12.3 Central force motion

Experts note that these problems do not use polar coordinates or any other fancy coordinate systems. Such descriptions come later in the text. At this point we want to lay out the basic equations and the qualitative features that can be found by numerical integration of the equations using Cartesian (xyz) coordinates.

Preparatory Problems

12.3.1 What exactly is meant by "central force motion"?

12.3.2 Under what circumstances is the angular momentum of a system, calculated relative to a point C which is fixed in a Newtonian frame, conserved?

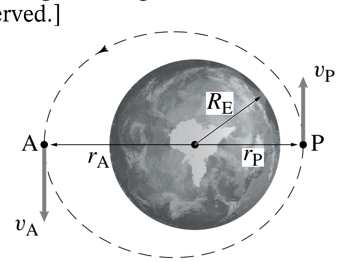
12.3.3 The mass of the earth is M, the mass of a satellite orbiting the earth is m, the radius of the earth is R, the force of gravity at the earth's surface is mg, the universal gravitational constant is G.

a) If the satellite is at distance r what is the force of the earth's gravity in terms of r, M, m and G?

b) If the satellite is at distance r what is the force of the earth's gravity in terms of r, R, m and g? (hint: evaluate the formula from the first part at $r = R$).

More-Involved Problems

12.3.4 A satellite is put into an elliptical orbit around the earth and has a speed v_P at position P. Find an expression for the speed v_A at position A (in terms of R_E, r_p, r_A, g, and v_P. The radii to A and P are, respectively, r_A and r_P. [Hint: both total energy and angular momentum are conserved.]



Problem 12.3.4

12.3.5 An intercontinental missile, modelled as a particle, is launched on a ballistic trajectory from the surface of the earth. The force on the missile from the earth's gravity is $F = mgR^2/r^2$ and is directed towards the center of the earth. When it is launched from the equator it has speed v_0 and in the direction shown, 45° from horizontal (both measured relative to a Newtonian reference frame). For the purposes of this calculation ignore the earth's rotation. You can think of this problem as two-dimensional in the plane shown. If you need numbers, use the following values:

$m = 1000\,\text{kg}$ = missile mass

$g = 10\,\text{m/s}^2$ at the earth's surface,

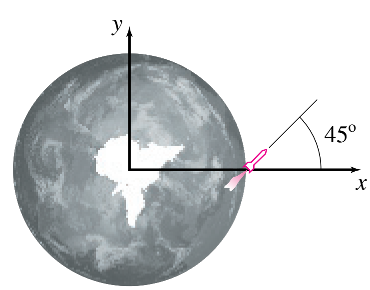
$R = 6,400,000\,\text{m}$ = earth's radius, and

$v_0 = 9000\,\text{m/s}$.

The distance of the missile from the center of the earth is $r(t)$.

a) Draw a free-body diagram of the missile. Write the linear momentum balance equation. Break this equation into x and y components. Rewrite these equations as a system of-4 first order ODE's suitable for computer solution. Write appropriate initial conditions for the ODE's.

b) Using the computer (or any other means) plot the trajectory of the rocket after it is launched for a time of 6670 seconds. [Hint: use a much shorter time when debugging your program.] On the same plot draw a (round) circle for the earth.



Problem 12.3.5: An intercontinental ballistic missile launch.

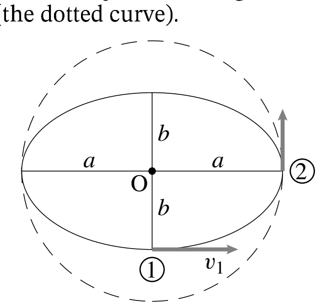
12.3.6 A particle of mass 2 kg moves in the horizontal xy-plane under the influence of a central force $\vec{F} = -k\vec{r}$ (attraction force proportional to distance from the origin), where $k = 200\,\text{N/m}$ and $\vec{r}$ is the position of the particle relative to the force center. Neglect all other forces.

a) Show that circular trajectories are possible, and determine the relation between speed v and circular radius r_o which must hold on a circular trajectory. [hint: Write $\vec{F} = m\vec{a}$, break into x and y components, solve the separate scalar equations, pick fortuitous values for the free constants in your solutions.]

b) It turns out that trajectories are in general elliptical, as depicted in the diagram.

