13.2 Collisions and explosions of particles in 2D and 3D

When two things bump into each other there is often a big interaction force. Think about a ball bouncing off the ground, two pool balls colliding, a baseball hitting a bat, two cars crashing, or the big forces when a satellite gravitationally slingshots around a planet it passes close by. Similarly there are big short-lived forces when things explode into two or more pieces. A big and short-lived force is often described by

$$\text{its net impulse } \vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}} dt$$

rather than its detailed time-history $\vec{\boldsymbol{F}}(t)$. The collision modeling assumption is that these interaction forces are so big that all other forces on the particles can be ignored. For a two-particle collision the impulses are $\vec{\boldsymbol{P}}_1 = \vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{P}}_2 = -\vec{\boldsymbol{P}}$ acting on m_1 and m_2. Rather than looking at the acceleration of mass during the collision one just calculates

$$\text{the net change in velocity} = \Delta\vec{\boldsymbol{v}}.$$

Before the collision two particles m_1 and m_2 have velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ (see fig. 13.9). The superscript "$^-$" (minus) means *just* before the collision. Then the particles collide. Even though we ignore the spatial extent of the particles for most of the mechanics analysis, we note that the two particles have a common tangent plane. The normal of that plane, pointing out of particle 1, say, is $\hat{\boldsymbol{n}}$. Just after the collision the particles have velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ with the superscript"$^+$" indicating *just* after the collision.

The general collision problem is

> Given some information about the motion before the collision, the motion after the collision, and the collisional impulse, find other information about these same quantities.

We find the unknowns using

- Momentum balance for each particle: $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}(t)\, dt = m\Delta\vec{\boldsymbol{v}}$; and
- Some information about the collisional impulse, usually a constitutive law for the collision.

For collisional modeling the constitutive law for interaction involves impulse and change of velocity. We only consider two such constitutive models:

- *plastic sticking* collisions where $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$.

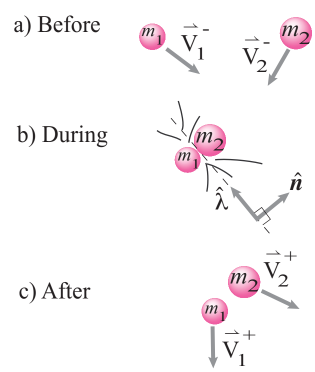

Figure 13.9: Two particles collide. Their velocities before the collision are $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$. When they collide their common tangent plane has normal $\hat{\boldsymbol{n}}$. After the collision their velocities are $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$.

- frictionless restitution with $(\vec{v}_1^+ - \vec{v}_2^+) \cdot \hat{n} = -e(\vec{v}_1^- - \vec{v}_2^-) \cdot \hat{n}$ and $\vec{P} \cdot \hat{\lambda} = 0$.

The constitutive models are discussed further below in the context of the three idealized collisions we treat here:

- sticking collisions
- frictionless collisions with restitution
- explosions.

The only expansion in this section over the 1D collisions in section 10.5 is the need for 2D and 3D geometry.

Sticking collisions

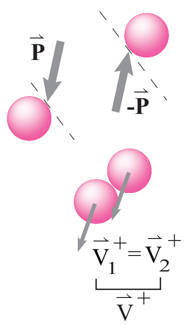

Figure 13.10: Two particles collide and stick together so in their subsequent motion $\vec{v}_1^+ = \vec{v}_2^+ = \vec{v}^+$. The action-reaction impulse pair is in whatever direction it needs to be to get this sticking; the impulses need not be in just the $\hat{n}$ direction.

The conceptually simplest collision is a *sticking* collision also called a *perfectly plastic no-slip* collision (see fig. 13.10). Here the word 'plastic' is used in its old latin meaning malleable or 'clay like'. Imagine two lumps of wet clay colliding in space and just sticking together. The plastic collision model applies, for example,

- when a projectile gets embedded in its target,
- when two cars crash and get entangled so they move together after the collision, or
- when two machine parts engage at contact because of a mechanism like a door catch.

