12.3 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe, characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on distance between attracting points.

If we define the position of the particle as $\vec{\boldsymbol{r}}$ with magnitude r, linear momentum balance for central-force motion is

$$
\begin{aligned}
\sum \vec{\boldsymbol{F}}_{\mathrm{i}} &= \dot{\vec{\boldsymbol{L}}} \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \qquad (12.12)
\end{aligned}
$$

where $-\vec{\boldsymbol{r}}/r$ is a unit vector pointed toward the origin and $F(r)$ is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of eqn. (12.12). We start with the most historically important example.

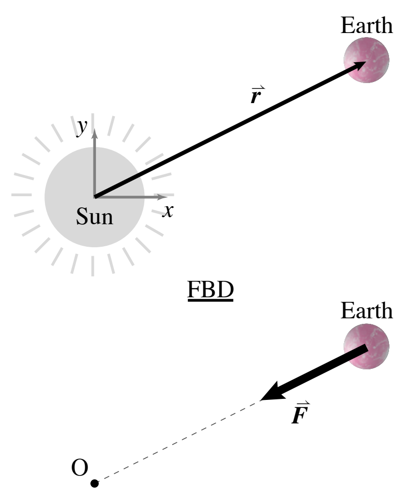

Figure 12.31: The earth moving around a fixed sun. The attraction force $\vec{\boldsymbol{F}}$ is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (fig. 12.31). Thus we have ①

$$F = \frac{Gm_e m_s}{r^2}$$

where m_e and m_s are the masses of the earth and sun, r is the distance between the earth and sun. 'Big G' is a universal constant $G \approx 6.67 \cdot 10^{-11}\,\mathrm{N\,m^2/\,kg^2}$. What is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= \left(\frac{Gm_e m_s}{r^2}\right)\left(\frac{-\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}\right) \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= -Gm_e m_s\left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= -Gm_e m_s\left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) \qquad (12.13)
\end{aligned}$$

where we have used that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, $r = |\vec{\boldsymbol{r}}| = \sqrt{x^2+y^2}$, and $\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}$. Now we can write the linear momentum balance equation for the earth in great detail.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad -Gm_e m_s\left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) &= m_e\,(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}) \qquad (12.14)
\end{aligned}$$

Taking the dot product of equation 12.14 with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (*i.e.,* taking x and y components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-Gm_s x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-Gm_s y}{(x^2+y^2)^{3/2}}. \qquad (12.15)$$

This pair of coupled second order differential equations describes the motion of the earth.② Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations eqn. (12.15) in the form of a set of coupled first order ordinary differential equations. If we define $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, and $z_4 = \dot{y}$. We can now write equations 12.15 as

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -Gm_s z_1/(z_1^2 + z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -Gm_s z_3/(z_1^2 + z_3^2)^{3/2}. \qquad (12.16)
\end{aligned}$$

To actually solve these numerically we need a value for Gm_s and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

① Soon after Newton, Cavendish found G in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

② Note that G appears in the product Gm_s. Newton didn't know the value of big G, but he could do a lot of figuring without it. All he needed was the product Gm_s which he could find from the period and radius of the earth's orbit. The entanglement of G with the mass of the sun is why some people call Cavendish's measurement of big G, "weighing the sun". From Newton's calculation of Gm_s and Cavendish's measurement of G you can find m_s. Naturally, the real history is a bit more complicated; Cavendish called his experiment 'weighing the earth'.

Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency ω and radius r_o obeys the parametric equation

$$\begin{aligned} \vec{\boldsymbol{r}} &= r_o\left(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}\right) \\ \text{differentiating twice} \quad \Rightarrow \quad \ddot{\vec{\boldsymbol{r}}} &= -\omega^2 r_o\left(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}\right) \\ &= -\omega^2 \vec{\boldsymbol{r}}. \end{aligned} \tag{12.17}$$

Comparing eqn. (12.17) with eqn. (12.12) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where $m = m_e$ we have $F(r) = Gm_s m_e/r^2$ so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{1}{Gm_s}}\;\; r^{\frac{3}{2}} \tag{12.18}$$

because angular frequency is inversely proportional to the period ($\omega = 2\pi/T$). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

Conservation of energy

Any force of the form

$$\vec{\boldsymbol{F}} = -F(r)\frac{\vec{\boldsymbol{r}}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_{\mathrm{P}} = \int F(r')\,dr'.$$

For the case of gravitational attraction, the potential energy is

$$E_{\mathrm{P}} = \frac{-Gm_s m_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned} \text{Constant} &= E_{\mathrm{K}} + E_{\mathrm{P}} \\ &= \frac{1}{2}mv^2 + \frac{-Gm_s m_e}{r} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_s m_e}{\sqrt{x^2 + y^2}}. \end{aligned} \tag{12.19}$$

If that constant is bigger than zero then the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than "escape velocity" and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for m_s and the mass of the satellite for m_e. Thus we can write the governing equation eqn. (12.14) as

$$-GMm\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) = m\left(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}}\right) \tag{12.20}$$

where now M is the mass of the earth and m is the mass of the satellite. At the surface of the earth $r = R$, the earth's radius, and $GM/R^2 = g$ so we can rewrite the governing equation for rockets and the like as

$$-gR^2\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) = \left(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}}\right). \tag{12.21}$$

Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}}\\
-k\vec{\boldsymbol{r}} &= m\ddot{\vec{\boldsymbol{r}}}\\
-k(x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}) &= m\left(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}}\right).
\end{aligned} \tag{12.22}$$

Dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x}+\frac{k}{m}x=0 \quad \text{and} \quad \ddot{y}+\frac{k}{m}y=0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x = A\cos(\lambda t)+B\sin(\lambda t) \quad \text{and} \quad y = C\cos(\lambda t)+D\sin(\lambda t) \tag{12.23}$$

where $A, B, C,$ and D are arbitrary constants which are determined by initial conditions. For all $A, B, C,$ and D eqn. (12.23) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation eqn. (12.12) and take the cross product of both sides with $\vec{\boldsymbol{r}}$ we get the following.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \\
\Rightarrow \quad \vec{\boldsymbol{r}} \times \left(F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right)\right) &= \vec{\boldsymbol{r}} \times \left(m\ddot{\vec{\boldsymbol{r}}}\right) \\
\Rightarrow \quad \vec{\boldsymbol{0}} &= \frac{d}{dt}\left(m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}\right) \qquad \text{(because } \dot{\vec{\boldsymbol{r}}} \times \dot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{0}}) \\
\Rightarrow \quad \text{constant} &= m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}. \qquad (12.24)
\end{aligned}$$

But this last quantity is proportional to the rate at which area is swept out by a moving particle (it equals twice the areal rate times the mass), which is what makes it constant. Thus Kepler's second law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.