13.2 Collisions and explosions of particles in 2D and 3D

When two things bump into each other there is often a big interaction force. Think about a ball bouncing off the ground, two pool balls colliding, a baseball hitting a bat, two cars crashing, or the big forces when a satellite gravitationally slingshots around a planet it passes close by. Similarly there are big short-lived forces when things explode into two or more pieces. A big and short-lived force is often described by

$$\text{its net impulse } \vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}} dt$$

rather than its detailed time-history $\vec{\boldsymbol{F}}(t)$. The collision modeling assumption is that these interaction forces are so big that all other forces on the particles can be ignored. For a two-particle collision the impulses are $\vec{\boldsymbol{P}}_1 = \vec{\boldsymbol{P}}$ and $\vec{\boldsymbol{P}}_2 = -\vec{\boldsymbol{P}}$ acting on m_1 and m_2. Rather than looking at the acceleration of mass during the collision one just calculates

$$\text{the net change in velocity} = \Delta\vec{\boldsymbol{v}}.$$

Before the collision two particles m_1 and m_2 have velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ (see fig. 13.9). The superscript "$^-$" (minus) means *just* before the collision. Then the particles collide. Even though we ignore the spatial extent of the particles for most of the mechanics analysis, we note that the two particles have a common tangent plane. The normal of that plane, pointing out of particle 1, say, is $\hat{\boldsymbol{n}}$. Just after the collision the particles have velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ with the superscript"$^+$" indicating *just* after the collision.

The general collision problem is

> Given some information about the motion before the collision, the motion after the collision, and the collisional impulse, find other information about these same quantities.

We find the unknowns using

- Momentum balance for each particle: $\vec{\boldsymbol{P}} = \int \vec{\boldsymbol{F}}(t)\, dt = m\Delta\vec{\boldsymbol{v}}$; and
- Some information about the collisional impulse, usually a constitutive law for the collision.

For collisional modeling the constitutive law for interaction involves impulse and change of velocity. We only consider two such constitutive models:

- *plastic sticking* collisions where $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$.

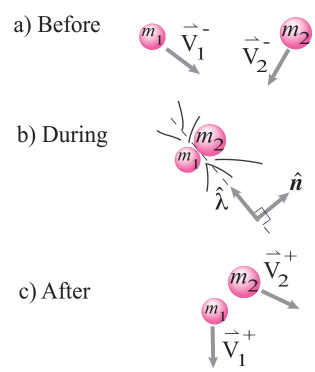


Figure 13.9: Two particles collide. Their velocities before the collision are $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$. When they collide their common tangent plane has normal $\hat{\boldsymbol{n}}$. After the collision their velocities are $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$.

- frictionless restitution with $(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+) \cdot \hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-) \cdot \hat{\boldsymbol{n}}$ and $\vec{\boldsymbol{P}} \cdot \hat{\boldsymbol{\lambda}} = 0$.

The constitutive models are discussed further below in the context of the three idealized collisions we treat here:

- sticking collisions
- frictionless collisions with restitution
- explosions.

The only expansion in this section over the 1D collisions in section 10.5 is the need for 2D and 3D geometry.

Sticking collisions

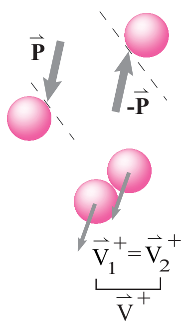


Figure 13.10: Two particles collide and stick together so in their subsequent motion $\vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+ = \vec{\boldsymbol{v}}^+$. The action-reaction impulse pair is in whatever direction it needs to be to get this sticking; the impulses need not be in just the $\hat{\boldsymbol{n}}$ direction.

The conceptually simplest collision is a *sticking* collision also called a *perfectly plastic no-slip* collision (see fig. 13.10). Here the word 'plastic' is used in its old latin meaning malleable or 'clay like'. Imagine two lumps of wet clay colliding in space and just sticking together. The plastic collision model applies, for example,

- when a projectile gets embedded in its target,
- when two cars crash and get entangled so they move together after the collision, or
- when two machine parts engage at contact because of a mechanism like a door catch.

In short, the constitutive law for plastic collisions is

$$\text{Plastic (sticking collision)} \quad \Rightarrow \quad \vec{\boldsymbol{v}}_1^+ = \vec{\boldsymbol{v}}_2^+$$

And the impulse is what it is, as determined by momentum balance for the two particles. Here's the simplest collision problem.

Example: A particle collides with an immovable object.
The impulse on the particle is

$$\vec{\boldsymbol{P}} = m\Delta\vec{\boldsymbol{v}} = m(\vec{\boldsymbol{0}} - \vec{\boldsymbol{v}}^-) = -m\vec{\boldsymbol{v}}^-.$$

And here is the general two-particle sticking collision problem.

Example: Two particles collide and stick.
There are three velocities to consider, the before-collision velocities $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ and the common after-collision velocity $\vec{\boldsymbol{v}}^+$. Also relevant is the interaction impulse $\vec{\boldsymbol{P}}$. That's 4 vector quantities (8 scalars in 2D, 12 in 3D). The governing equations are momentum balance for the two particles

$$\vec{\boldsymbol{P}} = m_1(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}_1^-) \qquad \text{and} \qquad \vec{\boldsymbol{P}} = m_2(\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}_2^-)$$

making up 2 vector equations (4 scalar equations in 2D, 6 in 3D). Thus to solve a problem in 2D, 4 scalar quantities need to be given so that the other quantities can be found from the momentum balance equations. In 3D, 6 scalar quantities have to be given.

There are all different ways to involute such problems, say by taking one of the masses as unknown. Here is the most straightforward example.

Example: Find the post-collision velocities for a sticking collision.
Given $m_1, m_2, \vec{v}_1^-$ and $\vec{v}_2^-$ we find by solving the momentum balance equations that

$$\vec{v}^+ = \frac{m_1\vec{v}_1^- + m_2\vec{v}_2^-}{m_1 + m_2} \qquad \text{and} \qquad \vec{P}_1 = -\vec{P}_2 = \frac{m_1 m_2}{m_1 + m_2}(\vec{v}_2^- - \vec{v}_1^-).$$

The answer can be interpreted like this. The final velocity is the same as the pre-collision average velocity. This is also the system's initial (and final) center of mass velocity. The impulsive interaction is associated with the change of velocity of $\vec{v}_1^- - \vec{v}_2^-$ of an effective 'reduced mass' with a value of $m_{red} = m_1 m_2/(m_1 + m_2)$ (see box 13.1 on page 3).

Frictionless collisions with restitution

This is the most common model used in elementary mechanics courses. It is originally due to Newton, at least in the 1D case we discussed in section 10.5. Two particles collide and then separate. There is no interaction force in their common contact tangent plane (hence 'frictionless'). See fig. 13.11. The impulse is such that the particles separate at a speed that is a fixed ratio e of the speed at which they approached. The speed of approach and separation are measured in the $\hat{\boldsymbol{n}}$ direction.

The speed of approach is the rate at which the distance between the particles decreases just before the collision. Really, this only makes precise sense if

- the masses are round, or
- the masses are not rotating.

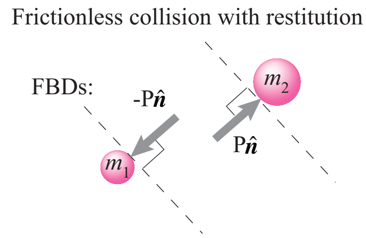


Figure 13.11: Two particles collide and bounce off each other frictionlessly. The assumption is that their relative separation speed is the coefficient of restitution e times their approach speed. Even though for momentum balance we treat the masses as particles, for considering the collision we look at the normal $\hat{\boldsymbol{n}}$ of the common contacting tangent plane. The separation speed is $(\vec{v}_1^+ - \vec{v}_2^+)\cdot\hat{\boldsymbol{n}}$ and the approach speed is $-(\vec{v}_1^- - \vec{v}_2^-)\cdot\hat{\boldsymbol{n}}$.

