13.1 Coupled motions of particles in space

Assume you know enough about a system so that you know the forces on each particle if someone tells you the time and the positions and velocities of all the particles. This means you can write the governing equations for the system of particles like this:

$$
\begin{aligned}
\vec{a}_1 &= \frac{1}{m_1}\vec{F}_1 \\
\vec{a}_2 &= \frac{1}{m_2}\vec{F}_2 \\
\vec{a}_3 &= \frac{1}{m_3}\vec{F}_3 \\
&\text{etc.}
\end{aligned}
\tag{13.1}
$$

where $\vec{F}_1$, $\vec{F}_2$ *etc.* are the total of the forces on the corresponding particles. The force on each particle may come from air-friction, from springs or dashpots connected here and there, or from gravity interactions with other particles, from known applied loads, *etc.*. One way or another, all the forces on all the particles are known given the time, the positions and velocities of the particles. Thus eqn. (13.1) can be written as a system of first order differential equations in standard form, ready for computer simulation. Given accurate initial conditions and a good computer then the motions of all the particles can be found accurately.

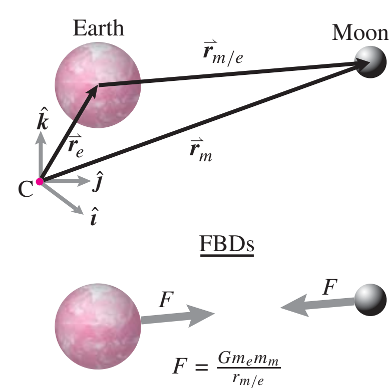

Figure 13.1: The earth and moon. Position is measured relative to some “fixed” point C.

Example: Coupled motion of the earth and moon in three dimensions.
Let’s neglect the sun and just look at the coupled motions of the earth and moon. They attract each other by the same law of gravity that we used for the sun and earth. The difference between this problem and a “central-force” problem is that we now need to look at the ‘absolute’ positions of the earth and the moon ($\vec{r}_e$ and $\vec{r}_m$), as well as the ‘relative’ position $\vec{r}_{m/e} \equiv \vec{r}_m - \vec{r}_e$ (fig. 13.1).

The linear momentum balance equations are now

$$
m_e \ddot{\vec{r}}_e = \frac{+G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3} \quad \text{and} \tag{13.2}
$$

$$
m_m \ddot{\vec{r}}_m = \frac{-G m_e m_m \vec{r}_{m/e}}{|\vec{r}_{m/e}|^3}, \tag{13.3}
$$

which, when broken into x, y, and z components give 6 second order ordinary differential equations. These equations can be written as 12 first order equations by defining a list of 12 z variables: $z_1 = x_e$, $z_2 = \dot{x}_e$, $z_3 = y_e$, $z_4 = \dot{y}_e$, *etc.*

After you find solutions, using various initial conditions you can check if the computer finds such truths (that is, features of the exact solution of the differential equations) as:

1. that the line between the earth and moon always lies on one fixed plane,
2. the center-of-mass moves at constant speed on a straight line,
3. relative to the center-of-mass both the earth and moon travel on paths that are conic sections (circle, ellipse, parabola, hyperbola or a straight line).
4. the total energy ($E_K + E_P$) of the system is constant,

5. and that the angular momentum of the system about the center-of-mass is a constant.

These facts are discussed further below in the subsection on ‘Two-particle central force motion’.

Momentum and energy of systems

There are a plethora of theorems about the momentum and energy of systems of particles. These are discussed in section B. The simplest of these are just the ones that you get from adding up the results for a single particle from section 12.2:

Linear momentum balance. $\sum_{\text{all forces}} \vec{\boldsymbol{F}}_j = \sum m_i \vec{\boldsymbol{a}}_i.$

Either because the forces between particles in a system are usually assumed to come in equal and opposite pairs or because it is an independent postulate of mechanics for general systems, the force sum can be replaced with a sum over all the external forces.

Angular momentum balance. $\sum_{\text{all forces}} \vec{\boldsymbol{r}}_{j/C} \times \vec{\boldsymbol{F}}_j = \sum \vec{\boldsymbol{r}}_{i/C} \times (m_i \vec{\boldsymbol{a}}_i).$

As for linear momentum, the force sum can be replaced with only the forces that act externally on the system.

