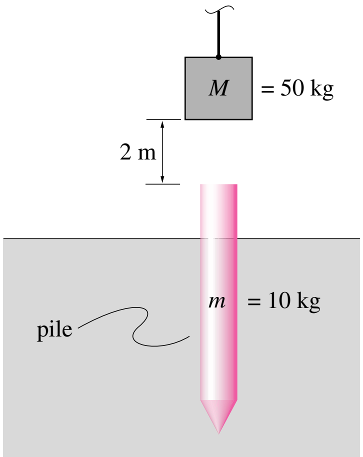


Figure 14.10

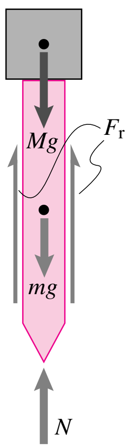


Figure 14.11: Free-body diagram of the hammer and pile system. F_r is the total resistance of the ground.

SAMPLE 14.2 Driving a pile into the ground. A cylindrical wooden pile of mass 10 kg and cross-sectional diameter 20 cm is driven into the ground with the blows of a hammer. The hammer is a block of steel with mass 50 kg which is dropped from a height of 2 m to deliver the blow. At the nth blow the pile is driven into the ground by an additional 5 cm. Assuming the impact between the hammer and the pile to be totally inelastic (*i.e.,* the two stick together), find the average resistance of the soil to penetration of the pile.

Solution Let F_r be the average (constant over the period of driving the pile by 5 cm) resistance of the soil. From the free-body diagram of the pile and hammer system, we have

$$\sum \vec{F} = -mg\hat{\boldsymbol{\jmath}} - Mg\hat{\boldsymbol{\jmath}} + N\hat{\boldsymbol{\jmath}} + F_r\hat{\boldsymbol{\jmath}}.$$

But N is the normal reaction of the ground, which from static equilibrium, must be equal to $mg + Mg$. Thus,

$$\sum \vec{F} = F_r\hat{\boldsymbol{\jmath}}.$$

Therefore, from linear momentum balance ($\sum \vec{F} = m\vec{a}$),

$$\vec{a} = \frac{F_r}{M+m}\hat{\boldsymbol{\jmath}}.$$

Now we need to find the acceleration from given conditions. Let v be the speed of the hammer just before impact and V be the combined speed of the hammer and the pile immediately after impact. Then, treating the hammer and the pile as one system, we can ignore all other forces *during* the impact (none of the external forces: gravity, soil resistance, ground reaction, is comparable to the impulsive impact force, see page 904). The impact force is internal to the system. Therefore, during impact, $\sum \vec{F} = \vec{0}$ which implies that linear momentum is conserved. Thus

$$\begin{aligned} -Mv\hat{\boldsymbol{\jmath}} &= -(m+M)V\hat{\boldsymbol{\jmath}} \\ \Rightarrow \quad V &= \left(\frac{M}{m+M}\right)v = \frac{50\,\text{kg}}{60\,\text{kg}}v = \frac{5}{6}v. \end{aligned}$$

The hammer speed v can be easily calculated, since it is the free fall speed from a height of 2 m:

$$v = \sqrt{2gh} = \sqrt{2\cdot(9.81\,\text{m/s}^2)\cdot(2\,\text{m})} \quad = 6.26\,\text{m/s} \quad \Rightarrow \quad V = \frac{5}{6}v = 5.22\,\text{m/s}.$$

The pile and the hammer travel a distance of $s = 5\,\text{cm}$ under the deceleration a. The initial speed $V = 5.22\,\text{m/s}$ and the final speed $= 0$. Plugging these quantities into the one-dimensional kinematic formula

$$v^2 \quad = \quad v_0^2 + 2as,$$

we get,

$$\begin{aligned} 0 &= V^2 - 2as \quad \text{(Note that } a \text{ is negative)} \\ \Rightarrow \quad a &= \frac{V^2}{2s} = \frac{(5.22\,\text{m/s})^2}{2\times 0.05\,\text{m}} = 272.48\,\text{m/s}^2. \end{aligned}$$

Thus $\vec{a} = 272.48\,\text{m/s}^2\,\hat{\boldsymbol{\jmath}}$. Therefore,

$$F_r = (m+M)a = (60\,\text{kg})\cdot(272.48\,\text{m/s}^2) = 1.635\times 10^4\,\text{N}$$

$$\boxed{F_r \approx 16.35\,\text{kN}}$$

SAMPLE 14.3 **Pulley kinematics.** For the masses and ideal-massless pulleys shown in figure 14.12, find the acceleration of mass A in terms of the acceleration of mass B. Pulley C is fixed to the ceiling and pulley D is free to move vertically. All strings are inextensible.