13.1 Effective mass

In two-particle collisions the forces of interest are in the action-reaction pair between the particles: $\vec{F}_1$ acts on m_1 and $\vec{F}_2 = -\vec{F}_1$ acts on particle 2. If we know one we know the other, so let's call $\vec{F}$ the force $\vec{F}_1$ on m_1. The two-particle system has no net acceleration, meaning the center of mass does not accelerate. All that the interaction force does is affect the relative motion of the particles.

So consider the relative acceleration of particle 1, say, relative to particle 2:

$$\vec{a}_{\rm rel} = \vec{a}_1 - \vec{a}_2 = \frac{\vec{F}_1}{m_1} - \frac{\vec{F}_2}{m_2} = \frac{\vec{F}_1}{m_1} - \frac{-\vec{F}_1}{m_2} = \vec{F}\left(\frac{1}{m_1} + \frac{1}{m_2}\right).$$

Thus we can write

$$\vec{F} = m_{\rm eff}\vec{a}_{\rm rel}$$

$$\text{with} \quad m_{\rm eff} = \left(\frac{1}{m_1} + \frac{1}{m_2}\right)^{-1} = \frac{m_1 m_2}{m_1 + m_2}$$

The 'reduced mass' or 'effective mass' $m_{\rm eff}$ is that which connects the interaction force with the relative acceleration $\vec{a}_{\rm rel} = \vec{a}_1 - \vec{a}_2$ of the masses. To personalize it, imagine you have negligible mass and are floating in space between two big masses holding a handle on each. Then the relation between the tension in your arms and the relative acceleration of the masses is determined by the reduced effective mass $m_{\rm eff}$. The effective mass is less than either of the masses separately (because the relative acceleration comes from the addition of the two accelerations). For two equal masses $m = m_1 = m_2$ the effective mass is $m_{\rm eff} = m/2$.

Integrating in time the effective mass also relates the interaction impulse with the change in relative velocity.

$$\vec{P} = m_{\rm eff}\Delta\vec{v}_{\rm rel},$$

where $\vec{P} = \int \vec{F}dt$ acts on m_1 and $\vec{v}_{\rm rel} = \vec{v}_1 - \vec{v}_2$.

The approach speed is the relative velocity dotted with the $\hat{\boldsymbol{n}}$ direction

$$v_{approach} = \left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}.$$

The separation speed, measured just after the collision, has the same definition but with a sign change

$$v_{\text{sep}} = -\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}}.$$

Newton's law of collisional restitution① is

$$v_{\text{sep}} = e v_{approach} \quad \text{or} \quad \left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} = -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}. \quad (13.4)$$

①Why the word *restitution*? The particles approach each other with some momentum relative to their common center of mass. At some point during the collision they have none. Then some of the momentum is restituted, paid back, and they bounce. No restitution, $e = 0$, and there is no bounce. Full restitution, $e = 1$, and they separate with the as much relative momentum as they had when they approached.

We use the coefficient of restitution for approximate collisional modeling but,

> the 'coefficient of restitution' equation is not an accurate law of nature.

Example: Two-particle elastic collision.
Two particles m_1 and m_2 have pre-collision velocities of $\vec{\boldsymbol{v}}_1^-$ and $\vec{\boldsymbol{v}}_2^-$ and collide frictionlessly with coefficient of restitution e on the tangent plane with normal $\hat{\boldsymbol{n}}$. The post collision velocities $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$, as well as the impulse $\vec{\boldsymbol{P}}$ are found by simultaneously solving these equations.

$$\begin{aligned}
\vec{\boldsymbol{P}} &= m_1\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_1^-\right)\\
-\vec{\boldsymbol{P}} &= m_2\left(\vec{\boldsymbol{v}}_2^+ - \vec{\boldsymbol{v}}_2^-\right)\\
\left(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+\right) \cdot \hat{\boldsymbol{n}} &= -e\left(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-\right) \cdot \hat{\boldsymbol{n}}
\end{aligned}$$

In 2D this makes up 5 scalar equations for 5 scalar unknowns. In 3D it's 7 equations for 7 unknowns. The most direct solution is to set up and solve these equations using a computer.

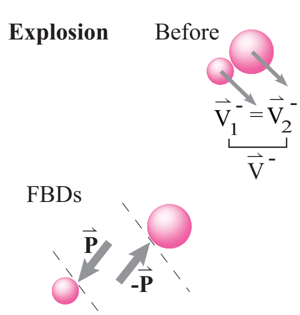


Figure 13.12: An explosion is like a sticking collision run backwards in time. The particles initially have the same velocity $\vec{\boldsymbol{v}}_1^- = \vec{\boldsymbol{v}}_2^- = \vec{\boldsymbol{v}}^-$ and then separate due to an action-reaction impulse pair in any direction.

Rather it is an approximate empirical observation. Or, to put it another way, the value of the coefficient of restitution e depends on the material, the shape, the orientation and the speed of the colliding particles. It is not a true constant. Nonetheless, eqn. (13.4) is a reasonable approximation for some engineering purposes. Just don't assume that predictions it makes will generally be highly accurate.

The 'frictionless' part of this collision law is expressed by the assumption that the net impulse of interaction is in the $\hat{\boldsymbol{n}}$ direction. So $\vec{\boldsymbol{P}} = P\hat{\boldsymbol{n}}$ with no component in the $\hat{\boldsymbol{\lambda}}$ direction.

Generally one assumes that the coefficient of restitution is between zero and one:

$$0 \leq e \leq 1.$$

For $e < 0$ the masses have to pass through each other. For $e > 1$ the collision would involve a gain in energy. This might happen if there was explosive gunpowder in the contact region of the collision.

In the case $e = 0$ the collision is perfectly plastic but still frictionless. This is generally not a sticking collision because the masses can enter and hence leave the collision with some relative velocity in the common contact plane (the $\hat{\boldsymbol{\lambda}}$ direction).

Explosions

If one particle explodes into pieces it's as if the pieces had a collision. It's just that the initial velocities of the pieces were all the same and the total kinetic energy of the system increases during the 'collision'. See fig. 13.12. The overall treatment is extremely similar to that for sticking collisions, but in some sense backwards. Instead of the particles entering the collision with different velocities and leaving with the same velocity, they enter with the same velocity and leave with different velocities. But the same momentum principles apply. There is no collision law or coefficient of restitution to apply, all of the post-collision relative velocity is restituted from nothing. Rather one just has to know (or find) the action-reaction impulse between the masses.

Example: An explosion.
Two particles m_1 and m_2 are stuck together and moving at $\vec{\boldsymbol{v}}^-$ when they explode

13.2 Energetics of collisions

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. We'd like to treat these cases in a unified way. First let's calculate the total kinetic energy.

$$\begin{aligned}
2E_{\mathrm{K}} &= m_1 v_1^2 + m_2 v_2^2 \\
&= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_1 |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_{\mathrm{cm}}|^2 + m_2 |\vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_{\mathrm{cm}}|^2 \\
&= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_{\mathrm{eff}} |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|^2 \\
&= m_{\mathrm{tot}} v_{\mathrm{cm}}^2 + m_{\mathrm{eff}} v_{\mathrm{rel}}^2
\end{aligned}$$

where $m_{\mathrm{tot}} = m_1 + m_2$, $m_{\mathrm{eff}} = m_1 m_2/(m_1 + m_2)$ and $v_{\mathrm{rel}} = |\vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_2|$. There are a few algebra steps needed to go from line to line above (see section B for related calculations). The concept of effective mass m_{eff} is introduced in box 13.1. The key result is that the kinetic energy of a two-particle system can be written as the sum of two terms, one involving center of mass velocity and one involving the relative velocity of the two masses.