In short, the constitutive law for plastic collisions is

$$\text{Plastic (sticking collision)} \quad \Rightarrow \quad \vec{v}_1^+ = \vec{v}_2^+$$

And the impulse is what it is, as determined by momentum balance for the two particles. Here's the simplest collision problem.

Example: A particle collides with an immovable object.
The impulse on the particle is

$$\vec{P} = m\Delta\vec{v} = m(\vec{0} - \vec{v}^-) = -m\vec{v}^-.$$

And here is the general two-particle sticking collision problem.

Example: Two particles collide and stick.
There are three velocities to consider, the before-collision velocities $\vec{v}_1^-$ and $\vec{v}_2^-$ and the common after-collision velocity $\vec{v}^+$. Also relevant is the interaction impulse $\vec{P}$. That's 4 vector quantities (8 scalars in 2D, 12 in 3D). The governing equations are momentum balance for the two particles

$$\vec{P} = m_1(\vec{v}^+ - \vec{v}_1^-) \qquad \text{and} \qquad \vec{P} = m_2(\vec{v}^+ - \vec{v}_2^-)$$

making up 2 vector equations (4 scalar equations in 2D, 6 in 3D). Thus to solve a problem in 2D, 4 scalar quantities need to be given so that the other quantities can be found from the momentum balance equations. In 3D, 6 scalar quantities have to be given.

There are all different ways to involute such problems, say by taking one of the masses as unknown. Here is the most straightforward example.

> **Example: Find the post-collision velocities for a sticking collision.**
> Given $m_1, m_2, \vec{v}_1^-$ and $\vec{v}_2^-$ we find by solving the momentum balance equations that
>
> $$\vec{v}^+ = \frac{m_1\vec{v}_1^- + m_2\vec{v}_2^-}{m_1+m_2} \qquad \text{and} \qquad \vec{P}_1 = -\vec{P}_2 = \frac{m_1 m_2}{m_1+m_2}\left(\vec{v}_2^- - \vec{v}_1^-\right).$$
>
> The answer can be interpreted like this. The final velocity is the same as the pre-collision average velocity. This is also the system's initial (and final) center of mass velocity. The impulsive interaction is associated with the change of velocity of $\vec{v}_1^- - \vec{v}_2^-$ of an effective 'reduced mass' with a value of $m_{red} = m_1 m_2/(m_1+m_2)$ (see box 13.1 on page 3).

Frictionless collision with restitution

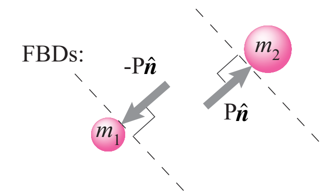

Figure 13.11: Two particles collide and bounce off each other frictionlessly. The assumption is that their relative separation speed is the coefficient of restitution e times their approach speed. Even though for momentum balance we treat the masses as particles, for considering the collision we look at the normal $\hat{n}$ of the common contacting tangent plane. The separation speed is $(\vec{v}_1^+ - \vec{v}_2^+)\cdot\hat{n}$ and the approach speed is $-(\vec{v}_1^- - \vec{v}_2^-)\cdot\hat{n}$.

Frictionless collisions with restitution

This is the most common model used in elementary mechanics courses. It is originally due to Newton, at least in the 1D case we discussed in section 10.5. Two particles collide and then separate. There is no interaction force in their common contact tangent plane (hence 'frictionless'). See fig. 13.11. The impulse is such that the particles separate at a speed that is a fixed ratio e of the speed at which they approached. The speed of approach and separation are measured in the $\hat{n}$ direction.

The speed of approach is the rate at which the distance between the particles decreases just before the collision. Really, this only makes precise sense if

- the masses are round, or
- the masses are not rotating.

13.1 Effective mass

In two-particle collisions the forces of interest are in the action-reaction pair between the particles: $\vec{F}_1$ acts on m_1 and $\vec{F}_2 = -\vec{F}_1$ acts on particle 2. If we know one we know the other, so let's call $\vec{F}$ the force $\vec{F}_1$ on m_1. The two-particle system has no net acceleration, meaning the center of mass does not accelerate. All that the interaction force does is affect the relative motion of the particles.

