12.3 Central-force motion and celestial mechanics

One of Isaac Newton's greatest achievements was the explanation of Kepler's laws of planetary motion. Kepler, using the meticulous observations of Tycho Brahe, characterized the orbits of the planets about the sun with his 3 famous laws:

- Each planet travels on an ellipse with the sun at one focus.
- Each planet goes faster when it is close to the sun and slower when it is further. It speeds and slows so that the line segment connecting the planet to the sun sweeps out area at a constant rate.
- Planets that are further from the sun take longer to go around. More exactly, the periods are proportional to the lengths of the ellipses to the 3/2 power.

Newton, using his equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and his law of universal gravitational attraction, was able to formulate a differential equation governing planetary motion. He was also able to solve this equation and found that it exactly predicts all three of Kepler's laws.

The Newtonian description of planetary motion is the most historically significant example of *central-force motion* where,

- the only force acting on a particle is directed towards the origin of a given coordinate system, and
- the magnitude of the force depends only on distance between attracting points.

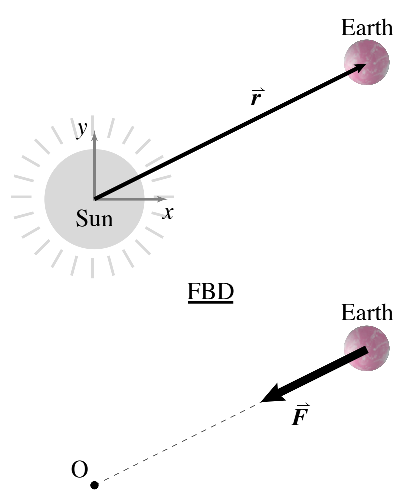

Figure 12.31: The earth moving around a fixed sun. The attraction force $\vec{\boldsymbol{F}}$ is directed "centrally" towards the sun and has magnitude proportional to both masses and inversely proportional to the distance squared.

If we define the position of the particle as $\vec{\boldsymbol{r}}$ with magnitude r, linear momentum balance for central-force motion is

$$
\begin{aligned}
\sum \vec{\boldsymbol{F}}_{\mathrm{i}} &= \dot{\vec{\boldsymbol{L}}} \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \qquad (12.12)
\end{aligned}
$$

where $-\vec{\boldsymbol{r}}/r$ is a unit vector pointed toward the origin and $F(r)$ is the magnitude of the origin-attracting force.

For the rest of this section we consider some of the consequences of eqn. (12.12). We start with the most historically important example.

Motion of the earth around a fixed sun

For simplicity let's assume that the sun does not move and that the motion of the earth lies in a plane. Newton's law of gravitation says that the attractive force of the sun on the earth is proportional to the masses of the sun and earth and inversely proportional to the distance between them squared (fig. 12.31). Thus we have ①

① Soon after Newton, Cavendish found G in his lab by delicately measuring the *small* attractive force between two balls. The gravitational attraction between two 1 kg balls a meter apart is about a ten-millionth of a billionth of a Newton (a Newton is about a fifth of a pound).

$$F = \frac{Gm_e m_s}{r^2}$$

where m_e and m_s are the masses of the earth and sun, r is the distance between the earth and sun. 'Big G' is a universal constant $G \approx 6.67 \cdot 10^{-11}\,\mathrm{N\,m^2/\,kg^2}$. What is the vector-valued force on the earth? It is its magnitude times a unit vector in the appropriate direction.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= \left(\frac{Gm_e m_s}{r^2}\right)\left(\frac{-\vec{\boldsymbol{r}}}{|\vec{\boldsymbol{r}}|}\right) \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= -Gm_e m_s \left(\frac{\vec{\boldsymbol{r}}}{r^3}\right) \\
\Rightarrow \quad \vec{\boldsymbol{F}} &= -Gm_e m_s \left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right)
\end{aligned} \tag{12.13}$$

where we have used that $\vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}$, $r = |\vec{\boldsymbol{r}}| = \sqrt{x^2+y^2}$, and $\vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}$. Now we can write the linear momentum balance equation for the earth in great detail.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad -Gm_e m_s \left(\frac{x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) &= m_e\,(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}})
\end{aligned} \tag{12.14}$$

Taking the dot product of equation 12.14 with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (*i.e.,* taking x and y components) gives two scalar second order ordinary differential equations:

$$\ddot{x} = \frac{-Gm_s x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-Gm_s y}{(x^2+y^2)^{3/2}}. \tag{12.15}$$

This pair of coupled second order differential equations describes the motion of the earth.② Pencil and paper solution is possible, Newton did it, but is a little too hard for this book. So we resort to computer solution. To set this up we put equations eqn. (12.15) in the form of a set of coupled first order ordinary differential equations. If we define $z_1 = x$, $z_2 = \dot{x}$, $z_3 = y$, and $z_4 = \dot{y}$. We can now write equations 12.15 as

② Note that G appears in the product Gm_s. Newton didn't know the value of big G, but he could do a lot of figuring without it. All he needed was the product Gm_s which he could find from the period and radius of the earth's orbit. The entanglement of G with the mass of the sun is why some people call Cavendish's measurement of big G, "weighing the sun". From Newton's calculation of Gm_s and Cavendish's measurement of G you can find m_s. Naturally, the real history is a bit more complicated; Cavendish called his experiment 'weighing the earth'.