This is a special result for two-particle systems. For any system the kinetic energy is a center of mass term $(mv_{\mathrm{cm}}^2/2)$ plus a term for motion relative to the center of mass. But generally the relative motion term is written as a sum of terms, one for each particle, and the motion of each particle is measured relative to the center of mass $(mv_{i/\mathrm{cm}})$. What is special for two-particle systems is that the relative motion part can be written in terms of the motion of the two particles relative to each other. Because that is not the velocity of any real thing, it only gives the right kinetic energy when used with the corrected effective mass (m_{eff}).

What about energy and collisions? The center of mass velocity and energy do not change in the collision. So the only change in kinetic energy is that associated with changes in $m_{\mathrm{eff}} v_{\mathrm{rel}}^2$.

$$\begin{aligned}
2\Delta E_{\mathrm{K}} &= m_{\mathrm{eff}} \left((v_{\mathrm{rel}}^+)^2 - (v_{\mathrm{rel}}^-)^2 \right) \\
&= 2\vec{\boldsymbol{v}}_{\mathrm{rel}}^- \cdot \vec{\boldsymbol{P}} + |\vec{\boldsymbol{P}}|^2/m_{\mathrm{eff}}
\end{aligned}$$

where we used that $\vec{\boldsymbol{P}} = m_{\mathrm{eff}}(\vec{\boldsymbol{v}}_{\mathrm{rel}}^+ - \vec{\boldsymbol{v}}_{\mathrm{rel}}^-)$. This formula applies for both sticking collisions, in which case $\vec{\boldsymbol{P}} = -m_{\mathrm{eff}}\vec{\boldsymbol{v}}_{\mathrm{rel}}^-$ and $2\Delta E_{\mathrm{K}} = -m_{\mathrm{eff}}(v_{\mathrm{rel}}^-)^2$, and to explosions where $\vec{\boldsymbol{P}} = m_{eff}\vec{\boldsymbol{v}}_{\mathrm{rel}}^+$ and $2\Delta E_{\mathrm{K}} = m_{\mathrm{eff}}(v_{\mathrm{rel}}^+)^2$. It also applies to interactions in-between.

All that enters the change-of-energy equations above is the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction. Thus the issue of energy loss or gain is determined by whether the projection of the relative velocity in the $\vec{\boldsymbol{P}}$ direction decreases or increases in magnitude. Thus a collision with $-1 < e < 1$ loses energy and a collision with $|e| > 1$ increases energy. We included $e < 0$ for completeness even though it is sometimes considered 'non-physical' in that it involves the particles passing by or passing through each other.

and an impulse $\vec{P}$ separates them. After the collision

$$\vec{v}_1^+ = \vec{v}^- + \vec{P}/m_1 \qquad \text{and} \qquad \vec{v}_2^+ = \vec{v}^- - \vec{P}/m_2.$$

The full range of behavior for sticking collisions to explosions can be captured with a single restitution coefficient e_g (see box 13.3).

Frictional collisions

Our avoiding of frictional collisions is not because there generally is no friction during collisions. Friction is a fact of the mechanical world. We avoid friction here because a host of special assumptions are needed to make frictional problems deterministic. And no given set of assumptions is known to yield accurate predictions. Frictional collision models have too dis-satisfyingly low a ratio of accuracy to complexity for inclusion in a book at this level.

Simultaneous collisions

If one particle is involved in two collisions at one time then we have not explained how to calculate the resulting motion. In an attempt to make the situation clear one is tempted to say “Let’s make it ideal and assume the collisions are exactly instantaneous and at *exactly* the same time.” Then, unfortunately, one is making the situation *exactly* ambiguous.

Unfortunately for our hope of making reliable predictions, simultaneous collisions are *not* rare events. Why? Imagine B is touching C and both are stationary. Then A comes and bangs into B. Because B and C are already touching one must assume that there are impulsive forces not just between A and B, but also between B and C. And we have no reliable rules for sorting out the result. Nor will we find such rules if we make it a life’s work.

Example: A triangular array of identical spheres.
Imagine 15 accurately-machined nominally-identical spheres laid out in a tight triangle (5 in one row, 4 in the next, then 3,2, and 1) on a very flat smooth surface. Then imagine a 16th ball rolls in and hits the apex of the triangle. How do the 15 balls move?

This experiment is performed in smoky rooms full of intoxicated people night after night. It’s the ‘break’ in a pool game. And the game depends on the result being unpredictable. Each time, due to tiny differences, the results are different.

And, according to theory, the more rigid and perfect the balls are, the more sensitive are the results to the smallest of differences in the initial conditions.

What is the source of the problem?

Example: Three balls in a line.
Consider the one-dimensional collision of three identical particles. B and C are in line, stationary and touching and then A comes along with $v_{A0} = v^-$. Let’s assume that the collision(s) whatever they are, are completely elastic and conserve energy ($e = 1, e_g = 0$). Here are two ways to predict the outcome:

- A hits B and C, being all the way at the other side of B, is oblivious to the interaction between A and B until it is complete. Thus A comes to rest and B is moving to the right with v^-. Then B collides elastically with C and B comes to rest and C shoots off with the v^-.
- B and C are touching and act as a single rigid object throughout the collision with A. Thus the result is like that between a particle with mass m and another with mass $2m$. Such an elastic collision would leave B and C going forwards at $2v^-/3$ and A going the other way with $v_A^+ = -v^-/3$. A different result.
- actually there is a one-parameter family of results that are consistent with energy conservation and momentum balance. We have three outcomes (the velocities of the three particles) and only 2 scalar equations restricting them (momentum and energy balance).

But what would *really* happen? That would depend on details that are not stated. Of course if the *exact* shape and configuration of the balls was known, and the exact rules for elastic and inelastic deformation, then one could calculate the resulting motion solving partial differential equations or with atomic simulations. In principle. But we generally do not know such details nor have such calculation abilities. And crowding that which we don't know into concepts like 'rigid-object' and 'exactly simultaneous' crowds the prediction of the outcome to dependence on infinitesimal things.

So, as an engineer, what are you supposed to do when calculating in situations involving simultaneous collisions?

- first relax and remember that no collision calculation is likely to be very accurate (unless the result only depends on balance of momentum). So simultaneous collisions, while philosophically worse in that even the equations are indeterminate, are not that much worse than the usual deterministic, but not accurate, collisional relations.
- do experiments, and
- take account of the range of outcomes depending on assumptions about the collision details.

Samples 13.4 and 13.5 starting on page 675 illustrate the ambiguity of simultaneous collisions in 2D.

Final comments

This is the second of three sections about collisions. Section 10.5 was about collisions in 1D, then this section about particles in 2D and 3D, and finally the ideas in this section will be extended from particles to rigid objects in section 17.5.

13.3 Coefficient of generation

Often one thinks of collisions as passive and energetically dissipative. However, as noted in the text, an explosion is a collision of sorts in which the system kinetic energy increases. For passive frictionless collisions one can characterize the collision by the coefficient of restitution

$$\begin{aligned} e &\equiv \frac{\text{(separation speed)}}{\text{(approach speed)}} \\ &= \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}}}{(\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \end{aligned} \tag{13.5}$$

However, for explosions the coefficient of restitution is $e = \infty$. If one is equally interested in energy absorbing or energy creating collisions one can use a more democratic coefficient of generation

$$\begin{aligned} e_g &\equiv \frac{\text{(separation speed)-(approach speed)}}{\text{(separation speed)+(approach speed)}} \\ &= \frac{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} - (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}}{-(\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} + (\vec{\boldsymbol{v}}_1^- - \vec{\boldsymbol{v}}_2^-)\cdot\hat{\boldsymbol{n}}} \end{aligned} \tag{13.6}$$

We can write the collisional coefficient of generation in terms of the restitution coefficient, and *vice versa*, as

$$e_g = \frac{e-1}{e+1} \qquad \text{and} \qquad e = \frac{1+e_g}{1-e_g}.$$

The generation coefficient is -1 for sticking collisions and 1 for explosions. This coefficient is zero for energetically neutral collisions (no gain, no loss, $e = 1$). And the coefficient of generation does not allow for passing-through or passing-by collisions ($e < 0$).