So consider the relative acceleration of particle 1, say, relative to particle 2:

$$\vec{a}_{\text{rel}} = \vec{a}_1 - \vec{a}_2 = \frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2} = \frac{\vec{F}_1}{m_1} - \frac{-\vec{F}_1}{m_2} = \vec{F}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

Thus we can write

$$\vec{F} = m_{\text{eff}}\vec{a}_{\text{rel}}$$

$$\text{with} \quad m_{\text{eff}} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1+m_2}$$

The 'reduced mass' or 'effective mass' m_{eff} is that which connects the interaction force with the relative acceleration $\vec{a}_{\text{rel}} = \vec{a}_1 - \vec{a}_2$ of the masses. To personalize it, imagine you have negligible mass and are floating in space between two big masses holding a handle on each. Then the relation between the tension in your arms and the relative acceleration of the masses is determined by the reduced effective mass m_{eff}. The effective mass is less than either of the masses separately (because the relative acceleration comes from the addition of the two accelerations). For two equal masses $m = m_1 = m_2$ the effective mass is $m_{\text{eff}} = m/2$.

Integrating in time the effective mass also relates the interaction impulse with the change in relative velocity.

$$\vec{P} = m_{\text{eff}}\Delta\vec{v}_{\text{rel}},$$

where $\vec{P} = \int \vec{F}\,dt$ acts on m_1 and $\vec{v}_{\text{rel}} = \vec{v}_1 - \vec{v}_2$.

The approach speed is the relative velocity dotted with the $\hat{\boldsymbol{n}}$ direction

$$v_{approach} = \left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}.$$

The separation speed, measured just after the collision, has the same definition but with a sign change

$$v_{\text{sep}} = -\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}}.$$

Newton's law of collisional restitution① is

$$v_{\text{sep}} = e v_{approach} \quad \text{or} \quad \left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} = -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}. \quad (13.4)$$

①Why the word *restitution*? The particles approach each other with some momentum relative to their common center of mass. At some point during the collision they have none. Then some of the momentum is restituted, paid back, and they bounce. No restitution, $e = 0$, and there is no bounce. Full restitution, $e = 1$, and they separate with the as much relative momentum as they had when they approached.

We use the coefficient of restitution for approximate collisional modeling but,

the 'coefficient of restitution' equation is not an accurate law of nature.

Example: Two-particle elastic collision.
Two particles m_1 and m_2 have pre-collision velocities of $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ and collide frictionlessly with coefficient of restitution e on the tangent plane with normal $\hat{\boldsymbol{n}}$. The post collision velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$, as well as the impulse $\vec{\boldsymbol{P}}$ are found by simultaneously solving these equations.

$$\begin{aligned}
\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_1^-\right) \\
-\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}_2^+ - \vec{\boldsymbol{v}}_2^-\right) \\
\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} &= -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}
\end{aligned}$$

In 2D this makes up 5 scalar equations for 5 scalar unknowns. In 3D it's 7 equations for 7 unknowns. The most direct solution is to set up and solve these equations using a computer.

Rather it is an approximate empirical observation. Or, to put it another way, the value of the coefficient of restitution e depends on the material, the shape, the orientation and the speed of the colliding particles. It is not a true constant. Nonetheless, eqn. (13.4) is a reasonable approximation for some engineering purposes. Just don't assume that predictions it makes will generally be highly accurate.

The 'frictionless' part of this collision law is expressed by the assumption that the net impulse of interaction is in the $\hat{\boldsymbol{n}}$ direction. So $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ with no component in the $\hat{\boldsymbol{\lambda}}$ direction.

Generally one assumes that the coefficient of restitution is between zero and one:

$$0 \leq e \leq 1.$$

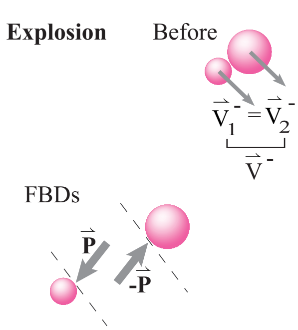

Figure 13.12: An explosion is like a sticking collision run backwards in time. The particles initially have the same velocity $\vec{\boldsymbol{v}}_1^- = \vec{\boldsymbol{v}}_2^- = \vec{\boldsymbol{v}}^-$ and then separate due to an action-reaction impulse pair in any direction.