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -Gm_s z_1/(z_1^2+z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -Gm_s z_3/(z_1^2+z_3^2)^{3/2}.
\end{aligned} \tag{12.16}$$

To actually solve these numerically we need a value for Gm_s and initial conditions. The solutions of these equations on the computer are all, within numerical error, consistent with Kepler's laws.

Without a full solution, there are some things we can figure out relatively easily.

Circular orbits

We generally think of the motions of the planets as being roughly circular orbits. In fact, for any attractive central force one of the possible motions is a circular orbit. Rather than trying to derive this, let's assume a circular solution and see if it solves the equations of motion. A constant speed circular orbit with angular frequency ω and radius r_o obeys the parametric equation

$$\begin{aligned} \vec{\boldsymbol{r}} &= r_o\left(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}\right) \\ \text{differentiating twice} \quad \Rightarrow \quad \ddot{\vec{\boldsymbol{r}}} &= -\omega^2 r_o\left(\cos(\omega t)\hat{\boldsymbol{\imath}} + \sin(\omega t)\hat{\boldsymbol{\jmath}}\right) \\ &= -\omega^2 \vec{\boldsymbol{r}}. \end{aligned} \tag{12.17}$$

Comparing eqn. (12.17) with eqn. (12.12) we see we have an identity (a solution to the equation) if

$$\omega^2 = \frac{F(r)}{mr}.$$

In the case of gravitational attraction where $m = m_e$ we have $F(r) = Gm_s m_e/r^2$ so we get circular motion with

$$\omega^2 = \frac{Gm_s}{r^3} \qquad \Rightarrow \qquad T = 2\pi\sqrt{\frac{1}{Gm_s}}\;\; r^{\frac{3}{2}} \tag{12.18}$$

because angular frequency is inversely proportional to the period ($\omega = 2\pi/T$). We have, for the special case of circular orbits, derived Kepler's third law. The orbital period is proportional to the orbital size to the 3/2 power.

Conservation of energy

Any force of the form

$$\vec{\boldsymbol{F}} = -F(r)\frac{\vec{\boldsymbol{r}}}{r}$$

is conservative and is associated with a potential energy given by the indefinite integral

$$E_{\mathrm{P}} = \int F(r')\,dr'.$$

For the case of gravitational attraction, the potential energy is

$$E_{\mathrm{P}} = \frac{-Gm_s m_e}{r}$$

where we could add an arbitrary constant. Thus, one of the features of planetary motion is that for a given orbit the energy is constant in time:

$$\begin{aligned} \text{Constant} &= E_{\mathrm{K}} + E_{\mathrm{P}} \\ &= \frac{1}{2}mv^2 + \frac{-Gm_s m_e}{r} \\ &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{-Gm_s m_e}{\sqrt{x^2+y^2}}. \end{aligned} \tag{12.19}$$

If that constant is bigger than zero then the orbit has enough energy to have positive kinetic energy even when infinitely far from the sun. Such orbits are said to have more than "escape velocity" and they do indeed have open hyperbola-shaped orbits, and only pass close to the sun at most once.

Motion of rockets and artificial satellites

Rockets and the like move around the earth much like planets, comets and asteroids move around the sun. All of the equations for planetary motion apply. But you need to substitute the mass of the earth for m_s and the mass of the satellite for m_e. Thus we can write the governing equation eqn. (12.14) as

$$-GMm\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) \quad = \quad m\,(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}}) \tag{12.20}$$

where now M is the mass of the earth and m is the mass of the satellite. At the surface of the earth $r=R$, the earth's radius, and $GM/R^2=g$ so we can rewrite the governing equation for rockets and the like as

$$-gR^2\left(\frac{x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}}{(x^2+y^2)^{3/2}}\right) \quad = \quad (\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}})\,. \tag{12.21}$$

Another central-force example: force proportional to radius

A less famous, but also useful, example of central force is where the attraction force is proportional to the radius. In this case the governing equations are:

$$\begin{aligned}\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}}\\ -k\vec{\boldsymbol{r}} &= m\ddot{\vec{\boldsymbol{r}}}\\ -k(x\hat{\boldsymbol{\imath}}+y\hat{\boldsymbol{\jmath}}) &= m\,(\ddot{x}\hat{\boldsymbol{\imath}}+\ddot{y}\hat{\boldsymbol{\jmath}})\,.\end{aligned} \tag{12.22}$$