As a replacement for the conventional coefficient of restitution the coefficient of generation e_g is more complex to use in simple calculations in that eqn. (13.6) is more complex than eqn. (13.5). On the other hand the coefficient of generation is convenient for describing situations which are a mix of passive ($e < 1$ and $e_g < 0$) and active ($e > 1$ and $e_g > 0$). Such is the case, for example, in simple models of legged locomotion (see box 13.4 on page 9).

Note that in all the collisional restitution formulas we could replace $\hat{\boldsymbol{n}}$ with $-\hat{\boldsymbol{n}}$ without affecting the validity of the equations. Similarly all the subscript 2's could be replaced with 1's and *vice versa* without affecting the validity of the equations. Knowing this relieves anxiety about the choice of normal $\hat{\boldsymbol{n}}$ (towards m_1 or towards m_2?) or which particle to call 1 and which to call 2.

13.4 A particle collision model of running

At every step a running person flies through the air, hits the ground with a foot and pushes on the ground. By action and reaction, the ground pushes back on the foot which pushes on the leg which pushes on the body which causes the body to slow its descent and then go from moving forward and somewhat down to moving forward and somewhat up. Then the foot leaves the ground and the person flies through the air again readying for the next foot contact.

Human bodies are somewhat bigger than human legs so one approximation is that the legs have negligible mass. Human bodies don't tumble about much during a running step, so a next approximation is to neglect all distortion and rotation of the body and think of it as a particle. Finally, one might imagine that the ground contact time is short, and that the step on the ground is like a bounce. Thus running is like a sequence of collisions between a body and the ground. Obviously a running person is not a bouncing particle in all regards. Nonetheless, this model gives a means for making various estimates about running.

In the flight phase of running, neglecting air friction, the body moves in a parabolic arc according to:

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \quad \Rightarrow \quad -mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}$$

This has solution that the time of flight is

$$t_f = 2v_{y0}/g$$

where v_{y0} is the vertical component of the velocity at the start of flight. The distance of flight is

$$d = v_x t_f = 2v_x v_y/g$$

where v_x is the constant horizontal component of velocity.

What happens in the 'collision' with the ground?

The horrible leap-frog model of running

We could think of each step as independent. Each running step would be a jump at the end of which the body would come to rest and then jump again. That is, each step would start with an explosion and, after a period of flight, end with a plastic no-slip collision. Then immediately after there would be another jump. How much energy would it take to run like that? Each jump would involve an impulse to get the body from zero velocity to $\vec{\boldsymbol{v}} = v_x\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}}$. The work of the legs would be the increase in kinetic energy.

$$W = mv^2/2 = m(v_x^2 + v_{y0}^2)/2$$

Then the legs would absorb that much energy at landing. Muscles, unlike generators, are not regenerative. If muscles were regenerative you would feel especially peppy after you walked down a long stair case. On the other hand, walking down stairs is not that tiring. So let's approximate that there is no metabolic cost for absorbing work. So the energetic cost of locomotion per unit time would be

$$P = W/t_f = \frac{m\left(v_x^2 + v_{y0}^2\right)/2}{2v_{y0}/g} = mgv_x\frac{\tan\theta + \cot\theta}{4}$$

where $\tan\theta = v_{y0}/v_x$ is the angle of the trajectory at liftoff. The function $\tan\theta + \cot\theta$ has its minimum value of 2 at $\theta = 45°$ so the cost of such locomotion, in terms of average power, is $mgv_x/2$. Muscles use about 4 times as much chemical energy as they can produce work (*i.e.,* about 25% efficient at best) so the chemical energy to run by jumping and landing, over and over again, would be about

$$\text{Metabolic Cost per unit time} = P_{\text{met}} = 2mgv_x,$$

that is twice the weight times the speed. The chemical energy needed per unit distance would be about $2 \cdot$ (weight).

Obviously this seems like a tiring way to run. You shouldn't stop and start your horizontal motion at every step. Real people don't do that. Furthermore, the energy cost we have just predicted is bigger than what people use by a factor of about 5; the rate at which people use chemical energy to run is more like $mgv_x/2$ or $mgv_x/3$. Notice that the energetic cost of this mode of 'running' does not depend on the step length or flight time but only on the initial angle of the trajectories. Smaller steps involve smaller collisions and hence smaller energy cost per collision. But with smaller jumps there are more collisions per unit distance. The two effects exactly cancel in this model. Only the angle of liftoff matters, not the length of the jumps.

Frictionless collision model of running

Although shoes generally have high friction, the legs pivot under the body during ground contact. The result is that the main force transmitted by the leg to the body is vertical. In effect the leg mediates an effectively frictionless collision. At least that's an extreme idealization of what a leg does. Perhaps a better model of running is then a sequence of vertical frictionless collisions.

At each step there is, in effect, a plastic frictionless collision which absorbs energy immediately followed by an energetically generative collision that sends the body back up again. Together they look like a single frictionless elastic collision, but in this model we want to take account of the work absorbed in landing and the work needed to take off again. To start we will neglect that humans do have springs in their legs (e.g., tendons).

Thus at each step the energy needed to take off is

$$W = mv_{y0}^2/2.$$

The time of flight is again $t_f = 2v_{y0}/g$ and so, for this model the average work per unit time is

$$P = W/t_f = \frac{mv_{y0}^2/2}{2v_{y0}/g} = mgv_{y0}/4$$

and the work per unit distance would be

$$\text{work per unit distance} = mg\frac{v_{y0}/v_x}{4} = mg\tan\theta/4$$

and the metabolic cost per unit distance, taking muscle efficiency as 25% again, would be the weight times $\tan\theta$. So, at a given horizontal speed, the energy cost per unit distance can be made arbitrarily small by having the flight angle small and there being, consequently, more and more small collisions. But for a person to try to save energy that way she would have to swing her legs in impossibly tiring small rapid steps. To complete this model so that it would not predict that people should choose infinite frequency and infinitesimal steps we would have to add in a formula for the cost of swinging the legs rapidly. If we evaluate this model with the step length of real human running, and the consequent launch angle θ we over-estimate the actual energetic cost of running by about a factor of 2. Why is that? Probably because people do use their springs to bounce. They don't just throw away all their energy at each ground landing and then jump vertically. Rather their tendons store energy and release it at each step, doing something like half of the work needed to get airborne again.

SAMPLE 13.3 Projectile hits a slanted floor: A ball of mass $m = 0.2\,\mathrm{kg}$ is thrown in the air at an angle $\theta = 60°$ with initial speed $v_0 = 10\,\mathrm{m/s}$. The ball lands on a hard, frictionless floor that is tilted at angle $\alpha = 20°$ with the horizontal. The coefficient of restitution between the floor and the ball is $e = 0.85$. Ignore air resistance. Find the height of the ball after the rebound from the floor.

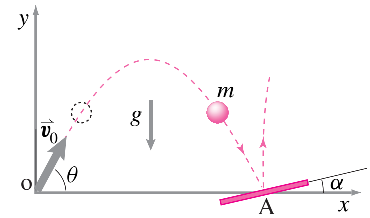


Figure 13.13

Solution This problem has two parts to it. In order to figure out the height after rebound, we need to find the rebound velocity. But to find the rebound velocity, we need to know the velocity of the ball before impact with the floor. Let the velocity just before the impact be $\vec{v}^-$ and the velocity of rebound (just after impact) be $\vec{v}^+$. Let us first find $\vec{v}^-$.

