SAMPLE 12.11 **Circular orbits of planets:** Refer to eqn. (12.15) in the text that governs the motion of planets around a fixed sun.

1. Let $x = A\cos(\lambda t)$ and $y = A\sin(\lambda t)$. Show that x and y satisfy the equations of planetary motion and that they describe a circular orbit.
2. Show that the solution assumed above satisfies Kepler's third law by showing that the orbital period $T = 2\pi/\lambda$ is proportional to the 3/2 power of the size of the orbit (which can be characterized by its radius).

Solution

1. The governing equation of planetary motion can be written as

$$\begin{aligned} \frac{\ddot{x}}{x} = \frac{-Gm_s}{(x^2+y^2)^{3/2}} &= \frac{\ddot{y}}{y} \\ \Rightarrow \quad \ddot{x}y - \ddot{y}x &= 0 \end{aligned} \tag{12.25}$$

Now,

$$\begin{aligned} x &= A\cos(\lambda t) \quad \Rightarrow \quad \ddot{x} = -\lambda^2 A\cos(\lambda t) \\ y &= A\sin(\lambda t) \quad \Rightarrow \quad \ddot{y} = -\lambda^2 A\sin(\lambda t) \end{aligned}$$

Substituting these values in eqn. (12.25), we get

$$-\lambda^2 A^2 \cos(\lambda t)\cdot\sin(\lambda t) + \lambda^2 A\sin(\lambda t)\cdot\cos(\lambda t) \stackrel{\surd}{=} 0$$

Thus the assumed form of x and y satisfy the governing equations of planetary motion, *i.e.*, $x(t) = A\cos(\lambda t)$ and $y(t) = A\sin(\lambda t)$ form a solution of planetary motion. Now, it is easy to show that

$$x^2 + y^2 = A^2\cos^2(\lambda t) + A^2\sin^2(\lambda t) = A^2,$$

i.e., x and y satisfy the equation of a circle with radius A. Thus, the assumed solution gives a circular orbit.

2. Substituting $x = A\cos(\lambda t)$ in eqn. (12.15), and noting that square of the radius of the orbit is $r^2 = x^2 + y^2 = A^2$, we get

$$\begin{aligned} -\lambda^2 A\cos(\lambda t) &= -Gm_s\frac{A\cos(\lambda t)}{r^3} \\ \Rightarrow \quad \lambda^2 &= \frac{Gm_s}{A^3} \\ \text{or} \quad \left(\frac{2\pi}{T}\right)^2 &= \frac{Gm_s}{A^3} \\ \Rightarrow \quad T^2 &= \frac{4\pi^2}{Gm_s}A^3 \\ \text{or} \quad T &= KA^{3/2} \end{aligned}$$

where $K = 2\pi/\sqrt{Gm_s}$ is a constant. Thus the orbital period T is proportional to the 3/2 power of the radius, or the size, of the circular orbit.

Of course, the same holds true for elliptic orbits too, but it is harder to show that analytically using cartesian coordinates, x and y.

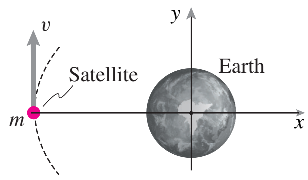

Figure 12.32

SAMPLE 12.12 Numerical computation of satellite orbits: The following data is known for an earth satellite: mass = 2000 kg, the distance to the closest point, the perigee, on its orbit from the earth's surface = 1100 km, and its velocity at perigee, which is purely tangential, is 9500 m/s. The radius of the earth is 6400 km and the acceleration due to gravity $g = 9.81\,\mathrm{m/s^2}$.

1. Solve the equations of motion of the satellite numerically with the given data and show that the orbit of the satellite is elliptical. Find the apogee of the orbit and the speed of the satellite at the apogee.
2. From the data at apogee and perigee show that the angular momentum and the energy of the satellite are conserved.
3. Find the orbital period of the satellite and show that it satisfies Kepler's third law (in equality form).