Dotting both sides with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get two uncoupled linear homogeneous constant coefficient differential equations:

$$\ddot{x}+\frac{k}{m}x=0 \quad \text{and} \quad \ddot{y}+\frac{k}{m}y=0.$$

These you recognize as the harmonic oscillator equations so we can pick off the general solutions immediately as:

$$x=A\cos(\lambda t)+B\sin(\lambda t) \quad \text{and} \quad y=C\cos(\lambda t)+D\sin(\lambda t) \tag{12.23}$$

where $A,B,C,$ and D are arbitrary constants which are determined by initial conditions. For all $A,B,C,$ and D eqn. (12.23) describes an ellipse (or a special case of an ellipse, like a circle or a straight line). In the case of planetary motion we also had ellipses. In this case, however, the center of attraction is at the center of the ellipse and not at one of the foci.

Conservation of angular momentum and Kepler's second law

If we take the linear momentum balance equation eqn. (12.12) and take the cross product of both sides with $\vec{\boldsymbol{r}}$ we get the following.

$$\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
\Rightarrow \quad F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right) &= m\ddot{\vec{\boldsymbol{r}}} \\
\Rightarrow \quad \vec{\boldsymbol{r}} \times \left(F(r)\left(\frac{-\vec{\boldsymbol{r}}}{r}\right)\right) &= \vec{\boldsymbol{r}} \times \left(m\ddot{\vec{\boldsymbol{r}}}\right) \\
\Rightarrow \quad \vec{\boldsymbol{0}} &= \frac{d}{dt}\left(m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}\right) \qquad \text{(because } \dot{\vec{\boldsymbol{r}}} \times \dot{\vec{\boldsymbol{r}}} = \vec{\boldsymbol{0}}\text{)} \\
\Rightarrow \quad \text{constant} &= m\vec{\boldsymbol{r}} \times \dot{\vec{\boldsymbol{r}}}. \qquad (12.24)
\end{aligned}$$

But this last quantity is proportional to the rate at which area is swept out by a moving particle (it equals twice the areal rate times the mass), which is what makes it constant. Thus Kepler's second law has been derived for all central-force motions (not just inverse square attractions). The last quantity is also the angular momentum of the particle. Thus for a particle in central force motion we have derived conservation of angular momentum from $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$.

SAMPLE 12.11 Circular orbits of planets: Refer to eqn. (12.15) in the text that governs the motion of planets around a fixed sun.

1. Let $x = A\cos(\lambda t)$ and $y = A\sin(\lambda t)$. Show that x and y satisfy the equations of planetary motion and that they describe a circular orbit.
2. Show that the solution assumed above satisfies Kepler's third law by showing that the orbital period $T = 2\pi/\lambda$ is proportional to the 3/2 power of the size of the orbit (which can be characterized by its radius).

Solution

1. The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} = \frac{-Gm_s}{(x^2+y^2)^{3/2}} &= \frac{\ddot{y}}{y} \\ \Rightarrow \quad \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \tag{12.25}$$

Now,

$$\begin{aligned} x &= A\cos(\lambda t) \quad \Rightarrow \quad \ddot{x} = -\lambda^2 A\cos(\lambda t) \\ y &= A\sin(\lambda t) \quad \Rightarrow \quad \ddot{y} = -\lambda^2 A\sin(\lambda t) \end{aligned}$$

Substituting these values in eqn. (12.25), we get

$$-\lambda^2 A^2\cos(\lambda t)\cdot\sin(\lambda t) + \lambda^2 A\sin(\lambda t)\cdot\cos(\lambda t) \stackrel{\surd}{=} 0$$

Thus the assumed form of x and y satisfy the governing equations of planetary motion, *i.e.*, $x(t) = A\cos(\lambda t)$ and $y(t) = A\sin(\lambda t)$ form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = A^2\cos^2(\lambda t) + A^2\sin^2(\lambda t) = A^2,$$

i.e., x and y satisfy the equation of a circle with radius A. Thus, the assumed solution gives a circular orbit.

2. Substituting $x = A\cos(\lambda t)$ in eqn. (12.15), and noting that square of the radius of the orbit is $r^2 = x^2 + y^2 = A^2$, we get

$$\begin{aligned} -\lambda^2 A\cos(\lambda t) &= -Gm_s\frac{A\cos(\lambda t)}{r^3} \\ \Rightarrow \quad \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or} \qquad \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow \quad T^2 &= \frac{4\pi^2}{Gm_s}A^3 \\ \text{or} \qquad T &= KA^{3/2} \end{aligned}$$

where $K = 2\pi/\sqrt{Gm_s}$ is a constant. Thus the orbital period T is proportional to the 3/2 power of the radius, or the size, of the circular orbit.