The ball undergoes projectile motion before it lands at A. Its initial (launch) velocity is $\vec{v}_0 = v_0(\cos\theta\hat{\imath} + \sin\theta\hat{\jmath})$. From energy conservation, we know that the kinetic energy just before impact at A, $m|\vec{v}^-|^2/2$, must be the same as kinetic energy at launch, $mv_0^2/2$. Thus $|\vec{v}^-| = v_0$. And, from the symmetry of the flight, we can conclude that $\vec{v}^-$ must make the same angle θ with the horizontal that $\vec{v}_0$ does. Thus, using fig. 13.14, we have

$$\vec{v}^- = v_0(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}).$$

Now we are ready to do collision mechanics at point A. We need to determine $\vec{v}^+$ given $\vec{v}^-$ and the coefficient of restitution for the collision at A. From collision law, we know that the velocity component normal to the floor changes because of the normal impulse during collision, while the tangential velocity remains the same because there is no force or impulse parallel to the floor. Thus,

$$\begin{aligned} \vec{v}^+ \cdot \hat{\boldsymbol{\lambda}} &= \vec{v}^- \cdot \hat{\boldsymbol{\lambda}} \\ \vec{v}^+ \cdot \hat{\boldsymbol{n}} &= -e(\vec{v}^- \cdot \hat{\boldsymbol{n}}). \end{aligned}$$

Writing out $\vec{v}^+ = v_x^+\hat{\imath} + v_y^+\hat{\jmath}$, and noting that $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ and $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, we get, from the equations above,

$$\begin{aligned} \cos\alpha v_x^+ + \sin\alpha v_y^+ &= v_0\cos\theta\cos\alpha - v_0\sin\theta\sin\alpha = v_0\cos(\theta+\alpha) \\ -\sin\alpha v_x^+ + \cos\alpha v_y^+ &= -e\,v_0(-\cos\theta\sin\alpha - \sin\theta\cos\alpha) = e\,v_0\sin(\theta+\alpha). \end{aligned}$$

These are two equations in two unknowns, v_x^+ and v_y^+. Writing them in a matrix form and solving the matrix equation, we get

$$\begin{pmatrix} v_x^+ \\ v_y^+ \end{pmatrix} = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{pmatrix} v_0\cos(\theta+\alpha) \\ e\,v_0\sin(\theta+\alpha) \end{pmatrix} = \begin{pmatrix} v_0\cos(\theta+2\alpha) \\ e\,v_0\sin(\theta+2\alpha) \end{pmatrix}.$$

Thus, we know the rebound velocity $\vec{v}+ = v_0[\cos(\theta+2\alpha)\hat{\imath} + e\sin(\theta+2\alpha)\hat{\jmath}]$.

To find the maximum height reached by the ball on the rebound, we only need the vertical component of the rebound velocity. Since the ball has a constant deceleration g, we can use the formula $(v_y)^2 = (v_{y_0})^2 - 2gh$ with $v_y = 0$ at the maximum height $h_{\max}$ to get,

$$h_{\max} = \frac{(v_y^+)^2}{2g} = \frac{e^2\,v_0^2\,\sin^2(\theta+2\alpha)}{2g}$$

Substituting the given values of v_0, e, θ, and α, and using $g = 9.81\,\mathrm{m/s^2}$, we get,

$$h_{\max} = 3.57\,\mathrm{m}.$$

$$\boxed{h_{\max} = 3.57\,\mathrm{m}}$$

You can see that the inclined plane helps in getting the ball to reach higher on the bounce. If the floor were flat ($\alpha = 0$), we would get $h_{\max} = 2.76\,\mathrm{m}$. It should be obvious that for maximum height, we should have $\sin(\theta+2\alpha) = 1$ which gives $\alpha = \frac{1}{2}(\frac{\pi}{2} - \theta)$.

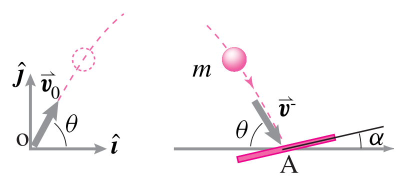


Figure 13.14: Trajectory of the ball through air: From the symmetry of the trajectory, we can conclude that the angle the velocity vector makes with the horizontal at point A is the same as the angle of launch, θ.

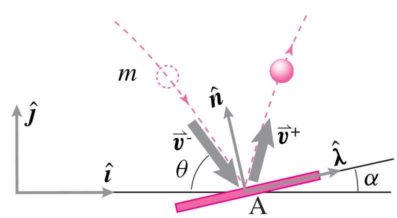


Figure 13.15: Collision of the ball with the tilted floor: The ball strikes the floor with velocity $\vec{v}^-$. It rebounds with velocity $\vec{v}^+$. The impulse during the collision acts in the normal direction to the floor at A. The unit normal and the unit tangent vectors at A are $\hat{\boldsymbol{n}} = -\sin\alpha\hat{\imath} + \cos\alpha\hat{\jmath}$, and $\hat{\boldsymbol{\lambda}} = \cos\alpha\hat{\imath} + \sin\alpha\hat{\jmath}$ respectively.

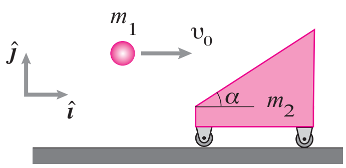


Figure 13.16

SAMPLE 13.4 Simultaneous collisions: This problem involves two simultaneous collisions. In general, such problems are hard to solve. We are going to show one way of solving such problems by treating the collisions successively. However, this leads to nonuniqueness of solution. Here we solve the problem in one way and in the next sample, we solve the same problem in another way.

A 12 kg cart with an inclined face rests on a frictionless floor. A ball of mass 3 kg is shot horizontally with speed 30 m/s at the inclined face of the cart. The coefficient of restitution between the cart and the ball is 0.9. The cart subsequently moves horizontally on the floor. Find the velocity of the ball and that of the cart after the collision.

Solution There are two simultaneous collisions in this problem. One collision is between the ball and the cart and the other is between the cart and the ground. Here, we will treat the two collisions one after the other, the one between the ball and the cart preceding the one between the cart and the ground. In Sample 13.5, we treat the ground collision first.

Collision between the ball and the cart: Here we assume that the ball hits the cart and both are free to move in any direction immediately after the collision. Let the mass of the ball be m_1 and that of the cart be m_2. Let their after collision velocities be $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$, respectively. Let the impulse during this collision be P_1.

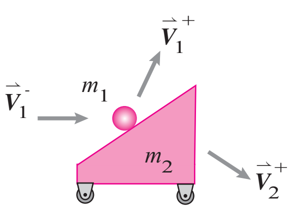


Figure 13.17: The ball and the cart as a system during the collision. There are no external forces or impulses acting on the system. Therefore, the linear momentum must be conserved during the collision; *i.e.*, $\vec{\boldsymbol{L}}^- = \vec{\boldsymbol{L}}^+$.