Solution

1. The equations of motion of a satellite around a fixed earth are

$$\ddot{x} = \frac{-gR^2x}{(x^2+y^2)^{3/2}} \quad \text{and} \quad \ddot{y} = \frac{-gR^2y}{(x^2+y^2)^{3/2}}.$$

where g is the acceleration due to gravity and R is the radius of the earth (see eqn. (12.20) in the text). From the given data at perigee, the initial conditions are

$$x(0) = -7500\,\mathrm{km}, \quad \dot{x}(0) = 0, \quad y(0) = 0, \quad \dot{y}(0) = 9500\,\mathrm{m/s}.$$

In order to solve the equations of motion by numerical integration, we first rewrite these equations as four first order equations:

$$\begin{aligned}
\dot{z}_1 &= z_2 \\
\dot{z}_2 &= -gR^2z_1/(z_1^2+z_3^2)^{3/2} \\
\dot{z}_3 &= z_4 \\
\dot{z}_4 &= -gR^2z_3/(z_1^2+z_3^2)^{3/2}.
\end{aligned}$$

Now the given initial conditions in terms of the new variables are

$$z1(0) = -7.5\times10^6\,\mathrm{m}, \quad z2(0) = 0, \quad z3(0) = 0, \quad z4(0) = 9500\,\mathrm{m/s}.$$

We are now ready to go to a computer. We implement the following pseudocode on the computer to solve the problem.

```
ODEs ={z1dot=z2,  z2dot=-g*R^2*z_1/(z_1^2+z_3^2)^{3/2},
   z3dot=z4,  z4dot=-g*R^2*z_3/(z_1^2+z_3^2)^{3/2}}
IC   ={z1(0)=-7.5E06, z2(0)=0, z3(0)=0, z4(0)=9500}
Set  g = 9.81, R = 6.4E06
Solve ODEs with IC for t=0 to t=4E04
Plot z1 vs z3
```

Results obtained from implementing the code above with a Runge-Kutta method based integrator is shown in fig. 12.33 where we have also plotted the earth centered at the origin to put the orbit in perspective. The orbit is clearly elliptical. From the computer output, we find the following data for the apogee.

$$x = 4.0049\times10^7\,\mathrm{m}, \quad \dot{x} = 0, \quad y = 0, \quad \dot{y} = -1.7791\times10^3\,\mathrm{m/s}$$

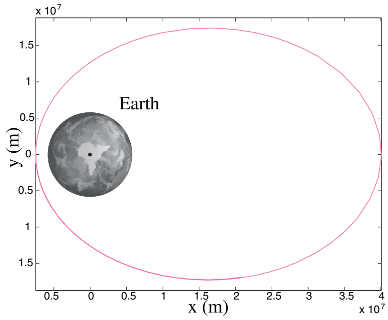

Figure 12.33: The elliptical orbit of the satellite, obtained from numerical integration of the equations of motion.

2. The expressions for energy E and angular momentum H for a satellite are,

$$\begin{aligned} E &= E_K + E_P = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - \frac{GMm}{r} \\ \vec{H}_o &= \vec{r} \times m\vec{v} = (x\hat{\imath} + y\hat{\jmath}) \times m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(x\dot{y} - y\dot{x})\hat{k} \end{aligned}$$

At both apogee and perigee, $y = 0$ and the velocity (which is tangential) is in the y direction, *i.e.*, $\dot{x} = 0$. Therefore, the expressions for energy and angular momentum become simpler:

$$\begin{aligned} E &= \frac{1}{2}m\dot{y}^2 - \frac{GMm}{r} = \frac{1}{2}m\dot{y}^2 - \frac{gR^2m}{|x|}, \\ \text{and} \quad H &= mx\dot{y}. \end{aligned}$$

Let E_1 and H_1 be the energy and the angular momentum of the satellite at the perigee, respectively, and E_2 and H_2 be the respective quantities at the apogee. Then, from the given data,

$$\begin{aligned} E_1 = \frac{1}{2}m\dot{y}_1^2 - \frac{gR^2m}{|x_1|} &= \frac{1}{2}2000\,\text{kg} \cdot (9500\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2 \cdot (6.4 \times 10^6\,\text{m})^2}{7.5 \times 10^6\,\text{m}} \\ &= -1.6901 \times 10^{10}\ \text{Joules} \\ H_1 = mx_1\dot{y}_1 &= 2000\,\text{kg} \cdot (-7.5 \times 10^6\,\text{m}) \cdot (9500\,\text{m/s}) \\ &= -1.4250 \times 10^{14}\,\text{N·m} \cdot \text{s} \\ E_2 = \frac{1}{2}m\dot{y}_2^2 - \frac{gR^2m}{|x_2|} &= \frac{1}{2}2000\,\text{kg} \cdot (-1779\,\text{m/s})^2 - \frac{9.81\,\text{m/s}^2 \cdot (6.4 \times 10^6\,\text{m})^2}{4.0049 \times 10^7\,\text{m}} \\ &= -1.6901 \times 10^{10}\text{Joules} \\ H_2 = mx_2\dot{y}_2 &= 2000\,\text{kg} \cdot (4.0049 \times 10^7\,\text{m}) \cdot (-1779\,\text{m/s}) \\ &= -1.4250 \times 10^{14}\,\text{N·m} \cdot \text{s} \end{aligned}$$

