SAMPLE 13.1 **Location of the center-of-mass.** A structure is made up of three point masses, $m_1 = 1\,\mathrm{kg}$, $m_2 = 2\,\mathrm{kg}$ and $m_3 = 3\,\mathrm{kg}$. At the instant of interest, the coordinates of the three masses are (1.25 m, 3 m), (2 m, 2 m), and (0.75 m, 0.5 m), respectively. At the same instant, the velocities of the three masses are $2\,\mathrm{m/s}\hat{\imath}$, $2\,\mathrm{m/s}(\hat{\imath} - 1.5\hat{\jmath})$ and $1\,\mathrm{m/s}\hat{\jmath}$, respectively.

1. Find the coordinates of the center-of-mass of the structure.
2. Find the velocity of the center-of-mass.

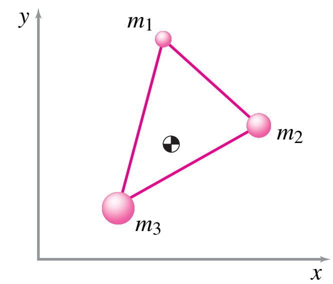

Figure 13.2

Solution

1. Let $(\bar{x}, \bar{y})$ be the coordinates of the mass-center. Then from the definition of mass-center

$$\begin{aligned}\bar{x} &= \frac{\sum m_i x_i}{\sum m_i} = \frac{m_1x_1 + m_2x_2 + m_3x_3}{m_1 + m_2 + m_3}\\ &= \frac{1\,\mathrm{kg}\cdot 1.25\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.75\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}}\\ &= \frac{7.5\,\cancel{\mathrm{kg}}\cdot\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.25\,\mathrm{m}.\end{aligned}$$

Similarly,

$$\begin{aligned}\bar{y} &= \frac{\sum m_i y_i}{\sum m_i}\\ &= \frac{1\,\mathrm{kg}\cdot 3\,\mathrm{m} + 2\,\mathrm{kg}\cdot 2\,\mathrm{m} + 3\,\mathrm{kg}\cdot 0.5\,\mathrm{m}}{1\,\mathrm{kg} + 2\,\mathrm{kg} + 3\,\mathrm{kg}}\\ &= \frac{8.5\,\cancel{\mathrm{kg}}\cdot\mathrm{m}}{6\,\cancel{\mathrm{kg}}} = 1.42\,\mathrm{m}.\end{aligned}$$

Thus the center-of-mass is located at the coordinates (1.25 m, 1.42 m).

$$\boxed{(1.25\,\mathrm{m},\ 1.42\,\mathrm{m})}$$

2. For a system of particles, the linear momentum

$$\begin{aligned}\vec{\boldsymbol{L}} &= \sum m_i\vec{\boldsymbol{v}}_i = m_{\mathrm{tot}}\vec{\boldsymbol{v}}_{cm}\\ \Rightarrow\quad \vec{\boldsymbol{v}}_{cm} &= \frac{\sum m_i\vec{\boldsymbol{v}}_i}{m_{\mathrm{tot}}}\\ &= \frac{1\,\mathrm{kg}\cdot(2\,\mathrm{m/s}\hat{\imath}) + 2\,\mathrm{kg}\cdot(2\hat{\imath} - 3\hat{\jmath})\,\mathrm{m/s} + 3\,\mathrm{kg}\cdot(1\,\mathrm{m/s}\hat{\jmath})}{6\,\mathrm{kg}}\\ &= \frac{(6\hat{\imath} - 3\hat{\jmath})\,\cancel{\mathrm{kg}}\cdot\mathrm{m/s}}{6\,\cancel{\mathrm{kg}}}\\ &= 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}}_{cm} = 1\,\mathrm{m/s}\hat{\imath} - 0.5\,\mathrm{m/s}\hat{\jmath}}$$

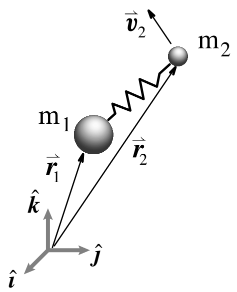

Figure 13.3

SAMPLE 13.2 A spring-mass system in space. A spring-mass system consists of two masses, $m_1 = 10\,\text{kg}$ and $m_2 = 1\,\text{kg}$, and a weak spring with stiffness $k = 1\,\text{N/m}$. The spring has zero relaxed length. The system is in 3-D space where there is no gravity. At the instant of observation, *i.e.,* at $t = 0, \vec{r}_1 = \vec{\mathbf{0}}, \vec{r}_2 = 1\,\text{m}(\hat{\imath} + \hat{\jmath} + \hat{k}), \dot{\vec{r}}_1 = \vec{\mathbf{0}}$, and $\dot{\vec{r}}_2 = \sqrt{6}\,\text{m/s}(-\hat{\imath} + \hat{\jmath})$. Track the motion of the system for the next 20 seconds. In particular,

1. Plot the trajectory of the two masses in space.
2. Plot the trajectory of the center-of-mass of the system.
3. Plot the trajectory of the two masses as seen by an observer sitting at the center-of-mass.
4. Compute and plot the total energy of the system and show that it remains constant during the entire motion.