Of course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates, x and y.

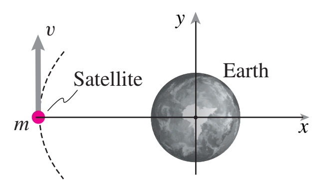

Figure 12.32

SAMPLE 12.12 Numerical computation of satellite orbits: The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point, the perigee, on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 m/s. The radius of the earth is 6400 km and the acceleration due to gravity $g = 9.81\,\mathrm{m/s^2}$.

1. Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.
2. From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.
3. Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

Solution

1. The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2y}{(x^2+y^2)^{3/2}}.$$

where g is the acceleration due to gravity and R is the radius of the earth (see eqn. (12.20) in the text). From the given data at perigee, the initial conditions are

$$x(0) = -7500\,\mathrm{km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500\,\mathrm{m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -gR^2 z_1/(z_1^2+z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -gR^2 z_3/(z_1^2+z_3^2)^{3/2}.
\end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z1(0) = -7.5\times 10^6\,\mathrm{m}, \quad z2(0) = 0, \quad z3(0) = 0, \quad z4(0) = 9500\,\mathrm{m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```
ODEs ={z1dot=z2,  z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
   z3dot=z4,  z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
IC   ={z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
Set  g = 9.81, R = 6.4E06
Solve ODEs with IC for t=0 to t=4E04
Plot z1 vs z3
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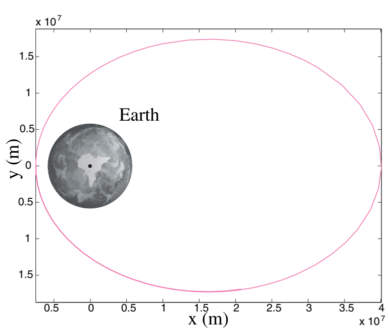


Figure 12.33: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in fig. 12.33 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptical. From the computer output, we find the following data for the apogee.

$$x = 4.0049\times 10^7\,\mathrm{m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791\times 10^3\,\mathrm{m/s}$$

2. The expressions for energy E and angular momentum H for a satellite are,

$$\begin{aligned} E &= E_K + E_P = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{GMm}{r} \\ \vec{\boldsymbol{H}}_o &= \vec{\boldsymbol{r}} \times m\vec{\boldsymbol{v}} = (x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) \times m(\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}) = m(x\dot{y} - y\dot{x})\hat{\boldsymbol{k}} \end{aligned}$$

At both apogee and perigee, $y = 0$ and the velocity (which is tangential) is in the y direction, *i.e.*, $\dot{x} = 0$. Therefore, the expressions for energy and angular momentum become simpler:

$$\begin{aligned} E &= \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|}, \\ \text{and} \quad H &= mx\dot{y}. \end{aligned}$$

Let E_1 and H_1 be the energy and the angular momentum of the satellite at the perigee, respectively, and E_2 and H_2 be the respective quantities at the apogee. Then, from the given data,

$$\begin{aligned} E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} &= \frac{1}{2}2000\,\text{kg}\cdot(9500\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{7.5\times10^6\,\text{m}} \\ &= -1.6901\times10^{10}\text{ Joules} \\ H_1 = mx_1\dot{y}_1 &= 2000\,\text{kg}\cdot(-7.5\times10^6\,\text{m})\cdot(9500\,\text{m/s}) \\ &= -1.4250\times10^{14}\,\text{N}\cdot\text{m}\cdot\text{s} \\ E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} &= \frac{1}{2}2000\,\text{kg}\cdot(-1779\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}{4.0049\times10^7\,\text{m}} \\ &= -1.6901\times10^{10}\text{Joules} \\ H_2 = mx_2\dot{y}_2 &= 2000\,\text{kg}\cdot(4.0049\times10^7\,\text{m})\cdot(-1779\,\text{m/s}) \\ &= -1.4250\times10^{14}\,\text{N}\cdot\text{m}\cdot\text{s} \end{aligned}$$

Clearly, the energy and the angular momentum are conserved.

3. From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be 3.6335×10^4 s $= 10.09$ hrs. Now let us compare this result with the analytical value of the orbital period.