Clearly, the energy and the angular momentum are conserved.

3. From the computer output, we find the time at which the satellite returns to the perigee for the first time. This is the orbital period. From the output data, we get the orbital period to be $3.6335 \times 10^4\,\text{s} = 10.09$ hrs. Now let us compare this result with the analytical value of the orbital period.

Let A be the semimajor axis of the elliptic orbit. Then the square of the orbital time period T is given by

$$T^2 = \frac{4\pi^2 A^3}{gR^2}.$$

For the orbit we have obtained by numerical integration,

$$\begin{aligned} 2A &= |x_1| + |x_2| \\ &= 7.5 \times 10^6\,\text{m} + 4.0049 \times 10^7\,\text{m} \\ &= 4.7549 \times 10^7\,\text{m} \\ \Rightarrow \quad A &= 2.3774 \times 10^7\,\text{m} \end{aligned}$$

Hence,

$$\begin{aligned} T &= \sqrt{\frac{4\pi^2 \cdot (2.3774 \times 10^7\,\text{m})^3}{9.81\,\text{m/s}^2 \cdot (6.4 \times 10^6\,\text{m})^2}} \\ &= 3.6335 \times 10^4\,\text{s}. \end{aligned}$$

which is the same value as obtained from numerical solution.

$$\boxed{T = 3.6335 \times 10^4\,\text{s} = 10.09\ \text{hrs}}$$

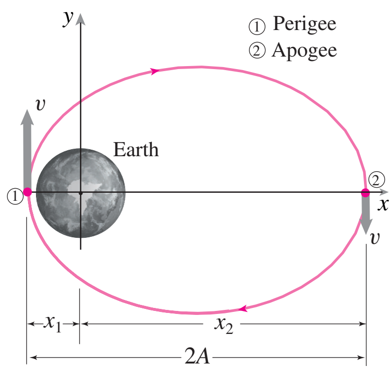

Figure 12.34: The elliptical orbit of the satellite. The perigee and apogee are marked as points 1 and 2 on the orbit.

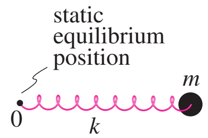

Figure 12.35

① No spring can have zero relaxed length, however, a spring can be configured in various ways to make it behave as if it has zero relaxed length. See box 7.1 on page 382

SAMPLE 12.13 Zero-length spring and central force motion: A zero-length spring ① (the relaxed length is zero) is tied to a mass $m = 1$ kg on one end and fixed on the other end. The spring stiffness is $k = 1\,\mathrm{N}/m$.

1. Find appropriate initial conditions for the mass so that its trajectory is a straight line along the y-axis.
2. Find appropriate initial conditions for the mass so that its trajectory is a circle.
3. Can you find any condition on initial conditions that guarantees elliptic orbits of the mass?
4. Let $\vec{r}(0) = 0.5\,\mathrm{m}\hat{\imath}$ and $\dot{\vec{r}}(0) = (0.5\hat{\imath} + 0.6\hat{\jmath})\,\mathrm{m/s}$. Describe the motion of the mass by plotting its trajectory for 12 s.

Solution Let the position of the mass be $\vec{r}$ at some instant t. Since the relaxed length of the spring is zero, the stretch in the spring is $|\vec{r}|$ and the spring force on the mass is $-k\vec{r}$. Then the equation of motion of the mass is

$$\begin{aligned} -k\vec{r} &= m\ddot{\vec{r}} \\ -k(x\hat{\imath} + y\hat{\jmath}) &= m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}) \\ \Rightarrow \quad \ddot{x} + \frac{k}{m}x &= 0 \\ \text{and} \quad \ddot{y} + \frac{k}{m}y &= 0. \end{aligned}$$

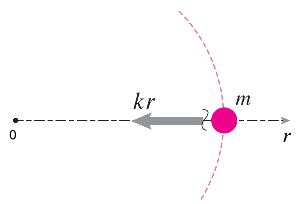

Figure 12.36: Free-body diagram of the mass.