Solution The free-body diagrams of the two masses are shown in fig. 13.4. The only force acting on each mass is the force due to the spring which is directed along the line joining the two masses. Thus, the system represents a central force problem. From the linear momentum balance of the two masses, we can write the equations of motion as follows.

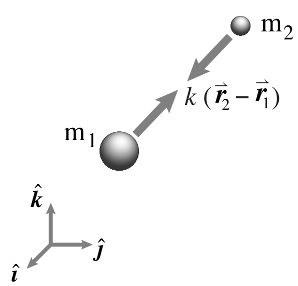

Figure 13.4: Free-body diagram of the two masses.

$$
\begin{aligned}
m_1\ddot{\vec{r}}_1 &= k(\vec{r}_2 - \vec{r}_1) \\
m_2\ddot{\vec{r}}_2 &= -k(\vec{r}_2 - \vec{r}_1)
\end{aligned}
$$

Let $\vec{r}_1 = x_1\hat{\imath} + y_1\hat{\jmath} + z_1\hat{k}$ and $\vec{r}_2 = x_2\hat{\imath} + y_2\hat{\jmath} + z_2\hat{k}$. Substituting above and dotting the two equations with $\hat{\imath}$, $\hat{\jmath}$, and $\hat{k}$, we get

$$
\begin{aligned}
\ddot{x}_1 &= \frac{k}{m_1}(x_2 - x_1); & \ddot{x}_2 &= -\frac{k}{m_2}(x_2 - x_1) \\
\ddot{y}_1 &= \frac{k}{m_1}(y_2 - y_1); & \ddot{y}_2 &= -\frac{k}{m_2}(y_2 - y_1) \\
\ddot{z}_1 &= \frac{k}{m_1}(z_2 - z_1); & \ddot{z}_2 &= -\frac{k}{m_2}(z_2 - z_1)
\end{aligned}
$$

Thus we get six second order coupled linear ODEs as equations of motion.