Let A be the semimajor axis of the elliptic orbit. Then the square of the orbital time period T is given by

$$T^2 = \frac{4\pi^2A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$\begin{aligned} 2A &= |x_1| + |x_2| \\ &= 7.5\times10^6\,\text{m} + 4.0049\times10^7\,\text{m} \\ &= 4.7549\times10^7\,\text{m} \\ \Rightarrow \quad A &= 2.3774\times10^7\,\text{m} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2\cdot(2.3774\times10^7\,\text{m})^3}{9.81\,\text{m/s}^2\cdot(6.4\times10^6\,\text{m})^2}} \\ &= 3.6335\times10^4\,\text{s}. \end{aligned}$$

which is the same value as obtained from numerical solution.

$$\boxed{T = 3.6335\times10^4\,\text{s} = 10.09\text{ hrs}}$$

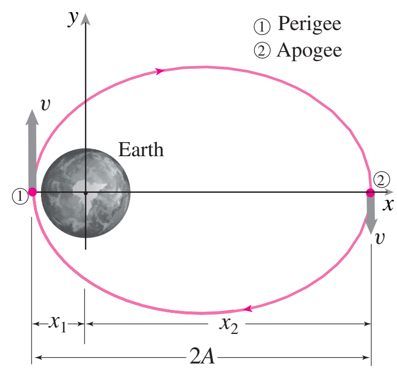


Figure 12.34: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

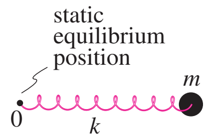


Figure 12.35

① No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. See box 7.1 on page 382

SAMPLE 12.13 Zero-length spring and central force motion: A zero-length spring ① (the relaxed length is zero) is tied to a mass $m = 1$ kg on one end and fixed on the other end. The spring stiffness is $k = 1\,\text{N}/m$.

1. Find appropriate initial conditions for the mass so that its trajectory is a straight line along the y-axis.
2. Find appropriate initial conditions for the mass so that its trajectory is a circle.
3. Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
4. Let $\vec{\boldsymbol{r}}(0) = 0.5\,\text{m}\hat{\boldsymbol{\imath}}$ and $\dot{\vec{\boldsymbol{r}}}(0) = (0.5\hat{\boldsymbol{\imath}} + 0.6\hat{\boldsymbol{\jmath}})\,\text{m/s}$. Describe the motion of the mass by plotting its trajectory for 12 s.

Solution Let the position of the mass be $\vec{\boldsymbol{r}}$ at some instant t. Since the relaxed length of the spring is zero, the stretch in the spring is $|\vec{\boldsymbol{r}}|$ and the spring force on the mass is $-k\vec{\boldsymbol{r}}$. Then the equation of motion of the mass is

$$\begin{aligned} -k\vec{\boldsymbol{r}} &= m\ddot{\vec{\boldsymbol{r}}} \\ -k(x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}}) &= m(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}}) \\ \Rightarrow \quad \ddot{x} + \frac{k}{m}x &= 0 \\ \text{and} \quad \ddot{y} + \frac{k}{m}y &= 0. \end{aligned}$$

Thus the equations of motion are decoupled in the x and y directions. The solutions, as discussed in the text (see eqn. (12.23)), are

$$\begin{aligned} x &= A\cos(\lambda t) + B\sin(\lambda t) \\ \text{and} \quad y &= C\cos(\lambda t) + D\sin(\lambda t) \end{aligned} \tag{12.26}$$

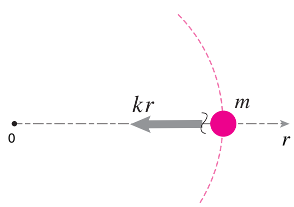


Figure 12.36: Free-body diagram of the mass.

where the constants A, B, C, D are determined from initial conditions. Let us take the most general initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0, y(0) = y_0$, and $\dot{y}(0) = \dot{y}_0$. By substituting these values in x and y equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$\begin{aligned} x &= x_0\cos(\lambda t) + \dot{x}_0/\lambda\sin(\lambda t) \\ \text{and} \quad y &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned} \tag{12.27}$$

1. For a straight line motion along the y-axis, we should have the x-component of motion identically zero. We can, therefore, set $x_0 = 0, \dot{x}_0 = 0$ and take any value for y_0 and $\dot{y}_0$ to give

$$\begin{aligned} x(t) &= 0 \\ \text{and} \quad y(t) &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned}$$

2. For a circular trajectory, we must pick initial conditions such that we get $x^2 + y^2 = (\text{a constant})^2$. We can easily achieve this by choosing, say, $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$, and $\dot{y}(0) = x_0\lambda$. Substituting these values in eqn. (12.27), we get

$$x^2 + y^2 = x_0^2\cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2\sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius x_0. Note that the initial position of the mass for this orbit is $\vec{\boldsymbol{r}}(0) = x_0\hat{\boldsymbol{\imath}})$, and the initial velocity is $(\vec{\boldsymbol{v}}(0) = x_0\lambda\hat{\boldsymbol{\jmath}})$, *i.e.*, the velocity is normal to the position vector $(\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{v}} = 0)$, and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact, it must be exactly equal to the product of the distance from the center and the orbital frequency λ.