For $e < 0$ the masses have to pass through each other. For $e > 1$ the collision would involve a gain in energy. This might happen if there was explosive gunpowder in the contact region of the collision.

In the case $e = 0$ the collision is perfectly plastic but still frictionless. This is generally not a sticking collision because the masses can enter and hence leave the collision with some relative velocity in the common contact plane (the $\hat{\boldsymbol{\lambda}}$ direction).

Explosions

If one particle explodes into pieces it's as if the pieces had a collision. It's just that the initial velocities of the pieces were all the same and the total kinetic energy of the system increases during the 'collision'. See fig. 13.12. The overall treatment is extremely similar to that for sticking collisions, but in some sense backwards. Instead of the particles entering the collision with different velocities and leaving with the same velocity, they enter with the same velocity and leave with different velocities. But the same momentum principles apply. There is no collision law or coefficient of restitution to apply, all of the post-collision relative velocity is restituted from nothing. Rather one just has to know (or find) the action-reaction impulse between the masses.

Example: An explosion.
Two particles m_1 and m_2 are stuck together and moving at $\vec{\boldsymbol{v}}^-$ when they explode

13.2 Energetics of collisions

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. We'd like to treat these cases in a unified way. First let's calculate the total kinetic energy.

$$\begin{aligned} 2E_\text{K} &= m_1 v_1^2 + m_2 v_2^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_1 |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_\text{cm}|^2 + m_2 |\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_\text{cm}|^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_\text{eff} |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|^2 \\ &= m_\text{tot} v_\text{cm}^2 + m_\text{eff} v_\text{rel}^2 \end{aligned}$$

where $m_\text{tot} = m_1 + m_2$, $m_\text{eff} = m_1 m_2/(m_1 + m_2)$ and $v_\text{rel} = |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|$. There are a few algebra steps needed to go from line to line above (see section B for related calculations). The concept of effective mass m_eff is introduced in box 13.1. The key result is that the kinetic energy of a two-particle system can be written as the sum of two terms, one involving center of mass velocity and one involving the relative velocity of the two masses.

This is a special result for two-particle systems. For any system the kinetic energy is a center of mass term $(mv_\text{cm}^2/2)$ plus a term for motion relative to the center of mass. But generally the relative motion term is written as a sum of terms, one for each particle, and the motion of each particle is measured relative to the center of mass $(mv_{i/\text{cm}})$. What is special for two-particle systems is that the relative motion part can be written in terms of the motion of the two particles relative to each other. Because that is not the velocity of any real thing, it only gives the right kinetic energy when used with the corrected effective mass (m_eff).

What about energy and collisions? The center of mass velocity and energy do not change in the collision. So the only change in kinetic energy is that associated with changes in $m_\text{eff} v_\text{rel}^2$.

$$\begin{aligned} 2\Delta E_\text{K} &= m_\text{eff}\left((v_\text{rel}^+)^2 - (v_\text{rel}^-)^2\right) \\ &= 2\vec{\boldsymbol{v}}_\text{rel}^- \cdot \vec{\boldsymbol{P}} + |\vec{\boldsymbol{P}}|^2/m_\text{eff} \end{aligned}$$

where we used that $\vec{\boldsymbol{P}} = m_\text{eff}(\vec{\boldsymbol{v}}_\text{rel}^+ - \vec{\boldsymbol{v}}_\text{rel}^-)$. This formula applies for both sticking collisions, in which case $\vec{\boldsymbol{P}} = -m_\text{eff}\vec{\boldsymbol{v}}_\text{rel}^-$ and $2\Delta E_\text{K} = -m_\text{eff}(v_\text{rel}^-)^2$, and to explosions where $\vec{\boldsymbol{P}} = m_{eff}\vec{\boldsymbol{v}}_\text{rel}^+$ and $2\Delta E_\text{K} = m_\text{eff}(v_\text{rel}^+)^2$. It also applies to interactions in-between.