Thus the equations of motion are decoupled in the x and y directions. The solutions, as discussed in the text (see eqn. (12.23)), are

$$\begin{aligned} x &= A\cos(\lambda t) + B\sin(\lambda t) \\ \text{and} \quad y &= C\cos(\lambda t) + D\sin(\lambda t) \end{aligned} \tag{12.26}$$

where the constants A, B, C, D are determined from initial conditions. Let us take the most general initial conditions $x(0) = x_0, \dot{x}(0) = \dot{x}_0, y(0) = y_0$, and $\dot{y}(0) = \dot{y}_0$. By substituting these values in x and y equations above and their derivatives, we get

$$A = x_0, \quad B = \dot{x}_0/\lambda, \quad C = y_0, \quad D = \dot{y}_0/\lambda.$$

Substituting these values we get

$$\begin{aligned} x &= x_0\cos(\lambda t) + \dot{x}_0/\lambda\sin(\lambda t) \\ \text{and} \quad y &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned} \tag{12.27}$$

1. For a straight line motion along the y-axis, we should have the x-component of motion identically zero. We can, therefore, set $x_0 = 0, \dot{x}_0 = 0$ and take any value for y_0 and $\dot{y}_0$ to give

$$\begin{aligned} x(t) &= 0 \\ \text{and} \quad y(t) &= y_0\cos(\lambda t) + \dot{y}_0/\lambda\sin(\lambda t). \end{aligned}$$

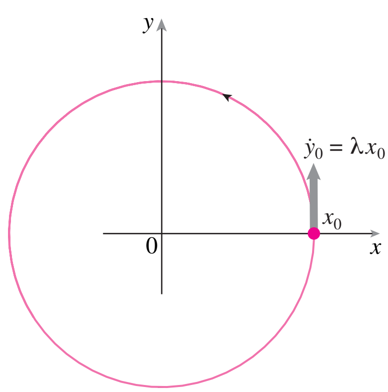

Figure 12.37: Circular trajectory of the mass.

2. For a circular trajectory, we must pick initial conditions such that we get $x^2 + y^2 = (\text{a constant})^2$. We can easily achieve this by choosing, say, $x(0) = x_0, \dot{x}(0) = 0, y(0) = 0$, and $\dot{y}(0) = x_0\lambda$. Substituting these values in eqn. (12.27), we get

$$x^2 + y^2 = x_0^2\cos^2(\lambda t) + \left(\frac{x_0\lambda}{\lambda}\right)^2 \sin^2(\lambda t) = x_0^2$$

which is a circular orbit of radius x_0. Note that the initial position of the mass for this orbit is $\vec{r}(0) = x_0\hat{\imath}$), and the initial velocity is $(\vec{v}(0) = x_0\lambda\hat{\jmath})$, *i.e.,* the velocity is normal to the position vector $(\vec{r} \cdot \vec{v} = 0)$, and the magnitude of the velocity is dependent on the magnitude of the position vector, in fact, it must be exactly equal to the product of the distance from the center and the orbital frequency λ.

3. In order to have elliptic orbits, the initial conditions should be selected such that x and y satisfy the equation of an ellipse. By examining the solutions in eqn. (12.27), we see that if we set $\dot{x}_0 = 0$ and $y_0 = 0$ and let the other two initial conditions have any arbitrary value, x_0 and $\dot{y}_0$, we get

$$\begin{aligned} x(t) &= x_0\cos(\lambda t), \\ \text{and} \quad y(t) &= (\dot{y}_0/\lambda)\sin(\lambda t), \\ \Rightarrow \quad \frac{x^2}{x_0^2} + \frac{y^2}{(\dot{y}_0/\lambda)^2} &= \cos^2(\lambda t) + \sin^2(\lambda t) \\ &= 1 \end{aligned}$$