1. To plot the trajectory of the two masses, we need to solve for $\vec{r}_1(t)$ and $\vec{r}_2(t)$, *i.e.,* for $x_1(t), y_1(t), z_1(t)$, and $x_2(t), y_2(t), z_2(t)$. We can do this by first writing the six second order equations as a set of 12 first order equations and then solving them using a numerical ODE solver. Here is a pseudocode to accomplish this task.

```
ODEs = {x1dot = u1,
        u1dot = k/m1*(x2-x1),
        y1dot = v1,
        v1dot = k/m1*(y2-y1),
        z1dot = w1,
        w1dot = k/m1*(z2-z1),
        x2dot = u2,
        u2dot = -k/m2*(x2-x1),
        y2dot = v2,
        v2dot = -k/m2*(y2-y1),
        z2dot = w2,
        w2dot = -k/m2*(z2-z1) }
IC   = {x1(0)=0, y1(0)=0, z1(0)=0,
        u1(0)=0, v1(0)=0, w1(0)=0,
        x2(0)=1, y2(0)=1, z2(0)=1,
        u2(0)=-sqrt(6), v2(0)=sqrt(6), w2(0)=0}
Set  k=1, m1=10, m2=1
Solve ODEs with IC for t=0 to t=20
Plot {x1,y1,z1} and {x2,y2,z2}
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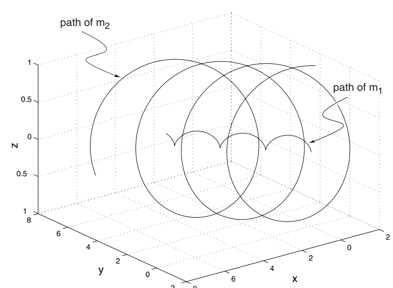

Figure 13.5: 3-D trajectory of m_1 and m_2 plotted from numerical solution of the equations of motion.

The 3-D plot showing the trajectory of the two masses obtained from the numerical solution is shown in fig. 13.5. From the plot, it seems like the smaller mass goes around the bigger mass as the bigger mass moves on its trajectory.

2. We can find the trajectory of the center-of-mass using the following relationships.

$$x_{\text{cm}} = \frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad y_{\text{cm}} = \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}, \quad z_{\text{cm}} = \frac{m_1 z_1 + m_2 z_2}{m_1 + m_2}.$$

Since there is no external force on the system if we consider the two masses and the spring together, the center-of-mass of the system has zero acceleration. Therefore, we expect the center-of-mass to move on a straight path with constant velocity. The center-of-mass coordinates x_{cm}, y_{cm}, and z_{cm} are plotted against time in fig. 13.6 which show that the center-of-mass moves on a straight line in a plane parallel to the xy-plane (z is constant). This is expected since the initial velocity of the center of mass has no z-component:

$$\begin{aligned}
\vec{\boldsymbol{v}}_{\text{cm}} &= \frac{m_1 \vec{\boldsymbol{v}}_1 + m_2 \vec{\boldsymbol{v}}_2}{m_1 + m_2} \\
&= \frac{m_1 \cdot \vec{\boldsymbol{0}} + 1\,\text{kg} \cdot \sqrt{(6)}\,\text{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}})}{10\,\text{kg} + 1\,\text{kg}} \\
&= 0.22\,\text{m/s}(-\hat{\boldsymbol{\imath}} + \hat{\boldsymbol{\jmath}}).
\end{aligned}$$

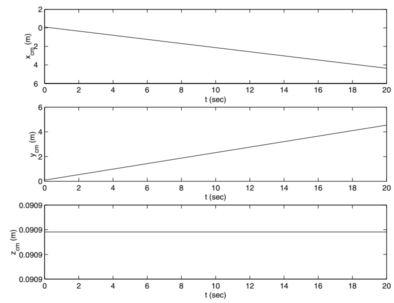

Figure 13.6: The center-of-mass coordinates $x_{\text{cm}}(t)$, $y_{\text{cm}}(t)$, and $z_{\text{cm}}(t)$. The center-of-mass moves on a straight line in a plane parallel to the xy-plane.

3. The trajectory of the two masses with respect to the center-of-mass can be easily obtained by the following relationships.

$$\begin{aligned}
x_{1/cm} &= x_1 - x_{\text{cm}}, \quad & y_{1/cm} &= y_1 - y_{\text{cm}}, \quad & z_{1/cm} &= z_1 - z_{\text{cm}} \\
x_{2/cm} &= x_2 - x_{\text{cm}}, \quad & y_{2/cm} &= y_2 - y_{\text{cm}}, \quad & z_{2/cm} &= z_2 - z_{\text{cm}}
\end{aligned}$$

The trajectories thus obtained are shown in fig. 13.7. It is clear that the two masses have closed orbits with respect to the center-of-mass. These closed orbits are actually conic sections as we would expect in a central force problem.

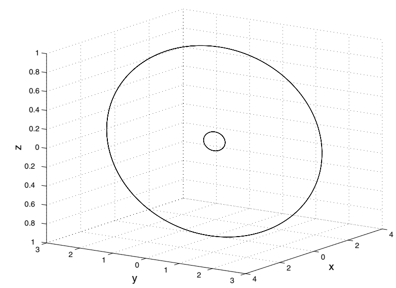

Figure 13.7: The paths of m_1 and m_2 as seen from the center-of-mass. The two masses are on closed orbits with respect to the center-of-mass.

4. We can calculate the kinetic energy of the two masses and the potential energy of the spring at each instant during the motion and add them up to find the total energy.

$$\begin{aligned}
(E_k)_{m_1} &= \frac{1}{2} m_1 (u_1^2 + v_1^2 + w_1^2) \\
(E_k)_{m_2} &= \frac{1}{2} m_2 (u_2^2 + v_2^2 + w_2^2) \\
E_p &= \frac{1}{2} k[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2] \\
E_{\text{total}} &= (E_k)_{m_1} + (E_k)_{m_2} + E_p
\end{aligned}$$

The energies so calculated are plotted in fig. 13.8. It is clear from the plot that the total energy remains constant during the entire motion.

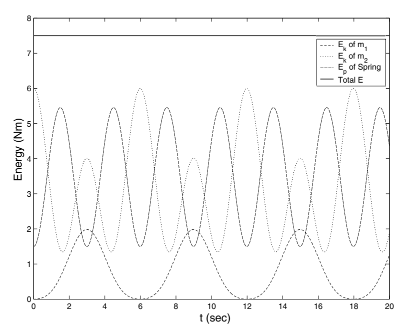

Figure 13.8: The kinetic energy of the two masses and the potential energy of the spring sum up to the constant total energy of the system.