All that enters the change-of-energy equations above is the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction. Thus the issue of energy loss or gain is determined by whether the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction decreases or increases in magnitude. Thus a collision with $-1 < e < 1$ loses energy and a collision with $|e| > 1$ increases energy. We included $e < 0$ for completeness even though it is sometimes considered 'non-physical' in that it involves the particles passing by or passing through each other.

and an impulse $\vec{\boldsymbol{P}}$ separates them. After the collision

$$\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}^- + \vec{\boldsymbol{P}}/m_1 \qquad \text{and} \qquad \vec{\boldsymbol{v}}_2^+ = \vec{\boldsymbol{v}}^- - \vec{\boldsymbol{P}}/m_2.$$

The full range of behavior for sticking collisions to explosions can be captured with a single restitution coefficient e_g (see box 13.3).

Frictional collisions

Our avoiding of frictional collisions is not because there generally is no friction during collisions. Friction is a fact of the mechanical world. We avoid friction here because a host of special assumptions are needed to make frictional problems deterministic. And no given set of assumptions is known to yield accurate predictions. Frictional collision models have too dis-satisfyingly low a ratio of accuracy to complexity for inclusion in a book at this level.

Simultaneous collisions

If one particle is involved in two collisions at one time then we have not explained how to calculate the resulting motion. In an attempt to make the situation clear one is tempted to say "Let's make it ideal and assume the collisions are exactly instantaneous and at *exactly* the same time." Then, unfortunately, one is making the situation *exactly* ambiguous.

Unfortunately for our hope of making reliable predictions, simultaneous collisions are *not* rare events. Why? Imagine B is touching C and both are stationary. Then A comes and bangs into B. Because B and C are already touching one must assume that there are impulsive forces not just between A and B, but also between B and C. And we have no reliable rules for sorting out the result. Nor will we find such rules if we make it a life's work.

Example: A triangular array of identical spheres.
Imagine 15 accurately-machined nominally-identical spheres laid out in a tight triangle (5 in one row, 4 in the next, then 3,2, and 1) on a very flat smooth surface. Then imagine a 16th ball rolls in and hits the apex of the triangle. How do the 15 balls move?

This experiment is performed in smoky rooms full of intoxicated people night after night. It's the 'break' in a pool game. And the game depends on the result being unpredictable. Each time, due to tiny differences, the results are different.

And, according to theory, the more rigid and perfect the balls are, the more sensitive are the results to the smallest of differences in the initial conditions.

What is the source of the problem?

Example: Three balls in a line.
Consider the one-dimensional collision of three identical particles. B and C are in line, stationary and touching and then A comes along with $v_{A0} = v^-$. Let's assume that the collision(s) whatever they are, are completely elastic and conserve energy ($e = 1, e_g = 0$). Here are two ways to predict the outcome:

- A hits B and C, being all the way at the other side of B, is oblivious to the interaction between A and B until it is complete. Thus A comes to rest and B is moving to the right with v^-. Then B collides elastically with C and B comes to rest and C shoots off with the v^-.
- B and C are touching and act as a single rigid object throughout the collision with A. Thus the result is like that between a particle with mass m and another with mass $2m$. Such an elastic collision would leave B and C going forwards at $2v^-/3$ and A going the other way with $v_A^+ = -v^-/3$. A different result.
- actually there is a one-parameter family of results that are consistent with energy conservation and momentum balance. We have three outcomes (the velocities of the three particles) and only 2 scalar equations restricting them (momentum and energy balance).

But what would *really* happen? That would depend on details that are not stated. Of course if the *exact* shape and configuration of the balls was known, and the exact rules for elastic and inelastic deformation, then one could calculate the resulting motion solving partial differential equations or with atomic simulations. In principle. But we generally do not know such details nor have such calculation abilities. And crowding that which we don't know into concepts like 'rigid-object' and 'exactly simultaneous' crowds the prediction of the outcome to dependence on infinitesimal things.

So, as an engineer, what are you supposed to do when calculating in situations involving simultaneous collisions?