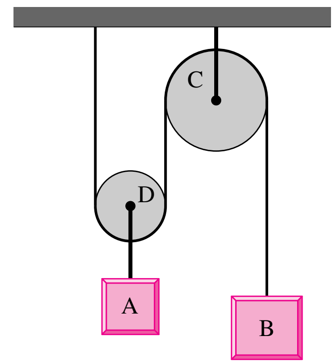


Figure 14.12

Solution Let us measure the position of the two masses from a fixed point, say the center of pulley C. (Since C is fixed, its center is fixed too.) Let y_A and y_B be the vertical distances of masses A and B, respectively, from the chosen reference (C). Then the position vectors of A and B are:

$$\vec{\boldsymbol{r}}_A = y_A\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \vec{\boldsymbol{r}}_B = y_B\hat{\boldsymbol{\jmath}}.$$

Therefore, the velocities and accelerations of the two masses are

$$\begin{aligned} \vec{\boldsymbol{v}}_A &= \dot{y}_A\hat{\boldsymbol{\jmath}}, & \vec{\boldsymbol{v}}_B = \dot{y}_B\hat{\boldsymbol{\jmath}}, \\ \vec{\boldsymbol{a}}_A &= \ddot{y}_A\hat{\boldsymbol{\jmath}}, & \vec{\boldsymbol{a}}_B = \ddot{y}_B\hat{\boldsymbol{\jmath}}. \end{aligned}$$

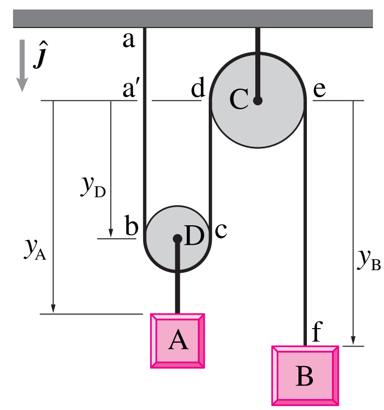


Figure 14.13

Since all quantities are in the same direction ($\hat{\boldsymbol{\jmath}}$), we can drop $\hat{\boldsymbol{\jmath}}$ from our calculations and just do scalar calculations. We are asked to relate $\ddot{y}_A$ to $\ddot{y}_B$.

In all pulley problems, the trick in doing kinematic calculations is to relate the variable positions to the fixed length of the string. Here, the length of the string ℓ_{tot} is: (1)

$$\begin{aligned} \ell_{tot} &= \text{ab} + \text{bc} + \text{cd} + \text{de} + \text{ef} = \text{constant} \\ \text{where} \quad \text{ab} &= \underbrace{\text{aa}'}_{\text{constant}} + \underbrace{\text{a}'\text{b}}_{(=\text{cd}=y_D)} \\ \text{bc} &= \text{string over the pulley D} = \text{constant} \\ \text{de} &= \text{string over the pulley C} = \text{constant} \\ \text{ef} &= y_B \\ \text{thus} \quad \ell_{tot} &= 2y_D + y_B + \overbrace{\text{constant}}^{(\text{aa}'+\text{bc}+\text{de})}. \end{aligned}$$

Taking the time derivative on both sides, we get

$$\overbrace{\frac{d}{dt}(\ell_{tot})}^{\boxed{=\ 0,\ \text{because } \ell_{tot} \text{ does not change with time}}} = 2\dot{y}_D + \dot{y}_B \quad \Rightarrow \quad \dot{y}_D = -\frac{1}{2}\dot{y}_B \tag{14.12}$$

$$\Rightarrow \quad \ddot{y}_D = -\frac{1}{2}\ddot{y}_B. \tag{14.13}$$

$$\begin{aligned} \text{But} \quad & y_D = y_A - \text{AD and AD} = \text{constant} \\ \Rightarrow \quad & \dot{y}_D = \dot{y}_A \quad \text{and} \quad \ddot{y}_D = \ddot{y}_A. \end{aligned}$$

Thus, substituting $\dot{y}_A$ and $\ddot{y}_A$ for $\dot{y}_D$ and $\ddot{y}_D$ in (14.12) and (14.13) we get

$$\dot{y}_A = -\frac{1}{2}\dot{y}_B \quad \text{and} \quad \ddot{y}_A = -\frac{1}{2}\ddot{y}_B$$

$$\ddot{y}_A = -\tfrac{1}{2}\ddot{y}_B$$

(1) We have done an elaborate calculation of ℓ_{tot} here. Usually, the constant lengths over the pulleys and some constant segments such as aa′ are ignored in calculating ℓ_{tot}. These constant length segments can be ignored because they drop out of the equation when we take time derivatives to relate velocities and accelerations of different points, such as B and D here.

SAMPLE 14.4 **A two-mass pulley system.** The two masses shown in Fig. 14.14 have frictionless bases and round frictionless pulleys. The inextensible cord connecting them is always taut. Given that $F = 130\,\text{N}$, $m_A = m_B = m = 40\,\text{kg}$, find the acceleration of the two blocks using:

1. linear momentum balance and
2. energy balance.

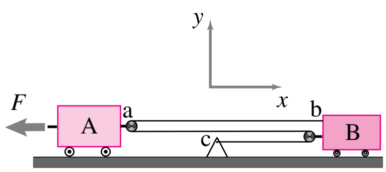


Figure 14.14: A two-mass pulley system.