Let us consider the cart and the ball as a single system during the collision. Then, the impulse becomes internal to this system and there is no net impulse on this system. Therefore, the linear momentum is conserved; that is, $\vec{\boldsymbol{L}}^+ = \vec{\boldsymbol{L}}^-$. From this relationship, we have,

$$\mathrm{m}_1\vec{\boldsymbol{v}}_1^+ + \mathrm{m}_2\vec{\boldsymbol{v}}_2^+ = \mathrm{m}_1\vec{\boldsymbol{v}}_1^- + \mathrm{m}_2\vec{\boldsymbol{v}}_2^- = m_1 v_0\hat{\boldsymbol{\imath}}.$$

Writing out the unknown velocities in terms of their x and y components and dotting the resulting equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ separately, we get the following two scalar equations:

$$m_1 v_{1x}^+ + m_2 v_{2x}^+ = m_1 v_0 \tag{13.7}$$
$$m_1 v_{1y}^+ + m_2 v_{2y}^+ = 0. \tag{13.8}$$

We have four unknowns here, v_{1x}^+, v_{1y}^+, v_{2x}^+, and v_{2y}^+. So far, we have just two equations. We need more equations. We can write restitution equation relating the relative velocities of the ball and the cart in the normal direction before and after the collision:

$$(\vec{\boldsymbol{v}}_2^+ - \vec{\boldsymbol{v}}_1^+)\cdot\hat{\boldsymbol{n}} = -e(\vec{\boldsymbol{v}}_2^- - \vec{\boldsymbol{v}}_1^-)\cdot\hat{\boldsymbol{n}} = e(v_0\hat{\boldsymbol{\imath}})\cdot\hat{\boldsymbol{n}}.$$

Now, writing $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$ and carrying out the dot products (after writing $\vec{\boldsymbol{v}}_1^+$ and $\vec{\boldsymbol{v}}_2^+$ in terms of their components), we get,

$$(v_{2x}^+ - v_{1x}^+)n_x + (v_{2y}^+ - v_{1y}^+)n_y = -e\, v_0\, n_x. \tag{13.9}$$

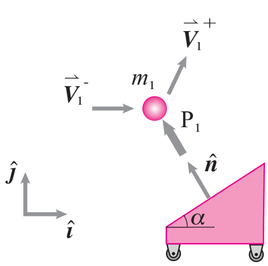


Figure 13.18: Free-body diagram of the ball during the collision. The impulse P_1 acts normal to the plane of collision which is the inclined surface of the cart.

We still need another equation. Let us now consider the impulse acting on the ball during the collision. From the free-body diagram shown in fig. 13.2, we can write the change in momentum of the ball as,

$$\begin{aligned} P_1\hat{\boldsymbol{n}} &= m_1\vec{\boldsymbol{v}}_1^+ - m_1\vec{\boldsymbol{v}}_1^- \\ \text{or} \qquad P_1(n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}) &= m_1(v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}} - v_0\hat{\boldsymbol{\imath}}). \end{aligned}$$

Again, separating out this equation in scalar equations (by dotting the equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ separately), we get,

$$P_1 n_x - m_1 v_{1x}^+ = -m_1 v_0 \tag{13.10}$$
$$P_1 n_y - m_1 v_{1y}^+ = 0. \tag{13.11}$$

Now, we have added another unknown P_1, but fortunately, we have got an extra equation too. We now have five unknowns and five independent equations. So we should be able to solve for all the unknowns.

For solving these equations, we first write them in matrix form and then use a computer to solve them. We write eqn. (13.7)–eqn. (13.11) as,

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & m_2 & 0 \\ -n_x & -n_y & n_x & n_y & 0 \\ -m_1 & 0 & 0 & 0 & n_x \\ 0 & -m_1 & 0 & 0 & n_y \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_{2x}^+ \\ v_{2y}^+ \\ P_1 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ -e v_0 n_x \\ -m_1 v_0 \\ 0 \end{pmatrix}.$$

Here is the pseudo computer code to solve this matrix equation:

```
m1 = 3, m2 = 12
theta = pi/6                    % angle in radians
nx = -sin(theta), ny = cos(theta) % components of the normal
v0 = 30
e =  0.9

A = [ m1   0   m2  0    0    % x comp of lin mom bal
      0    m1  0   m2   0    % y comp of lin mom bal
     -nx  -ny  nx  ny   0    % restitution equation
     -m1   0   0   0   -nx   % impulse-momentum for m1, x comp
      0   -m1  0   0   -ny]  % impulse-momentum for m1, y comp
b = [m1*v0  0  -e*v0*nx  -m1*v0  0]' % the known right hand side
solve A*x = b for x
```

The solution thus computed gives us

$$\begin{aligned} v_{1x}^+ &= 18.60\,\mathrm{m/s}, & v_{1y}^+ &= 19.74\,\mathrm{m/s}, \\ v_{2x}^+ &= 2.85\,\mathrm{m/s}, & v_{2y}^+ &= -4.94\,\mathrm{m/s}, \\ P_1 &= -68.40\,\mathrm{N\cdot s}. \end{aligned}$$

$$\begin{aligned} \vec{\boldsymbol{v}}_1^+ &= (18.60\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (19.74\,\mathrm{m/s})\hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{v}}_2^+ &= (2.85\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (-4.94\,\mathrm{m/s})\hat{\boldsymbol{\jmath}} \end{aligned}$$

Collision between the cart and the ground: Now, we consider the collision between the cart and the ground, taking $\vec{\boldsymbol{v}}_2^+$ as the velocity of the cart just before the collision. Figure 13.19 shows the impulse from the ground acting on the cart. We know the final velocity of the cart has to be in the $\hat{\boldsymbol{\imath}}$ direction. Just to keep our notations straight, let us denote the velocity of the cart after collision as $\vec{\boldsymbol{v}}_2^{++}$ (after the second collision) and keep the incoming velocity as $\vec{\boldsymbol{v}}_2^+$. Then, from impulse momentum, we have,

$$P_2\hat{\boldsymbol{\jmath}} = m_2\vec{\boldsymbol{v}}_2^{++} - m_2\vec{\boldsymbol{v}}_2^+.$$

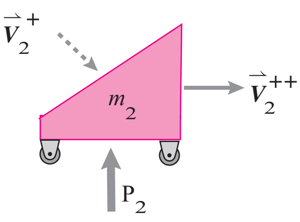


Figure 13.19

This is a vector equation which we can write as two scalar equations in the $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ directions. Note that $\vec{\boldsymbol{v}}_2^{++} = v_2^{++}\hat{\boldsymbol{\imath}}$ and we already know $\vec{\boldsymbol{v}}_2^+ = (2.85\,\mathrm{m/s})\hat{\boldsymbol{\imath}} + (-4.94\,\mathrm{m/s})\hat{\boldsymbol{\jmath}}$ as found before. Thus,

$$\begin{aligned} v_2^{++} &= \vec{\boldsymbol{v}}_2^+\cdot\hat{\boldsymbol{\imath}} = 2.85\,\mathrm{m/s} \\ P_2 &= -m_2\vec{\boldsymbol{v}}_2^+\cdot\hat{\boldsymbol{\jmath}} = -(12\,\mathrm{kg})(-4.94\,\mathrm{m/s}) = 59.24\,\mathrm{kg\cdot m/s} = 59.24\,\mathrm{N\cdot s}. \end{aligned}$$

$$\vec{\boldsymbol{v}}_2^{++} = 2.85\,\mathrm{m/s}\hat{\boldsymbol{\imath}}$$

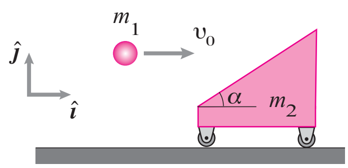


Figure 13.20

SAMPLE 13.5 Simultaneous collisions again: Consider the ball and the cart collision problem of Sample 13.4 again. This time, consider the ball and the cart together to have a collision with the ground first. Then consider the collision between the cart and the ball. Once again, you are to find the final horizontal velocity of the cart. The problem parameters are the same — mass of the ball $m_1 = 3\,\text{kg}$, mass of the cart $m_2 = 12\,\text{kg}$, $e = 0.9$ between the ball and the cart, and the velocity of the ball before impact, $v_0 = 30\,\text{m/s}$.