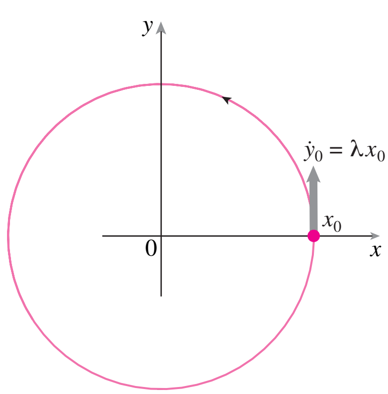


Figure 12.37: Circular trajectory of the mass.

3. In order to have elliptic orbits, the initial conditions should be selected such that x and y satisfy the equation of an ellipse. By examining the solutions in eqn. (12.27), we see that if we set $\dot{x}_0 = 0$ and $y_0 = 0$ and let the other two initial conditions have any arbitrary value, x_0 and $\dot{y}_0$, we get

$$\begin{aligned} x(t) &= x_0 \cos(\lambda t), \\ \text{and} \quad y(t) &= (\dot{y}_0/\lambda)\sin(\lambda t), \\ \Rightarrow \quad \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}_0/\lambda)^2} &= \cos^2(\lambda t) + \sin^2(\lambda t) \\ &= 1 \end{aligned}$$

which is the equation of an ellipse with semimajor axis x_0 and semiminor axis $\dot{y}/\lambda$. Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting $x_0 = 0$ and $\dot{y}_0 = 0$, and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector, *e.g.*,

$$\begin{aligned} \vec{\boldsymbol{r}}(0) = x_0\hat{\boldsymbol{\imath}} \quad &\text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = \dot{y}_0\hat{\boldsymbol{\jmath}}, \\ \text{or} \quad \vec{\boldsymbol{r}}(0) = y_0\hat{\boldsymbol{\jmath}} \quad &\text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = \dot{x}_0\hat{\boldsymbol{\imath}}, \end{aligned}$$

or, more generally,

$$\vec{\boldsymbol{r}}(0) = r_0\hat{\boldsymbol{\lambda}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector along the position vector of the mass and $\hat{\boldsymbol{n}}$ is normal to $\hat{\boldsymbol{\lambda}}$. Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep x_0 fixed and vary $\dot{y}_0$ we can get different elliptic orbits, including a circular one, based on the same major axis. Taking $x_0 = 1\,\text{m}$, we show different orbits obtained for the mass by varying $\dot{y}_0$ in fig. 12.38.

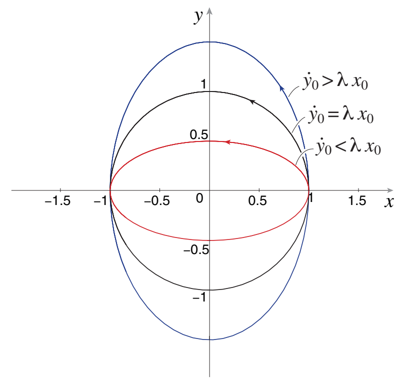


Figure 12.38: Elliptic orbits of the mass obtained from the initial conditions $x_0 = 1\,\text{m}, \dot{x}_0 = 0, y_0 = 0$, and various values of $\dot{y}_0$.

4. By substituting the given initial values $x_0 = 0.5\,\text{m}, \dot{x}(0) = 0.5\,\text{m/s}, y(0) = 0$ and $\dot{y} = 0.6\,\text{m/s}$ in eqn. (12.27) and and noting that $\lambda \equiv \sqrt{k/m} = \sqrt{(1\,\text{N/m})/(1\,\text{kg})} = (1/\,\text{s})$, we get

$$\begin{aligned} x(t) &= (0.5\,\text{m}) \cdot \cos\left(\frac{1}{s}\cdot t\right) + \left(\frac{0.5\,\text{m/s}}{s}\right)\cdot \sin\left(\frac{1}{s}\cdot t\right) \\ y(t) &= \left(\frac{0.6\,\text{m/s}}{s}\right)\cdot \sin\left(\frac{1}{s}\cdot t\right) \end{aligned}$$

The functions $x(t)$ and $y(t)$ do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between x and y that we can recognize. Instead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate x and y at, say, 100 values of t between 0 and 10 s and then plot x vs y.

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in fig. 12.39.