which is the equation of an ellipse with semimajor axis x_0 and semiminor axis $\dot{y}/\lambda$. Of course, the symmetry of the equations implies that we could also get elliptic orbits by setting $x_0 = 0$ and $\dot{y}_0 = 0$, and letting the other two initial conditions be arbitrary. Thus the condition for elliptic orbits is to have the initial velocity normal to the position vector, *e.g.*,

$$\begin{aligned} \vec{\boldsymbol{r}}(0) &= x_0\hat{\boldsymbol{\imath}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = \dot{y}_0\hat{\boldsymbol{\jmath}}, \\ \text{or} \quad \vec{\boldsymbol{r}}(0) &= y_0\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = \dot{x}_0\hat{\boldsymbol{\imath}}, \end{aligned}$$

or, more generally,

$$\vec{\boldsymbol{r}}(0) = r_0\hat{\boldsymbol{\lambda}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v\hat{\boldsymbol{n}},$$

where $\hat{\boldsymbol{\lambda}}$ is a unit vector along the position vector of the mass and $\hat{\boldsymbol{n}}$ is normal to $\hat{\boldsymbol{\lambda}}$. Note that the condition obtained in (b) for circular orbits is just a special case of the condition for elliptic orbits (well, a circle is just a special case of an ellipse). Therefore, if we keep x_0 fixed and vary $\dot{y}_0$ we can get different elliptic orbits, including a circular one, based on the same major axis. Taking $x_0 = 1\,\text{m}$, we show different orbits obtained for the mass by varying $\dot{y}_0$ in fig. 12.38.

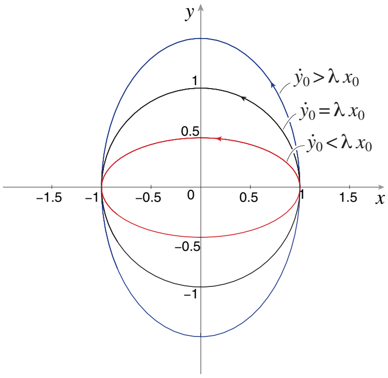

Figure 12.38: Elliptic orbits of the mass obtained from the initial conditions $x_0 = 1\,\text{m}, \dot{x}_0 = 0, y_0 = 0$, and various values of $\dot{y}_0$.

4. By substituting the given initial values $x_0 = 0.5\,\text{m}, \dot{x}(0) = 0.5\,\text{m/s}, y(0) = 0$ and $\dot{y} = 0.6\,\text{m/s}$ in eqn. (12.27) and and noting that $\lambda \equiv \sqrt{k/m} = \sqrt{(1\,\text{N/m})/(1\,\text{kg})} = (1/\,\text{s})$, we get

$$\begin{aligned} x(t) &= (0.5\,\text{m})\cdot\cos\left(\frac{1}{s}\cdot t\right) + \left(\frac{0.5\,\text{m/s}}{s}\right)\cdot\sin\left(\frac{1}{s}\cdot t\right) \\ y(t) &= \left(\frac{0.6\,\text{m/s}}{s}\right)\cdot\sin\left(\frac{1}{s}\cdot t\right) \end{aligned}$$

The functions $x(t)$ and $y(t)$ do not seem to describe any simple geometric path immediately. We could, perhaps, do some mathematical manipulations and try to get a relationship between x and y that we can recognize. Instead, let us plot the orbit on a computer to see the path that the mass takes during its motion with these initial conditions. To plot this orbit, we evaluate x and y at, say, 100 values of t between 0 and 10 s and then plot x vs y.

```
t = [0 0.1 0.2 ... 9.9 10]
x = 0.5 * cos(t) + 0.6 * sin(t)
y = 0.6 * sin(t)
plot x vs y
```

The plot obtained by performing these operations on a computer is shown in fig. 12.39.

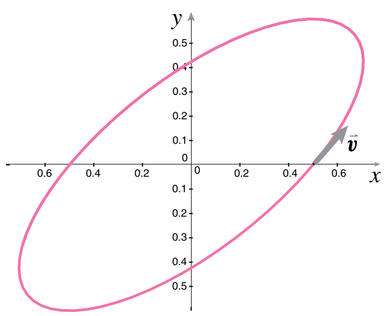

Figure 12.39: The orbit of the mass obtained from the initial conditions $x_0 = 0.5\,\text{m}, \dot{x}_0 = 0.5\,\text{m/s}, y_0 = 0$, and $\dot{y}_0 = 0.6\,\text{m/s}$.