Power balance. $\sum_{\text{all forces}} \vec{\boldsymbol{F}}_j \cdot \vec{\boldsymbol{v}}_j = \frac{d}{dt} \sum m_i v_i^2/2.$

In this case the sum is over *all* the forces, internal and external. The simplification to just external forces doesn’t apply to system kinetic energy like it does for momentum and angular momentum.

The one-body problem

Let’s review one special problem from the previous section. The ‘one-body’ problem should properly be about the mechanics of a single particle interacting with nothing else. Such a particle moves at constant velocity and is too boring to get a name. Instead, when people refer to the ‘one-body’ problem they are talking about a particle flying around a stationary point mass to which it is attracted. That stationary point mass is held in place by, well, who knows what. It’s just an idealized thing anchored by a massless structure. The ‘one-body’ problem is to find the motion of the particle flying around.

As we discussed in the previous sections, if the gravitational attraction follows an inverse square law then the particle moves on a plane on a curve which is either an ellipse, a circle, a parabola or a hyperbola. These are, quite accurately, the trajectories of the planets and comets around the sun.

The two-body problem: two mutually attracting particles

If two particles are attracted equally to each other by mutually central forces, and no other forces act, this is called 'the two-body problem'①. Assume the two particles are m_1 and m_2 with positions $\vec{\boldsymbol{r}}_1$ and $\vec{\boldsymbol{r}}_2$ (relative to the origin of a coordinate system fixed in a Newtonian frame). The force on particle 1 from particle 2 is

①This is such a famous problem in the history of science that people use it for word play to describe certain social situations. For example if two people in a couple are having trouble finding jobs in the same city they are said to have a 'two-body problem'.

$$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}}$$

where $\vec{\boldsymbol{r}}_{12} = \vec{\boldsymbol{r}}_{2/1}$ is the position of particle 2 relative to particle 1, r_{12} is the distance $|\vec{\boldsymbol{r}}_{12}|$ between the particles and F is the magnitude of the attractive force. We assume the force on particle 2 is the opposite of this

$$\vec{\boldsymbol{F}}_{21} = -\vec{\boldsymbol{F}}_{12}.$$

The instantaneous velocities are $\vec{\boldsymbol{v}}_1$ and $\vec{\boldsymbol{v}}_2$. We can find the center of mass G of the pair of particles as

$$m_{tot}\vec{\boldsymbol{r}}_G = m_1\vec{\boldsymbol{r}}_1 + m_2\vec{\boldsymbol{r}}_2$$

with $m_{tot} = m_1 + m_2$.

Either by system linear momentum balance or by adding up $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ for each of the particles it is easy to see that

$$\vec{\boldsymbol{a}}_G = \vec{\boldsymbol{0}} \qquad \text{and} \qquad \vec{\boldsymbol{v}}_G = \text{constant}.$$

Thus we could put the origin of a good Newtonian reference frame at the center of mass $\vec{\boldsymbol{r}}_G$. The positions, velocities and accelerations relative to G, indicated with a prime ('), are

$$\begin{aligned}
\vec{\boldsymbol{r}}_1' &= \vec{\boldsymbol{r}}_1 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{r}}_2' &= \vec{\boldsymbol{r}}_2 - \vec{\boldsymbol{r}}_G \\
\vec{\boldsymbol{v}}_1' &= \vec{\boldsymbol{v}}_1 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{v}}_2' &= \vec{\boldsymbol{v}}_2 - \vec{\boldsymbol{v}}_G \\
\vec{\boldsymbol{a}}_1' &= \vec{\boldsymbol{a}}_1 \\
\vec{\boldsymbol{a}}_2' &= \vec{\boldsymbol{a}}_2
\end{aligned}$$

where we can skip use of the prime for the acceleration. Now some facts.