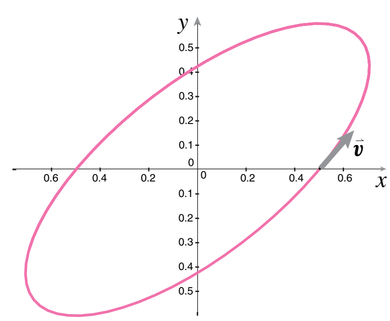


Figure 12.39: The orbit of the mass obtained from the initial conditions $x_0 = 0.5\,\text{m}, \dot{x}_0 = 0.5\,\text{m/s}, y_0 = 0$, and $\dot{y}_0 = 0.6\,\text{m/s}$.

12.3 Central force motion

Experts note that these problems do not use polar coordinates or any other fancy coordinate systems. Such descriptions come later in the text. At this point we want to lay out the basic equations and the qualitative features that can be found by numerical integration of the equations using Cartesian (xyz) coordinates.

Preparatory Problems

12.3.1 What exactly is meant by "central force motion"?

12.3.2 Under what circumstances is the angular momentum of a system, calculated relative to a point C which is fixed in a Newtonian frame, conserved?

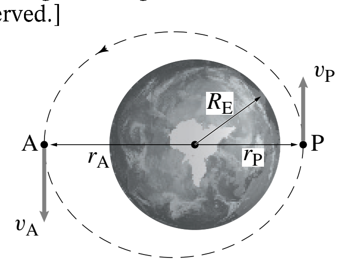
12.3.3 The mass of the earth is M, the mass of a satellite orbiting the earth is m, the radius of the earth is R, the force of gravity at the earth's surface is mg, the universal gravitational constant is G.

a) If the satellite is at distance r what is the force of the earth's gravity in terms of r, M, m and G?

b) If the satellite is at distance r what is the force of the earth's gravity in terms of r, R, m and g? (hint: evaluate the formula from the first part at $r = R$).

More-Involved Problems

12.3.4 A satellite is put into an elliptical orbit around the earth and has a speed v_P at position P. Find an expression for the speed v_A at position A (in terms of R_E, r_p, r_A, g, and v_P. The radii to A and P are, respectively, r_A and r_P. [Hint: both total energy and angular momentum are conserved.]



Problem 12.3.4

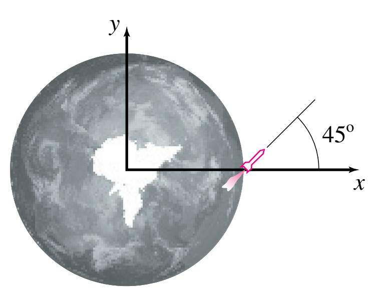
12.3.5 An intercontinental missile, modelled as a particle, is launched on a ballistic trajectory from the surface of the earth. The force on the missile from the earth's gravity is $F = mgR^2/r^2$ and is directed towards the center of the earth. When it is launched from the equator it has speed v_0 and in the direction shown, 45° from horizontal (both measured relative to a Newtonian reference frame). For the purposes of this calculation ignore the earth's rotation. You can think of this problem as two-dimensional in the plane shown. If you need numbers, use the following values:

$m = 1000\,\text{kg}$ = missile mass
$g = 10\,\text{m/s}^2$ at the earth's surface,
$R = 6{,}400{,}000\,\text{m}$ = earth's radius, and
$v_0 = 9000\,\text{m/s}$.

The distance of the missile from the center of the earth is $r(t)$.

a) Draw a free-body diagram of the missile. Write the linear momentum balance equation. Break this equation into x and y components. Rewrite these equations as a system of-4 first order ODE's suitable for computer solution. Write appropriate initial conditions for the ODE's.

b) Using the computer (or any other means) plot the trajectory of the rocket after it is launched for a time of 6670 seconds. [Hint: use a much shorter time when debugging your program.] On the same plot draw a (round) circle for the earth.



Problem 12.3.5: An intercontinental ballistic missile launch.

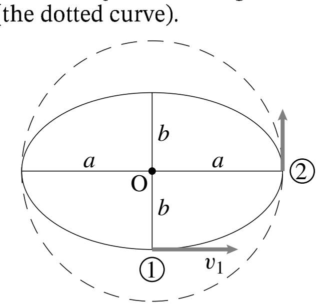
12.3.6 A particle of mass 2 kg moves in the horizontal xy-plane under the influence of a central force $\vec{F} = -k\vec{r}$ (attraction force proportional to distance from the origin), where $k = 200\,\text{N/m}$ and $\vec{r}$ is the position of the particle relative to the force center. Neglect all other forces.

a) Show that circular trajectories are possible, and determine the relation between speed v and circular radius r_o which must hold on a circular trajectory. [hint: Write $\vec{F} = m\vec{a}$, break into x and y components, solve the separate scalar equations, pick fortuitous values for the free constants in your solutions.]

b) It turns out that trajectories are in general elliptical, as depicted in the diagram.