- first relax and remember that no collision calculation is likely to be very accurate (unless the result only depends on balance of momentum). So simultaneous collisions, while philosophically worse in that even the equations are indeterminate, are not that much worse than the usual deterministic, but not accurate, collisional relations.
- do experiments, and
- take account of the range of outcomes depending on assumptions about the collision details.

Samples 13.4 and 13.5 starting on page 675 illustrate the ambiguity of simultaneous collisions in 2D.

Final comments

This is the second of three sections about collisions. Section 10.5 was about collisions in 1D, then this section about particles in 2D and 3D, and finally the ideas in this section will be extended from particles to rigid objects in section 17.5.

13.3 Coefficient of generation

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. For passive frictionless collisions one can characterize the collision by the coefficient of restitution

$$\begin{aligned} e &\equiv \frac{\text{(separation speed)}}{\text{(approach speed)}} \\ &= \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}}}{(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \end{aligned} \tag{13.5}$$

However, for explosions the coefficient of restitution is $e = \infty$. If one is equally interested in energy absorbing or energy creating collisions one can use a more democratic coefficient of generation

$$\begin{aligned} e_g &\equiv \frac{\text{(separation speed)-(approach speed)}}{\text{(separation speed)+(approach speed)}} \\ &= \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} - (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}}{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} + (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \end{aligned} \tag{13.6}$$

We can write the collisional coefficient of generation in terms of the restitution coefficient, and *vice versa*, as

$$e_g = \frac{e-1}{e+1} \qquad \text{and} \qquad e = \frac{1+e_g}{1-e_g}.$$

The generation coefficient is -1 for sticking collisions and 1 for explosions. This coefficient is zero for energetically neutral collisions (no gain, no loss, $e = 1$). And the coefficient of generation does not allow for passing-through or passing-by collisions ($e < 0$).

As a replacement for the conventional coefficient of restitution the coefficient of generation e_g is more complex to use in simple calculations in that eqn. (13.6) is more complex than eqn. (13.5). On the other hand the coefficient of generation is convenient for describing situations which are a mix of passive ($e < 1$ and $e_g < 0$) and active ($e > 1$ and $e_g > 0$). Such is the case, for example, in simple models of legged locomotion (see box 13.4 on page 9).

Note that in all the collisional restitution formulas we could replace $\hat{\boldsymbol{n}}$ with $-\hat{\boldsymbol{n}}$ without affecting the validity of the equations. Similarly all the subscript 2's could be replaced with 1's and *vice versa* without affecting the validity of the equations. Knowing this relieves anxiety about the choice of normal $\hat{\boldsymbol{n}}$ (towards m_1 or towards m_2?) or which particle to call 1 and which to call 2.

13.4 A particle collision model of running

At every step a running person flies through the air, hits the ground with a foot and pushes on the ground. By action and reaction, the ground pushes back on the foot which pushes on the leg which pushes on the body which causes the body to slow its descent and then go from moving forward and somewhat down to moving forward and somewhat up. Then the foot leaves the ground and the person flies through the air again readying for the next foot contact.

Human bodies are somewhat bigger than human legs so one approximation is that the legs have negligible mass. Human bodies don't tumble about much during a running step, so a next approximation is to neglect all distortion and rotation of the body and think of it as a particle. Finally, one might imagine that the ground contact time is short, and that the step on the ground is like a bounce. Thus running is like a sequence of collisions between a body and the ground. Obviously a running person is not a bouncing particle in all regards. Nonetheless, this model gives a means for making various estimates about running.

In the flight phase of running, neglecting air friction, the body moves in a parabolic arc according to:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \qquad \Rightarrow \qquad -mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}$$

This has solution that the time of flight is

$$t_f = 2v_{y0}/g$$

where v_{y0} is the vertical component of the velocity at the start of flight. The distance of flight is

$$d = v_x t_f = 2v_x v_y/g$$

where v_x is the constant horizontal component of velocity.

What happens in the 'collision' with the ground?