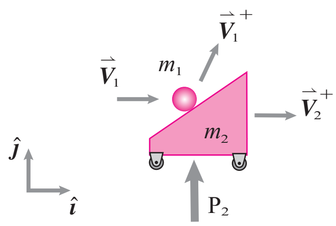


Figure 13.21: Free-body diagram of the ball-cart system during the collision with the ground. The ground impulse is P, the velocities of the ball and the cart before the collision are given — $\vec{\boldsymbol{v}}_1^- = v_0\hat{\boldsymbol{\imath}}$, $\vec{\boldsymbol{v}}_2^- = \vec{\boldsymbol{0}}$, and the velocities after the collision $\vec{\boldsymbol{v}}_2^+ = v_2\hat{\boldsymbol{\imath}}$ and $\vec{\boldsymbol{v}}_1^+ = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ are unknown.

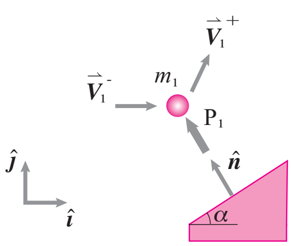


Figure 13.22: Free-body diagram of the ball during the collision with the cart. The cart impulse P_1 acts along the normal $\hat{\boldsymbol{n}}$. From geometry, we find that $\hat{\boldsymbol{n}} \equiv n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}} = -\sin\alpha\,\hat{\boldsymbol{\imath}} + \cos\alpha\,\hat{\boldsymbol{\jmath}}$.

Solution Let us consider the ball and the cart as a system colliding with the ground as shown in fig. 13.21. There is an unknown external impulse P_2 from the ground acting on this system in the $\hat{\boldsymbol{\jmath}}$ direction. Using this information, we now write impulse-momentum equation for this system:

$$P_2\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 = m_1\vec{\boldsymbol{v}}_1^+ + m_2\vec{\boldsymbol{v}}_2^+ - m_1\vec{\boldsymbol{v}}_1^-$$

Assuming that $\vec{\boldsymbol{v}}_1^+ = v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}$ and $\vec{\boldsymbol{v}}_2^+ = v_2^+\hat{\boldsymbol{\imath}}$, and using the given information $\vec{\boldsymbol{v}}_1^- = v_0\hat{\boldsymbol{\imath}}$, we obtain the following two scalar equations from the vector impulse-momentum equation:

$$\begin{aligned} m_1 v_{1x}^+ + m_2 v_2^+ &= m_1 v_0 && (13.12)\\ m_1 v_{1y}^+ - P_2 &= 0 && (13.13)\end{aligned}$$

So far, we have two equations and four unknowns — v_{1x}^+, v_{1y}^+, v_2^+ and P_2. Obviously, we need more equations. Now, let us consider the collision between the cart and the ball. Let the impulse of this collision be P_1. Then the impulse-momentum equation for the ball gives us,

$$P_1\hat{\boldsymbol{n}} = m_1(v_{1x}^+\hat{\boldsymbol{\imath}} + v_{1y}^+\hat{\boldsymbol{\jmath}}) - m_1 v_0\hat{\boldsymbol{\imath}}.$$

Once again, we separate out the scalar equations from this vector equation, using the information $\hat{\boldsymbol{n}} = n_x\hat{\boldsymbol{\imath}} + n_y\hat{\boldsymbol{\jmath}}$:

$$\begin{aligned} m_1 v_{1x}^+ - Pn_x &= m_1 v_0 && (13.14)\\ m_1 v_{1y}^+ - Pn_y &= 0 && (13.15)\end{aligned}$$

Thus, we have now four equations; we still need one more. We now use the restitution equation to relate the normal components of the relative velocities of approach and departure of the ball and the cart:

$$\begin{aligned} (\vec{\boldsymbol{v}}_1^+ - \vec{\boldsymbol{v}}_2^+)\cdot\hat{\boldsymbol{n}} &= -e(v_1^- - v_2^-)\cdot\hat{\boldsymbol{n}} \\ \Rightarrow\quad v_{1x}^+ n_x + v_{1y}^+ n_y - v_2 n_x &= -e\,v_0 n_x. && (13.16)\end{aligned}$$

Now we have five equations in five unknowns. All we need to do now is to solve these linear equations for all the unknowns. We do so by first writing the five equations (eqn. (13.12) to eqn. (13.16) in matrix form and then solving the matrix equation on a computer. The matrix equation is:

$$\begin{bmatrix} m_1 & 0 & m_2 & 0 & 0 \\ 0 & m_1 & 0 & 0 & -1 \\ m_1 & 0 & 0 & -n_x & 0 \\ 0 & m_1 & 0 & -n_y & 0 \\ n_x & n_y & -n_x & 0 & 0 \end{bmatrix} \begin{pmatrix} v_{1x}^+ \\ v_{1y}^+ \\ v_2^+ \\ P_1 \\ P_2 \end{pmatrix} = \begin{pmatrix} m_1 v_0 \\ 0 \\ m_1 v_0 \\ 0 \\ -e\,v_0 n_x. \end{pmatrix}.$$

Solving this equation as in the previous sample, we get,

$$v_{1x}^+ = 16.59\,\text{m/s},\; v_{1y}^+ = 23.23\,\text{m/s},\; v_2^+ = 3.35\,\text{m/s},\; P_1 = 80.47\,\text{N}\cdot\text{s},\; P_2 = 69.69\,\text{N}\cdot\text{s}.$$

$$\boxed{\vec{\boldsymbol{v}}_2^+ = (3.35\,\text{m/s})\hat{\boldsymbol{\imath}}}$$

Note that the answer obtained here is not the same as that found in Sample 13.4; the cart moves a bit faster to the right in this answer. Depending on the mass ratios and the angle of impact, the two methods can give very different answers or very close answers. Welcome to the world of modeling!

13.2 Collisions and explosions

Preparatory Problems

13.2.1 Assuming θ, v_0, and e to be known quantities, write the following equations in matrix form set up to solve for v'_{Ax} and v'_{Ay}:

$$\sin\theta v'_{Ax} + \cos\theta v'_{Ay} = e v_0 \cos\theta$$
$$\cos\theta v'_{Ax} - \sin\theta v'_{Ay} = v_0 \sin\theta.$$

*

13.2.2 The equation $(\vec{v}'_1 - \vec{v}'_2)\cdot\hat{\boldsymbol{n}} = e(\vec{v}_2 - \vec{v}_1)\cdot\hat{\boldsymbol{n}}$ relates relative velocities of two point masses before and after frictionless impact in the normal direction $\hat{\boldsymbol{n}}$ of the impact. If $\vec{v}'_1 = v_{1_x}\hat{\imath} + v_{1_y}\hat{\jmath}$, $\vec{v}'_2 = -v_0\hat{\imath}$, $e = 0.5$, $\vec{v}_2 = \vec{0}$, $\vec{v}_1 = 2\,\text{ft/s}\hat{\imath} - 5\,\text{ft/s}\hat{\jmath}$, and $\hat{\boldsymbol{n}} = \frac{1}{\sqrt{2}}(\hat{\imath} + \hat{\jmath})$, find the scalar equation relating the velocities in the normal direction. *

13.2.3 The following three equations are obtained by applying the principle of conservation of linear momentum on some system.

$$m_0 v_0 = 24\,\text{m/s}\,m_A - .67 m_B v_B - .58 m_C v_C$$
$$0 = 36\,\text{m/s}\,m_A + .33 m_B v_B + 0.3 m_C v_C$$
$$0 = 23\,\text{m/s}\,m_A - .67 m_B v_B - .58 m_C v_C.$$

Assume v_0, v_B, and v_C are the only unknowns. Write the equations in matrix form set up to solve for the unknowns. *