Solution

1. **Using Linear Momentum Balance:** The free-body diagrams of the two masses A and B are shown in Fig. 14.15.

Linear momentum balance for mass A gives (assuming $\vec{\boldsymbol{a}}_A = a_A\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{a}}_B = a_B\hat{\boldsymbol{\imath}}$):

$$\begin{aligned}
(2T-F)\hat{\boldsymbol{\imath}} + (2N_A - mg)\hat{\boldsymbol{\jmath}} &= m\vec{\boldsymbol{a}}_A = ma_A\hat{\boldsymbol{\imath}}\\
(\text{dotting with } \hat{\boldsymbol{\jmath}}) \quad \Rightarrow \quad 2N_A &= mg\\
(\text{dotting with } \hat{\boldsymbol{\imath}}) \quad \Rightarrow \quad 2T - F &= ma_A \qquad (14.14)
\end{aligned}$$

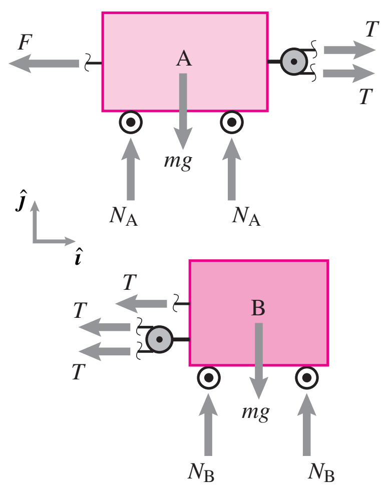


Figure 14.15: Free-body diagrams of the two masses.

Similarly, linear momentum balance for mass B gives:

$$\begin{aligned}
-3T\hat{\boldsymbol{\imath}} + (2N_B - mg)\hat{\boldsymbol{\jmath}} &= m\vec{\boldsymbol{a}}_B = ma_B\hat{\boldsymbol{\imath}}\\
\Rightarrow \quad 2N_B &= mg\\
\text{and} \quad -3T &= ma_B. \qquad (14.15)
\end{aligned}$$

From (14.14) and (14.15) we have three unknowns: T, a_A, a_B, but only 2 equations!. We need an extra equation to solve for the three unknowns. 2

2 You may be tempted to use angular momentum balance (AMB) to get an extra equation. In this case AMB could help determine the vertical reactions, but offers no help in finding the rope tension or the accelerations.

We can get the extra equation from kinematics. Since A and B are connected by a string of fixed length, their accelerations must be related. For simplicity, and since these terms drop out anyway, we neglect the radius of the pulleys and the lengths of the little connecting cords. We use the left wall as the reference position to get

$$\begin{aligned}
\ell_{tot} &\equiv \text{length of the string connecting A and B}\\
&= 3(x_B - X_C) + 2(X_C - x_A)\\
&= 3x_B - X_C - 2x_A
\end{aligned}$$

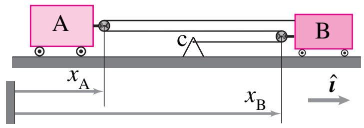


Figure 14.16: Pulley kinematics. The left wall is added as a reference point, off to the side of all pulleys and masses, to help prevent sign errors.

Now, differentiating the expression for ℓ_{tot} with time and noting that the total length of the string remains constant and that X_C is a fixed location in space, we get

$$\begin{aligned}
\overbrace{\frac{d}{dt}(\ell_{tot})}^{0} &= 3\dot{x}_B - \overset{0}{\cancel{\dot{X}_C}} - 2\dot{x}_A\\
\Rightarrow \quad \ddot{x}_B &= \frac{2}{3}\ddot{x}_A \qquad (14.16)\\
& \qquad (14.17)
\end{aligned}$$

Since

$$\begin{aligned}
\vec{\boldsymbol{v}}_A &= v_A\hat{\boldsymbol{\imath}} = \dot{x}_A\hat{\boldsymbol{\imath}},\\
\vec{\boldsymbol{a}}_A &= a_A\hat{\boldsymbol{\imath}} = \ddot{x}_A\hat{\boldsymbol{\imath}},\\
\vec{\boldsymbol{v}}_B &= v_B\hat{\boldsymbol{\imath}} = \dot{x}_B\hat{\boldsymbol{\imath}}, \text{ and}\\
\vec{\boldsymbol{a}}_B &= a_B\hat{\boldsymbol{\imath}} = \ddot{x}_B\hat{\boldsymbol{\imath}},
\end{aligned}$$

we get

$$a_B = \frac{2}{3} a_A. \tag{14.18}$$

Substituting (14.18) into (14.15), we get

$$9T = -2ma_A. \tag{14.19}$$

Now solving (14.14) and (14.19) for T, we get

$$T = \frac{2F}{13} = \frac{2 \cdot 130\,\text{N}}{13} = 20\,\text{N}.$$

Therefore,

$$\begin{aligned} a_A &= -\frac{9T}{2m} = -\frac{9 \cdot 20\,\text{N}}{2 \cdot 40\,\text{kg}} = -2.25\,\text{m/s}^2 \\ a_B &= \frac{2}{3} a_A = -1.5\,\text{m/s}^2 \end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}}_A = -2.25\,\text{m/s}^2\hat{\boldsymbol{\imath}},\ \vec{\boldsymbol{a}}_B = -1.5\,\text{m/s}^2\hat{\boldsymbol{\imath}}.}$$