The horrible leap-frog model of running

We could think of each step as independent. Each running step would be a jump at the end of which the body would come to rest and then jump again. That is, each step would start with an explosion and, after a period of flight, end with a plastic no-slip collision. Then immediately after there would be another jump. How much energy would it take to run like that? Each jump would involve an impulse to get the body from zero velocity to $\vec{\boldsymbol{v}} = v_x\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}}$. The work of the legs would be the increase in kinetic energy.

$$W = mv^2/2 = m(v_x^2 + v_{y0}^2)/2$$

Then the legs would absorb that much energy at landing. Muscles, unlike generators, are not regenerative. If muscles were regenerative you would feel especially peppy after you walked down a long stair case. On the other hand, walking down stairs is not that tiring. So let's approximate that there is no metabolic cost for absorbing work. So the energetic cost of locomotion per unit time would be

$$P = W/t_f = \frac{m\left(v_x^2 + v_{y0}^2\right)/2}{2v_{y0}/g} = mgv_x\frac{\tan\theta + \cot\theta}{4}$$

where $\tan\theta = v_{y0}/v_x$ is the angle of the trajectory at liftoff. The function $\tan\theta + \cot\theta$ has its minimum value of 2 at $\theta = 45°$ so the cost of such locomotion, in terms of average power, is $mgv_x/2$. Muscles use about 4 times as much chemical energy as they can produce work (*i.e.,* about 25% efficient at best) so the chemical energy to run by jumping and landing, over and over again, would be about

$$\text{Metabolic Cost per unit time} = P_{\text{met}} = 2mgv_x,$$

that is twice the weight times the speed. The chemical energy needed per unit distance would be about $2 \cdot$ (weight).

Obviously this seems like a tiring way to run. You shouldn't stop and start your horizontal motion at every step. Real people don't do that. Furthermore, the energy cost we have just predicted is bigger than what people use by a factor of about 5; the rate at which people use chemical energy to run is more like $mgv_x/2$ or $mgv_x/3$. Notice that the energetic cost of this mode of 'running' does not depend on the step length or flight time but only on the initial angle of the trajectories. Smaller steps involve smaller collisions and hence smaller energy cost per collision. But with smaller jumps there are more collisions per unit distance. The two effects exactly cancel in this model. Only the angle of liftoff matters, not the length of the jumps.

Frictionless collision model of running

Although shoes generally have high friction, the legs pivot under the body during ground contact. The result is that the main force transmitted by the leg to the body is vertical. In effect the leg mediates an effectively frictionless collision. At least that's an extreme idealization of what a leg does. Perhaps a better model of running is then a sequence of vertical frictionless collisions.

At each step there is, in effect, a plastic frictionless collision which absorbs energy immediately followed by an energetically generative collision that sends the body back up again. Together they look like a single frictionless elastic collision, but in this model we want to take account of the work absorbed in landing and the work needed to take off again. To start we will neglect that humans do have springs in their legs (e.g., tendons).

Thus at each step the energy needed to take off is

$$W = mv_{y0}^2/2.$$

The time of flight is again $t_f = 2v_{y0}/g$ and so, for this model the average work per unit time is

$$P = W/t_f = \frac{mv_{y0}^2/2}{2v_{y0}/g} = mgv_{y0}/4$$

and the work per unit distance would be

$$\text{work per unit distance} = mg\frac{v_{y0}/v_x}{4} = mg\tan\theta/4$$

and the metabolic cost per unit distance, taking muscle efficiency as 25% again, would be the weight times $\tan\theta$. So, at a given horizontal speed, the energy cost per unit distance can be made arbitrarily small by having the flight angle small and there being, consequently, more and more small collisions. But for a person to try to save energy that way she would have to swing her legs in impossibly tiring small rapid steps. To complete this model so that it would not predict that people should choose infinite frequency and infinitesimal steps we would have to add in a formula for the cost of swinging the legs rapidly. If we evaluate this model with the step length of real human running, and the consequent launch angle θ we over-estimate the actual energetic cost of running by about a factor of 2. Why is that? Probably because people do use their springs to bounce. They don't just throw away all their energy at each ground landing and then jump vertically. Rather their tendons store energy and release it at each step, doing something like half of the work needed to get airborne again.