13.1 Coupled motions of particles in space

Preparatory Problems

13.1.1 Linear momentum balance applied to the whole of a system consisting of multiple interacting particles reduces to $\vec{F} = m\vec{a}$ *if you interpret the terms correctly.* What are the correct interpretations of $\vec{F}$, m and $\vec{a}$? *

13.1.2 A particle of mass $m_1 = 6\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in the xy-plane. At a particular instant of interest, particle 1 has position $\vec{r}_1 = 3\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}$, velocity $\vec{v}_1 = -16\,\text{m/s}\hat{\imath} + 6\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_1 = 10\,\text{m/s}^2\hat{\imath} - 24\,\text{m/s}^2\hat{\jmath}$; and particle 2 has position $\vec{r}_2 = -6\,\text{m}\hat{\imath} - 4\,\text{m}\hat{\jmath}$, velocity $\vec{v}_2 = 8\,\text{m/s}\hat{\imath} + 4\,\text{m/s}\hat{\jmath}$, and acceleration $\vec{a}_2 = 5\,\text{m/s}^2\hat{\imath} - 16\,\text{m/s}^2\hat{\jmath}$.

a) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of each particle at the instant of interest. *

b) Find the linear momentum $\vec{L}$ and its rate of change $\dot{\vec{L}}$ of the system of the two particles at the instant of interest. *

c) Find the center of mass of the system at the instant of interest. *

d) Find the velocity and acceleration of the center of mass. *

13.1.3 A particle of mass $m_1 = 5\,\text{kg}$ and a particle of mass $m_2 = 10\,\text{kg}$ are moving in space. At a particular instant of interest, particle 1 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_1 &= 1\,\text{m}\hat{\imath} + 1\,\text{m}\hat{\jmath}\\ \vec{v}_1 &= 2\,\text{m/s}\hat{\jmath}\\ \vec{a}_1 &= 3\,\text{m/s}^2\hat{k}\end{aligned}$$

respectively, and particle 2 has position, velocity, and acceleration

$$\begin{aligned}\vec{r}_2 &= 2\,\text{m}\hat{\imath}\\ \vec{v}_2 &= 1\,\text{m/s}\hat{k}\\ \vec{a}_2 &= 1\,\text{m/s}^2\hat{\jmath}\end{aligned}$$

respectively. For the system of particles at the instant of interest, find its

a) linear momentum $\vec{L}$, *

b) rate of change of linear momentum $\dot{\vec{L}}$, *

c) angular momentum about the origin $\vec{H}_{/O}$, *

d) rate of change of angular momentum about the origin $\dot{\vec{H}}_{/O}$, *

e) kinetic energy E_K, and *

f) rate of change of kinetic energy. *

13.1.4 If you are given the total mass, the position, the velocity, and the acceleration of the center of mass of a system of particles can you find the angular momentum $\vec{H}_{/O}$ of the system, where O is not at the center of mass? If so, how and why? If not, then give a reason and/or a counter example. *

13.1.5 Seventeen particles are interacting with the force on particle i from particle j being $\vec{F}_{ij}$ with all $\vec{F}_{ij}$ known.

a) What is the commonly assumed assumption about the relation between, say, $\vec{F}_{36}$ and $\vec{F}_{63}$? *

b) What is the total force on particle 5? *

More-Involved Problems

13.1.6 Two particles each of mass m are connected by a massless elastic spring of spring constant k and unextended length $2R$. The system slides without friction on a horizontal table, so that no net external forces act.

a) Is the total linear momentum conserved? Justify your answer. *

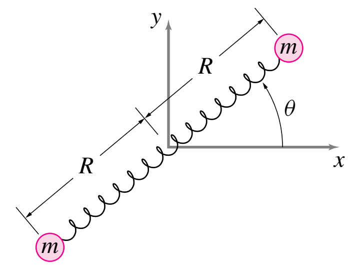
b) Can the center of mass accelerate? Justify your answer. *

c) Draw free-body diagrams for each mass.

d) Derive the equations of motion for each mass in terms of cartesian coordinates.

e) What are the total kinetic and potential energies of the system? *

f) For constant values and initial conditions of your choosing, plot the trajectories of the two particles and of the center of mass (on the same plot).