2. **Using Power Balance (III):** We have,

$$P = \dot{E}_\text{K}.$$

The power balance equation becomes

$$\sum \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} = m\, a_A\, v_A \; + m_B\, a_B\, v_B.$$

Because the force at A is the only force that does work on the system, 3 when we apply power balance to the whole system (see the FBD in fig. 14.17), we get,

$$\begin{aligned} -Fv_A - T\overbrace{v_q}^{=v_c=0} &= m_A v_A a_A + m_B v_B a_B \\ \text{or} \qquad F &= -ma_A - m\frac{v_B}{v_A} a_B \\ &= -a_A\left(m + m\frac{v_B}{v_A}\frac{a_B}{a_A}\right). \end{aligned}$$

Substituting $a_B = 2/3a_A$ and $v_B = 2/3v_A$ from eqn. (14.18),

$$\begin{aligned} a_A &= \frac{-F}{m + \frac{4}{9}m} \\ &= \frac{-130\,\text{N}}{40\,\text{kg}(1 + \frac{4}{9})} \\ &= -2.25\,\text{m/s}^2, \end{aligned}$$

and since $a_B = 2/3a_A$,

$$a_B = -1.5\,\text{m/s}^2,$$

which are the same accelerations as found before.

$$\boxed{a_A = -2.25\,\text{m/s}^2\hat{\boldsymbol{\imath}},\ a_B = -1.5\,\text{m/s}^2\hat{\boldsymbol{\imath}}}$$

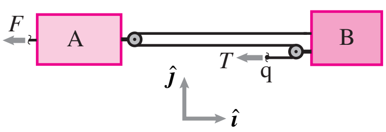


Figure 14.17: 1D free-body diagram of the whole system. Note that except F, no other forces do any work.

3 It may not be obvious why T shown in the FBD in fig. 14.17 does no work. T acts on a material point q' somewhere on the inextensible string from the fixed point 'c'. Even though mass B moves, point 'q' on the string has no displacement. Because the material point q on which it acts does not move, the work done by T is zero.

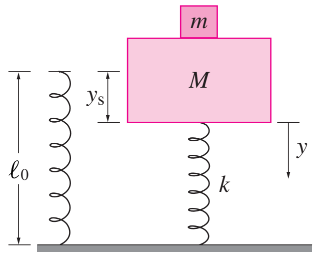


Figure 14.18

SAMPLE 14.5 In static equilibrium the spring in fig. 14.18 is compressed by y_s from its unstretched length ℓ_0. Now, the spring is compressed by an additional amount y_0 and released with no initial velocity.

1. Find the force on the top mass m exerted by the lower mass M.
2. When does this force become minimum? Can this force become zero?
3. Can the force on m due to M ever be negative?

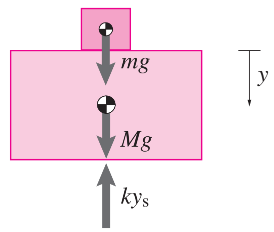


Figure 14.19: Free-body diagram of the two masses as one system in static equilibrium (this special case could be skipped as it follows from the free-body diagram below).

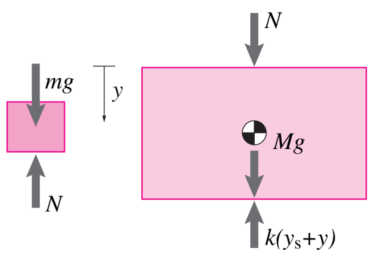


Figure 14.20: Free-body diagrams of the individual masses.

Solution

1. The free-body diagram of the two masses is shown in Figure 14.19 when the system is in static equilibrium. From linear momentum balance we have

$$\sum \vec{F} = \vec{\mathbf{0}} \qquad \Rightarrow \qquad ky_s = (m+M)g. \tag{14.20}$$

The free-body diagrams of the two masses at an arbitrary position y during motion are shown in Figure 14.20. Since the two masses oscillate together, they have the same acceleration. From linear momentum balance for mass m we get (note that we have chosen y to be positive downwards),

$$mg - N = m\ddot{y}. \tag{14.21}$$

We are interested in finding the normal force N. Clearly, we need to find $\ddot{y}$ to calculate N. Now, from linear momentum balance for mass M we get

$$Mg + N - k(y + y_s) = M\ddot{y}. \tag{14.22}$$

Adding eqn. (14.21) with eqn. (14.22) we get

$$(m+M)g - ky - ky_s = (m+M)\ddot{y}.$$

But $ky_s = (m+M)g$ from eqn. (14.20). Therefore, the equation of motion of the system is

$$\begin{aligned} -ky &= (m+M)\ddot{y} \\ \text{or} \quad \ddot{y} + \frac{k}{(m+M)}y &= 0. \end{aligned} \tag{14.23}$$