For a particular elliptical trajectory with $a = 1\,\text{m}$ and $b = 0.8\,\text{m}$, the velocity of the particle at point 1 is observed to be perpendicular to the radial direction, with magnitude v_1, as shown. When the particle reaches point 2, its velocity is again perpendicular to the radial direction.

Determine the speed increment Δv which would have to be added (instantaneously) to the particle's speed at point 2 to transfer it to the circular trajectory through point 2 (the dotted curve).



Problem 12.3.6

12.3.7 Circular motion. Generally when people talk about central force motion they not only mean that the only force is directed at the origin but that the magnitude of the force only depends on the distance from the origin. Thus in 2D

$$\vec{F} = \underbrace{-\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2 + y^2}}}_{\text{Unit vector}} \cdot \underbrace{F\left(\sqrt{x^2 + y^2}\right)}_{\text{Magnitude of force}}$$

where the scalar function $F(r)$ expresses the dependence of the central attractive force on distance $r = \sqrt{x^2 + y^2}$. Consider a particle with mass m on a candidate circular orbit

$$\vec{r} = R\cos\lambda t\hat{\imath} + R\sin\lambda t\hat{\jmath}$$

with constant speed $v = |\vec{v}| = |\dot{\vec{r}}| = R\lambda$. For each of the cases below find the speed v for circular motion at radius R. Find this by plugging the circular motion equation into $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ using the form of $F(r)$ given. Answer in terms of other constants given (e.g., k, m, M, G)

a) $F(r) = kr$ (zero-rest-length attractive spring)

b) $F(r) = GMm/r^2$ (inverse-square gravitational attraction)

c) $F(r) = r^n$ (arbitrary power law attraction)

d) $F(r) = F(r)$ (arbitrary function). In this case you need to find how the speed depends on F in general.

e) Can you find a function $F(r)$ for which there are two or more circular orbits at the same speed v?

12.3.8 Circular motion, numerical solution. For each of the cases in problem 12.3.7 pick values for the physical constants. Then pick initial conditions which, according to theory, should give circular orbits. Then numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits. How big is the discrepancy between your numerical solution and an exact circle?

12.3.9 Two equal mass satellites have circular orbits at two different radii. The one that is closer to the earth has smaller potential energy and bigger kinetic energy. Which satellite has bigger total energy?

12.3.10 Find initial conditions corresponding to circular motion for a central force problem and simulate this motion on the computer. Use any central force attraction law you like (e.g., zero-length spring, inverse square,...) Check that you get closed circular orbits by plotting several revolutions. Now, in your simulation, apply a slight drag force opposing motion $\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}$. Pick a value for c so that the orbit slowly spirals in (say, less than 10% per orbit).

a) Make a plot of the spiraling orbit.

b) Plot the speed $|\vec{\boldsymbol{v}}|$ vs time as it spirals in.

c) How is it that a drag force causes the satellite to speed up? Is that numerical error? An approximation in our formulation of the governing equations? A relativistic effect? What?

12.3.11 Circular motion, numerical solution. For each of the cases in problem 12.3.7 pick values for the physical constants.

a) Pick initial conditions which, according to theory, should give circular orbits.

b) Numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits.

c) How big is the discrepancy between your numerical solution and an exact circle? Between the theoretically predicted period and the actual period?

12.3.12 Conic sections, numerical solution. Newton discovered that with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and a central attractive force of $F = C/r^2$ that all motions were conic sections. In particular, consider this problem, all in consistent units: $m = 1, C = 1, x_0 = 1, y_0 = 0, \dot{x}_0 = 0, \dot{y}_0 = v_0$. Newton claimed that there are special values for v_0, let's call them v_0^c and v_0^p with $v_0^c < v_0^p$ so that

- for $v_0 < v_0^c$ all orbits are ellipses with maximum distance from the origin being 1;
- for $v_0 = v_0^c$ the orbit is a circle of radius 1;
- for $v_0^c < v_0 < v_0^p$ the orbit is an ellipse with minimum distance from the origin being 1;
- for $v_0 = v_0^p$ the orbit is a left-opening parabola with base at the initial condition. (to draw the complete parabola you need also to use all the same initial conditions but with $\dot{y}_0 = -v_0$).
- for $v_0^p < v_0$ the orbit is a left-opening hyperbola, asymptoting to a straight line. (again you need to use $\dot{y}_0 = -v_0$ to draw the complete hyperbola.)

a) By a sequence of more or less systematic numerical guesses find as accurately, as you can, v_0^c and v_0^p.

b) Numerically solve the 4 coupled first order ODEs for initial conditions that correspond to each of the 5 cases above.

c) Plot all 5 cases on one plot showing the 5 shapes clearly.