Problem 13.1.6

13.1.7 Two ice skaters whirl around one another. They are connected by a linear elastic cord whose center is stationary in space. We wish to consider the motion of one of the skaters by modeling her as a mass m held by a cord that exerts k Newtons for each meter it is extended from the central position.

a) Draw a free-body diagram showing the forces that act on the mass at an arbitrary position.

b) Write the differential equations that describe the motion. *

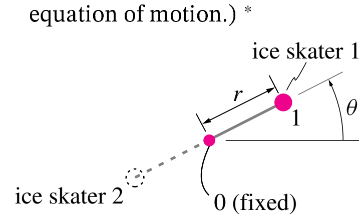
c) Describe in physical and mathematical terms the nature of the motion for the three cases

a) $\omega < \sqrt{k/m}$;

b) $\omega = \sqrt{k/m}$;

c) $\omega > \sqrt{k/m}$.

(You are not asked to solve the equation of motion.) *



Problem 13.1.7

13.1.8 n identical particles with mass m are on the vertices of an n sided regular polygon. Equivalently, n particles are equally spaced on a circle with radius R. At $t = 0$ they all have velocities tangent to the circle and equal in magnitude v_0. All the particles are attracted to each other with an inverse square gravitational attraction. For the numerical simulations below pick values of n, m, G and R any way that pleases you.

a) Find an initial value for v_0 so that all the masses spiral in and then bounce out again. Plot the trajectories of all the masses on one plot for a long-enough time so the plot is pleasing to the eye.

b) Find a value for v_0 so all the particles travel on circular trajectories.

c) Can you find a formula for v_0 above in terms of the other parameters in the problem? *

13.1.9 Two masses, both with $M = 1000m$ travel in circles on the xy plane according to

$$\vec{\boldsymbol{r}}_1 = -\vec{\boldsymbol{r}}_2 = R(\cos \omega t \hat{\boldsymbol{\imath}} + \sin \omega t \hat{\boldsymbol{\jmath}})$$

a) Assume inverse square attraction and find a set of values for m, G, R and ω so the assumed circular path is a solution of the equations of motion. *

b) A third mass m is introduced which is gravitationally attracted to the other two. Pick initial conditions for the two big masses that are consistent with their circular motion solution. For the third mass use initial conditions $[\vec{\boldsymbol{r}}_0]_{xyz} = [00z_0]$. Run a simulation for some time.

- Does the third mass stay *exactly* on the z axis for all time in the simulation? Would it if the simulation was exact? If so, why? If not, why not? *
- Is the motion of the third mass exactly periodic in the computer simulation? Would it be if the solution was exact? If so, why? If not, why not? *

13.1.10 The amazing eight. Three equal masses, say $m = 1$, are attracted by an inverse-square gravity law with $G = 1$. That is, each mass is attracted to the other by $F = Gm_1m_2/r^2$ where r is the distance between them. Use these unusual and special initial positions:

$$\begin{aligned}
(x1, y1) &= (-0.97000436, 0.24308753)\\
(x2, y2) &= (-x1, -y1)\\
(x3, y3) &= (0, 0)
\end{aligned}$$

and initial velocities

$$\begin{aligned}
(vx3, vy3) &= (0.93240737, 0.86473146)\\
(vx1, vy1) &= -(vx3, vy3)/2\\
(vx2, vy2) &= -(vx3, vy3)/2.
\end{aligned}$$

For each of the problems below show accurate computer plots and explain any curiosities. (Note, this solution was discovered independently, and nearly simultaneously, by Richard Montgomery and Chris Moore.)

a) Use computer integration to find and plot the motions of the particles. Plot each with a different color. Run the program for 2.1 time units. *

b) Same as above, but run for 10 time units. *

c) Same as above, but change the initial conditions slightly. *

d) Same as above, but change the initial conditions more and run for a much longer time. *