As you recall from your study of the harmonic oscillator, the general solution of this differential equation is

$$y(t) \quad = \quad A\sin\lambda t + B\cos\lambda t \tag{14.24}$$

$$\text{where} \quad \lambda \quad = \quad \sqrt{\frac{k}{m+M}}. \tag{14.25}$$

The constants A and B are to be determined from the initial conditions. From eqn. (14.24) we obtain

$$\dot{y}(t) = A\lambda\cos\lambda t - B\lambda\sin\lambda t. \tag{14.26}$$

Substituting the given initial conditions $y(0) = y_0$ and $\dot{y}(0) = 0$ in eqns. (14.24) and (14.26), respectively, we get

$$\begin{aligned} \overbrace{y(0)}^{y_0} &= A\sin(\lambda\cdot 0) + B\cos(\lambda\cdot 0) &\Rightarrow&\quad B = y_0 \\ \underbrace{\dot{y}(0)}_{0} &= A\lambda\cos(\lambda\cdot 0) - B\lambda\sin(\lambda\cdot 0) &\Rightarrow&\quad A = 0. \end{aligned}$$

Thus,

$$y(t) = y_0 \cos \lambda t. \tag{14.27}$$

Now we can find the acceleration by differentiating eqn. (14.27) twice :

$$\ddot{y} = -y_0 \lambda^2 \cos \lambda t.$$

Substituting this expression in eqn. (14.21) we get the force applied by mass M on the smaller mass m:

$$\begin{aligned} mg - N &= m \overbrace{(-y_0 \lambda^2 \cos \lambda t)}^{\ddot{y}} \\ \Rightarrow \quad N &= mg + m y_0 \lambda^2 \cos \lambda t \\ &= mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t \right) \qquad (14.28) \end{aligned}$$

$$\boxed{N = mg \left(1 + \frac{y_0 \lambda^2}{g} \cos \lambda t \right)}$$

2. Since $\cos \lambda t$ varies between ± 1, the value of the force N varies between $mg \pm y_0 \lambda^2$. Clearly, N attains its minimum value when $\cos \lambda t = -1$, *i.e.,* when $\lambda t = \pi$. This condition is met when the spring is fully stretched and the mass is at its highest vertical position. At this point,

$$N \equiv N_{\text{min}} = mg \left(1 - \frac{y_0 \lambda^2}{g} \right).$$

If y_0, the initial displacement from the static equilibrium position, is chosen such that $y_0 \lambda^2 = g$ (that is, the amplitude of the harmonically varying acceleration equals g), then $N = 0$ when $\cos \lambda t = -1$, *i.e.,* at the topmost point in the vertical motion. This condition, $N = 0$, means that the two masses momentarily lose contact with each other; and it happens precisely when they are about to begin their downward motion.

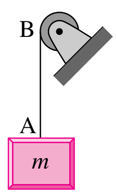
3. From eqn. (14.28) we can get a negative value of N when $\cos \lambda t = -1$ and $y_0 \lambda^2 > g$. However, a negative value for N is nonsense unless the blocks are glued. Without glue the bigger mass M cannot apply a negative force (or a compression) on m, *i.e.,* it cannot "suck" m. When $y_0 \lambda^2 > g$ then N becomes zero before $\cos \lambda t$ decreases to -1. That is, assuming no bonding, the two masses lose contact on their way to the highest vertical position but before reaching the highest point. Beyond that point, the equations of motion derived above are no longer valid for unglued blocks because the equations assume contact between m and M. Equation (14.28) is inapplicable when $N \leq 0$.

14.1 1D constrained motion and pulleys

For all problems, unless stated otherwise, treat all strings as inextensible, flexible and massless. Treat all pulleys and wheels as round, frictionless and massless. Assume all massive objects are prevented from rotating (e.g., wheels stay on the ground, *etc.*). When numbers are called for use $g = 10\,\mathrm{m/s^2}$ or $g = 32\,\mathrm{ft/s^2}$.

Preparatory Problems

14.1.1 A motor at B allows the block of mass $m = 3\,\mathrm{kg}$ shown in the figure to accelerate downwards at $2\,\mathrm{m/s^2}$. There is gravity. What is the tension in the string AB?