13.2.4 The following three equations are obtained to solve for v'_{Ax}, v'_{Ay}, and v'_{Bx}:

$$(v'_{Bx} - v'_{Ax})\cos\theta = v'_{Ay}\sin\theta - 10\,\text{m/s}$$
$$v'_{Ax}\sin\theta = v'_{Ay}\cos\theta - 36\,\text{m/s}$$
$$m_B v'_{Bx} + m_A v'_{Ax} = (-60\,\text{m/s})\,m_A.$$

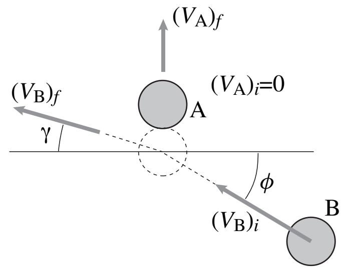
Set up these equations in matrix form. *

13.2.5 Solve for the unknowns v'_{A_x}, v'_{A_y}, and v'_{B_x} in problem 13.2.4 taking $\theta = 50^\circ$, $m_A = 1.5\,m_B$ and $m_B = 0.8\,\text{kg}$. Use any computer program. *

13.2.6 Using the matrix form of equations in Problem 13.2.1, solve for v'_{Ax} and v'_{Ay} if $\theta = 20^\circ$ and $v_0 = 5\,\text{ft/s}$. *

More-Involved Problems

13.2.7 Two frictionless equal-mass pucks sliding on a plane collide as shown below. Puck A is initially at rest. Given that $(V_B)_i = 1.0\,\text{m/s}$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the approach angle ϕ and rebound angle γ. The coefficient of restitution is $e = 0.9$.



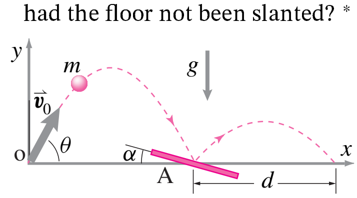
Problem 13.2.7

13.2.8 Reconsider problem 13.2.7. Given instead that $\gamma = 30^\circ$, $(V_A)_i = 0$, and $(V_A)_f = 0.5\,\text{m/s}$, find the initial velocity of puck B.

13.2.9 A ball of mass $m = 0.5\,\text{kg}$ is thrown up in the air with initial speed $v_0 = 50\,\text{m/s}$ at an angle $\theta = 60^\circ$. The ball lands on and bounces off a slanted floor that makes an angle $\alpha = 15^\circ$ with the horizontal. Assume the collision with the floor to be elastic and ignore air drag on the ball.

a) Find the impulse of the collision of the ball, *

b) After bouncing off the slanted floor, how much horizontal distance does the ball travel before landing on the ground again? Is this distance more, less, or the same as it would have travelled had the floor not been slanted? *



Problem 13.2.9

13.2.10 Solve the general two-particle frictionless collision problem. For example, write computer code that has lines like this near the start :

`m1=3; m2=19`	Set values of masses
`v1zero=[10 20]`	Initial velocity of mass 1
`v2zero=[-5 3]`	Initial velocity of mass 2
`e=.5`	Set coefficient of restitution
`theta=pi/4`	Angle that the normal to contact plane makes, measured CCW from +x axis, in radians

Your program (function, code, script) should calculate the impulse of mass 1 on mass 2, and the velocities of the two masses after the collision. Your program should assume consistent units for all quantities.

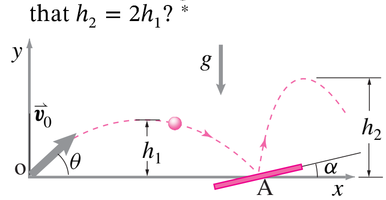
a) You should demonstrate that your program works by solving at least 4 different problems for which you can check your answer by simple pencil-and-paper calculations. These problems should have as much variety as possible. Sketch these problems clearly, show their analytic solution, and show that the computer agrees. *

b) Solve the problem given in the sample text given in the initial problem statement.

13.2.11 A projectile is launched at $\theta = 40^\circ$ with speed $v_0 = 25\,\text{m/s}$. The projectile lands on a steel plate that can be adjusted to make any angle α with the horizontal. The projectile bounces off the steel plate without losing any energy. The projectile is required to reach a height after rebound twice as much as it did during its flight before hitting the plate. Ignore air resistance.

a) Find the required angle α of the plate. *

b) Can you always find some α for any launch angle $\theta < \pi/2$ such that $h_2 = 2h_1$? *

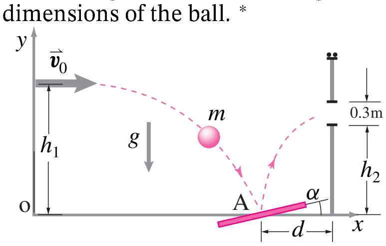


Problem 13.2.11

13.2.12 Two equal mass cars approach an intersection at right angles. They crash and stick together. One of the cars was going at 30 mph before the crash. The other car's path gets deflected by 15°. How fast was it going? *

13.2.13 A ball m is thrown horizontally at height h and speed v_0. It then has a sequence of bounces on the horizontal ground. Treating each collision as frictionless with restitution coefficient e how far has the ball travelled horizontally when it just finishes bouncing? Answer in terms of some or all of m, g, h, v_0 and e. *

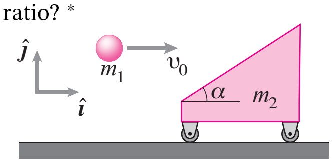
13.2.14 A game involves using a pedal to direct a falling ball into a fixed vertical slot by simply rotating the pedal when the ball hits the pedal. A model of this game is shown in the figure. The ball is thrown horizontally with an initial speed $v = 10\,\text{m/s}$ from a height $h_1 = 3\,\text{m}$. The pedal is located at $d = 2\,\text{m}$ from the wall that houses the slot at height $h_2 = 2\,\text{m}$. The slot itself is 0.3 m in extent. The coefficient of restitution between the pedal and the ball is $e = 0.9$. The air resistance is negligible. Find the angle α or the range of this angle, so that the ball makes it through the slot. You can ignore the dimensions of the ball. *



Problem 13.2.14

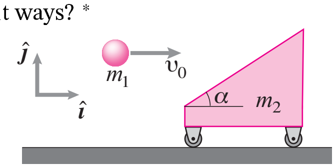
13.2.15 An airplane is flying steadily at an altitude of 30,000 ft at a speed of 500 mph. It explodes into two equal pieces. One piece is found to the right of the airplane's initial trajectory and 8 miles forward of the explosion point. Where should you look for the other piece? Assume the interaction impulse is in the horizontal plane and make the approximation that the two pieces fly in frictionless parabolic trajectories. *

13.2.16 Consider the simultaneous collision of a ball with a ramp and the ramp with the ground. Consider the ball to be much more massive than the cart; $m_1 = 20\,\text{kg}$ and $m_2 = 1\,\text{kg}$. The angle of the inclined face is very shallow, $\alpha = 2°$. The ball hits the cart with the velocity $\vec{v}_1 = 50\,\text{m/s}\hat{\imath}$. The impact of the ball with the ramp is elastic and frictionless. The ramp ends up moving in the $\hat{\imath}$ direction. Find the subsequent velocities of the ball and the cart using the two methods discussed in Sample 13.4 and Sample 13.5. Comment on the answers you get. How will your answers change if you reversed the mass ratio? *



Problem 13.2.16

13.2.17 Consider the simultaneous collisions problem of problem 13.2.16 again. Now assume that $m_1 = 5\,\text{kg}$, $m_2 = 10\,\text{kg}$, $\vec{v}_1 = 50\,\text{m/s}$, $e = 0.75$, and the angle $\alpha = 88°$. What is the net loss of energy in the impacts as calculated the two different ways? *



Problem 13.2.17