- $m_1\vec{\boldsymbol{r}}_1' + m_2\vec{\boldsymbol{r}}_2' = \vec{\boldsymbol{0}}$ so $\vec{\boldsymbol{r}}_1' = -(m_2/m_1)\vec{\boldsymbol{r}}_2'$. For all time the two positions (relative to the center of mass) are in the opposite direction and proportional. Similarly $\vec{\boldsymbol{v}}_1' = -(m_2/m_1)\vec{\boldsymbol{v}}_2'$ and $\vec{\boldsymbol{a}}_1' = -(m_2/m_1)\vec{\boldsymbol{a}}_2'$.
- At a given instant there is a single plane defined by $\vec{\boldsymbol{r}}_1', \vec{\boldsymbol{r}}_2', \vec{\boldsymbol{v}}_1', \vec{\boldsymbol{v}}_2', \vec{\boldsymbol{a}}_1'$ and $\vec{\boldsymbol{a}}_2'$ because the positions and accelerations are all parallel (or antiparallel) and the two velocities are (anti) parallel.

- The plane above is constant in time. This is because neither the velocity nor the acceleration has a component orthogonal to the plane, thus there is no tendency to leave the plane.
- Each particle moves as if it was a single particle attracted to a central force at G. Why? Let's look at the force on mass 1

  $$\vec{\boldsymbol{F}}_{12} = F(r_{12})\frac{\vec{\boldsymbol{r}}_{12}}{r_{12}} = -F(r_{12})\frac{\vec{\boldsymbol{r}}'_1}{r'_1}$$

  because the relative position of the masses passes through the origin G.
- In the special case of inverse-square gravitational attraction

  $$F = \frac{Gm_1m_2}{r_{12}^2} = \frac{Gm_1m_2}{(r'_1 + r'_2)^2} = \frac{Gm_1M}{{r'_1}^2}$$

  where $M = m_2/(1+2(m_1/m_2)+(m_1/m_2)^2)$ is a fictitious mass at G we find using the substitution $r'_2 = (m_1/m_2)r'_1$.

What we have found here is somewhat remarkable. Two particles are flying around in space attracted to each other by inverse-square gravitational attraction. Instead of doing something wild, they each move, relative to their joint center of mass, as if they were in central force motion with a fixed mass. That is, the 3D two-body problem reduces, exactly, to the 2D one body problem. You just have to use a coordinate system that is on the plane of motion and whose origin is at the center of mass.

Thus, the moon doesn't really go around the earth. Rather the moon and earth go around their common center of mass (a point about 3/4 of the way out towards the earth's surface from its center). And Jupiter doesn't go around the sun, the sun and Jupiter go around their combined center of mass just outside the sun. But both of these examples are, in detail, wrong. Because the earth-moon system is affected by the sun and jupiter. And the Jupiter-sun system is affected by the earth and moon.

The three-body problem

With inverse-square attraction, one body goes around a fixed point on one or another conic section. Two bodies go around each other in exactly the same way as one body about a fixed point. The two-body problem reduced to the one-body problem. What about lots of bodies? Let's start with three. How, in general, do three bodies move that are all mutually attracted with inverse-square gravitation? Great question. Lots of people have asked it. And no-one knows the answer. Given any three masses and their initial conditions we could use a computer program to find out their subsequent positions and velocities. But no-one knows how to categorize all the possible motions of such systems.

Some things are known about 'the three body problem.' One is that it is hard, the best minds haven't been able to solve it in general. Another is that the solutions can be pretty wild. For example, three particles might tumble around each other for a long time and, with no change in the equations, all of a sudden one of the particles will be ejected at high speed and never return (as if on a hyperbolic trajectory relative to the other two particles). A few special solutions of the three-body problem are known. For example, with the right initial conditions, three identical particles can move in either a circle or in Montgomery's figure 8.

Despite the difficulty of analytic description, there is no special impediment to finding solutions to any 3-body problem with computer simulation.

The n-body problem

With many particles all manner of complicated motion is possible. And there are few solutions which are known analytically. One solution has the n particles chasing each-other around in a circle, with the particles forming a regular polygon. Another amazing approximate solution, the Buck solution, is that a string of thousands of particles will all chase each-other around an arbitrary curve in 3-dimensional space. At least approximately, for a while.

By applying $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ to 3 or 1000 interacting particles you can see all manner of n-body solutions on your computer.