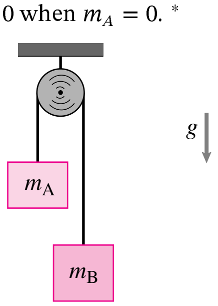


Problem 14.1.1

14.1.2 Two masses connected by an inextensible string hang from an ideal pulley.

a) Find the downward acceleration of mass B. Answer in terms of any or all of m_A, m_B, g, and the present velocities of the blocks. As a check, your answer should give $a_B = g$ when $m_A = 0$ and $a_B = 0$ when $m_A = m_B$. *.

b) Find the tension in the string. As a check, your answer should give $T = m_B g = m_A g$ when $m_A = m_B$ and $T = 0$ when $m_A = 0$. *

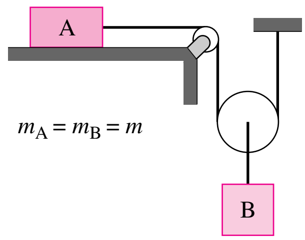


Problem 14.1.2

14.1.3 The blocks shown are released from rest.

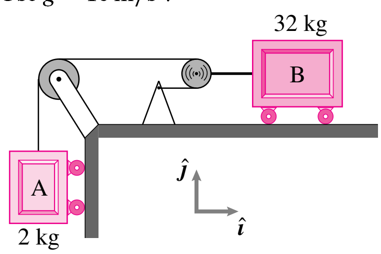
a) What is the acceleration of block A at $t = 0^+$ (just after release)?

b) What is the speed of block B after it has fallen 2 meters?



Problem 14.1.3

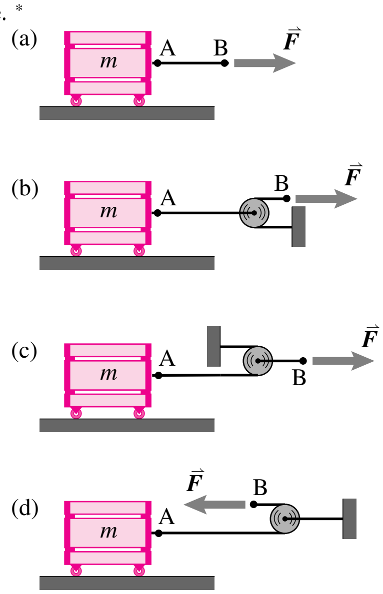
14.1.4 What is the acceleration of block A? Use $g = 10\,\mathrm{m/s^2}$. *



Problem 14.1.4

14.1.5 For the system shown in problem 14.1.2, find the acceleration of mass B using energy balance ($P = \dot{E}_K$).

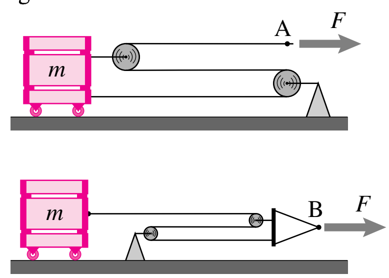
14.1.6 For the various situations pictured, find the acceleration of mass A and point B. Clearly define any variables, coordinates or sign conventions that you use. *



Problem 14.1.6: Four different ways to pull a mass.

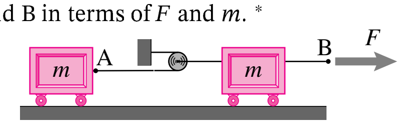
14.1.7 For each of the situations in problem 14.1.6 find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

14.1.8 What is the ratio of the acceleration of point A to that of point B in each configuration? $m = m$ and $F = F$. *



Problem 14.1.8

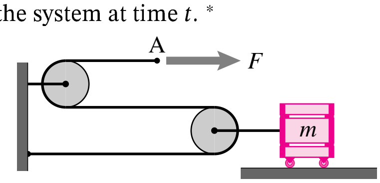
14.1.9 Find the acceleration of points A and B in terms of F and m. *



Problem 14.1.9

14.1.10 For the situation pictured in problem 14.1.9 find the accelerations of the two masses using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

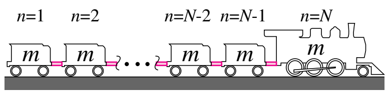
14.1.11 The point of application A of the force moves twice as fast as the mass. At some instant in time t, the speed of the mass is $\dot{x}$ to the left. Find the input power to the system at time t. *



Problem 14.1.11

More-Involved Problems

14.1.12 A train engine of mass m pulls and accelerates N cars, each of mass m. The power of the engine is P_t and its speed is v_t. Find the tension T_n between car n and car $n+1$. Assume there is no resistance and the ground is level. Assume the cars are connected with rigid links. *



Problem 14.1.12

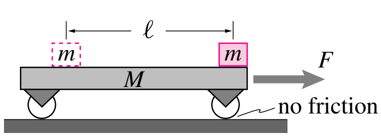
14.1.13 A cart of mass M, initially at rest, can move horizontally along a frictionless track. When $t = 0$, a force F is applied as shown to the cart. During the acceleration of M by the force F, a small box of mass m slides along the cart from the front to the rear. The coefficient of friction between the cart and the box is μ, and it is assumed that the acceleration of the cart is sufficient to cause sliding.

a) Draw free-body diagrams of the cart, the box, and the cart and box together.

b) Write the equation of linear momentum balance for the cart, the box, and the system of cart and box.

c) Show that the equations of motion for the cart and box can be combined to give the equation of motion of the mass center of the system of two bodies.

d) Find the displacement of the cart at the time when the box has moved a distance ℓ along the cart. *



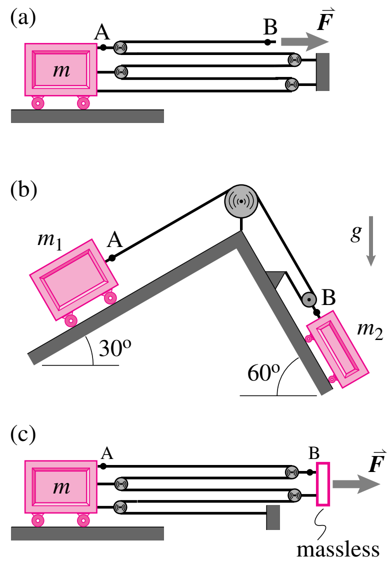
Problem 14.1.13

14.1.14 For the situations pictured, find the accelerations of mass A and of point B. Clearly define any variables, coordinates or sign conventions that you use.

a) A single mass and four pulleys. *

b) Two masses and two pulleys. *

c) A single mass and four pulleys. *



Problem 14.1.14: Various pulley arrangements.

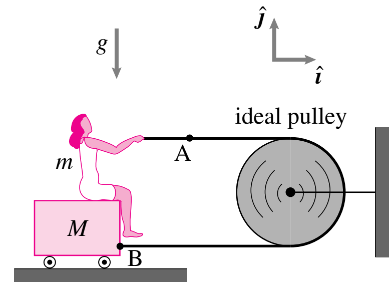
14.1.15 For the situations pictured in problem 14.1.14, find the acceleration of the mass using energy balance ($P = \dot{E}_K$).

14.1.16 A person of mass m, modeled as a rigid object, is sitting on a cart of mass $M > m$ and pulling the string towards herself. The coefficient of friction between her seat and the cart is μ. Point B is attached to the cart and point A is attached to the rope.

a) If you are given that she is pulling rope in with acceleration a_0 relative to herself (that is, $\vec{\boldsymbol{a}}_{A/B} \equiv \vec{\boldsymbol{a}}_A - \vec{\boldsymbol{a}}_B = -a_0\hat{\boldsymbol{\imath}}$) and that she is not slipping relative to the cart, find $\vec{\boldsymbol{a}}_A$. (Answer in terms of some or all of $m, M, g, \mu, \hat{\boldsymbol{\imath}}$ and a_0.) *

b) Find the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

c) If instead, $M < m$, what is the largest possible value of a_0 without the person slipping off the cart? (Answer in terms of some or all of m, M, g and μ. You may assume her legs get out of the way if she slips backwards.)

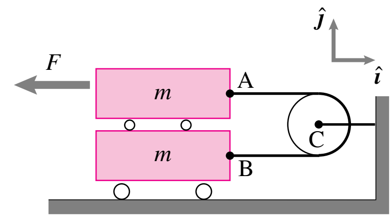


Problem 14.1.16: **Pulley.**

14.1.17 Two blocks and a pulley. Two identical blocks are stacked and tied together by the pulley as shown. Find

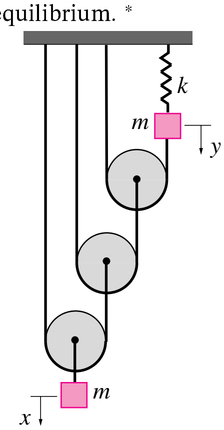
a) the acceleration of point A, and

b) the tension in the line.



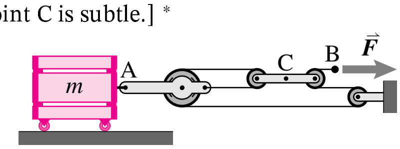
Problem 14.1.17

14.1.18 What is the natural frequency of vibration of this system? Include gravity. x measures the vertical position of the lower mass from equilibrium. y measures the vertical position of the upper mass from equilibrium. *



Problem 14.1.18

14.1.19 For the situation pictured, find the acceleration of mass A and points B and C shown. [Hint: the situation with point C is subtle.] *



Problem 14.1.19

14.1.20 For the situation pictured in problem 14.1.19, find the acceleration of point A using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.

14.1.21 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). You must design the system so that mass B accelerates to the left with $\frac{F}{2m_B}$ (i.e., $\vec{a}_B = -\frac{F}{2m_B}\hat{\imath}$).

a) Draw the system clearly. Justify your answer with enough words or equations so that a reasonable person, say a grader, can tell that you understand your solution.

b) Find the acceleration of point A.

14.1.22 Design a pulley system. You are to design a pulley system to move a mass. There is no gravity. Point A has a force $\vec{F} = F\hat{\imath}$ pulling it to the right. Mass B has mass m_B. You can connect the point A to the mass with any number of ideal strings and ideal pulleys. You can make use of rigid walls or supports anywhere you like (say, to the right or left of the mass). Draw the system clearly. Justify your answer with enough words or equations to convince a skeptical person that your solution is correct. You must design the system so that the mass B accelerates .

a) to the left with $\frac{F}{m_B}$ (i.e., $\vec{a}_B = -\frac{F}{m_B}\hat{\imath}$)

b) to the left with $\frac{2F}{m_B}$

c) to the left with $\frac{F}{2m_B}$

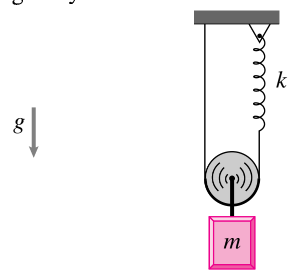
d) to the right with $\frac{2F}{m_B}$

e) to the right with $\frac{F}{2m_B}$

f) to the left with $\frac{8F}{m_B}$

g) to the right with $\frac{F}{5m_B}$

14.1.23 Pulley and spring. For the hanging mass find the period of oscillation. Only vertical motion is of interest. There is gravity.

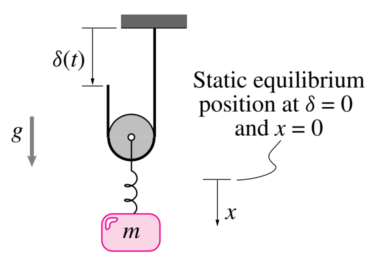


Problem 14.1.23

14.1.24 The spring-mass system shown ($m = 10$ slugs ($\equiv$ lb $\cdot$ sec^2/ft), $k = 10\,\mathrm{lb}/ft$) is excited by moving the free end of the cable vertically according to $\delta(t) = 4\sin(\omega t) \in$, as shown in the figure.

a) Derive the equation of motion for the block in terms of the displacement x from the static equilibrium position, as shown in the figure.

b) If $\omega = 0.9\,\mathrm{rad/s}$, check to see if the pulley is always in contact with the cable (ignore the transient solution).

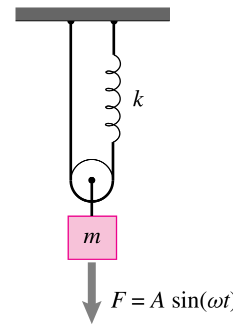


Problem 14.1.24

14.1.25 The block of mass m hanging on the spring with constant k and a string shown in the figure is forced by $F = A\sin(\omega t)$. Do not neglect gravity. The pulley has negligible mass.

a) What is the differential equation governing the motion of the block? You may assume that the only motion is vertical motion. *

b) Given A, m and k, for what values of ω would the string go slack at some point in the cyclical motion? (The common assumption in such problems, which you can use, is to neglect the homogeneous solution to the differential equation. It is assumed that the damping, small enough to be neglected in the governing equations is large enough so that the particular solution will have damped out at the time of observation.) *

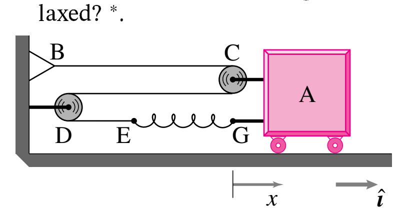


Problem 14.1.25

14.1.26 Block A, with mass m_A, is pulled to the right a distance d from the position it would have if the spring were relaxed. It is then released from rest. Assume ideal string, pulleys and wheels. The spring has constant k.

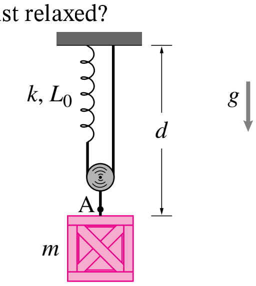
a) What is the acceleration of block A just after it is released (in terms of k, m_A, and d)? *

b) What is the speed of the mass when the mass passes through the position where the spring is relaxed? *.



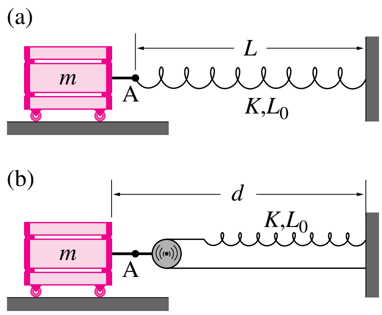
Problem 14.1.26

14.1.27 What is the static displacement of the mass from the position where the spring is just relaxed?



Problem 14.1.27

14.1.28 For the two situations pictured, find the acceleration of point A shown using balance of linear momentum ($\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$). Assuming both masses are deflected an equal distance from the position where the spring is just relaxed, how much smaller or bigger is the acceleration of block (b) than that of block (a). Define any variables, coordinate system origins, coordinates or sign conventions that you need to do your calculations and to define your solution.



Problem 14.1.28

14.1.29 For each of the situations pictured in problem 14.1.28, find the acceleration of the mass using energy balance ($P = \dot{E}_K$). Define any variables, coordinates, or sign conventions that you need to do your calculations and to define your solution.