For a particular elliptical trajectory with $a = 1\,\text{m}$ and $b = 0.8\,\text{m}$, the velocity of the particle at point 1 is observed to be perpendicular to the radial direction, with magnitude v_1, as shown. When the particle reaches point 2, its velocity is again perpendicular to the radial direction.

Determine the speed increment Δv which would have to be added (instantaneously) to the particle's speed at point 2 to transfer it to the circular trajectory through point 2 (the dotted curve).



Problem 12.3.6

12.3.7 Circular motion. Generally when people talk about central force motion they not only mean that the only force is directed at the origin but that the magnitude of the force only depends on the distance from the origin. Thus in 2D

$$\vec{F} = -\underbrace{\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}}}_{\text{Unit vector}} \cdot \underbrace{F\left(\sqrt{x^2+y^2}\right)}_{\text{Magnitude of force}}$$

where the scalar function $F(r)$ expresses the dependence of the central attractive force on distance $r = \sqrt{x^2+y^2}$. Consider a particle with mass m on a candidate circular orbit

$$\vec{r} = R\cos\lambda t\hat{\imath} + R\sin\lambda t\hat{\jmath}$$

with constant speed $v = |\vec{v}| = |\dot{\vec{r}}| = R\lambda$. For each of the cases below find the speed v for circular motion at radius R. Find this by plugging the circular motion equation into $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ using the form of $F(r)$ given. Answer in terms of other constants given (e.g., k, m, M, G)

a) $F(r) = kr$ (zero-rest-length attractive spring)

b) $F(r) = GMm/r^2$ (inverse-square gravitational attraction)

c) $F(r) = r^n$ (arbitrary power law attraction)

d) $F(r) = F(r)$ (arbitrary function). In this case you need to find how the speed depends on F in general.

e) Can you find a function $F(r)$ for which there are two or more circular orbits at the same speed v?

12.3.8 Circular motion, numerical solution. For each of the cases in problem 12.3.7 pick values for the physical constants. Then pick initial conditions which, according to theory, should give circular orbits. Then numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits. How big is the discrepancy between your numerical solution and an exact circle?

12.3.9 Two equal mass satellites have circular orbits at two different radii. The one that is closer to the earth has smaller potential energy and bigger kinetic energy. Which satellite has bigger total energy?

12.3.10 Find initial conditions corresponding to circular motion for a central force problem and simulate this motion on the computer. Use any central force attraction law you like (e.g., zero-length spring, inverse square,...) Check that you get closed circular orbits by plotting several revolutions. Now, in your simulation, apply a slight drag force opposing motion $\vec{\boldsymbol{F}} = -c\vec{\boldsymbol{v}}$. Pick a value for c so that the orbit slowly spirals in (say, less than 10% per orbit).

a) Make a plot of the spiraling orbit.

b) Plot the speed $|\vec{\boldsymbol{v}}|$ vs time as it spirals in.

c) How is it that a drag force causes the satellite to speed up? Is that numerical error? An approximation in our formulation of the governing equations? A relativistic effect? What?

12.3.11 Circular motion, numerical solution. For each of the cases in problem 12.3.7 pick values for the physical constants.

a) Pick initial conditions which, according to theory, should give circular orbits.

b) Numerically solve the 4 coupled first order ODEs that describe planar motion, make a plot, and show that you do indeed get circular orbits.

c) How big is the discrepancy between your numerical solution and an exact circle? Between the theoretically predicted period and the actual period?

12.3.12 Conic sections, numerical solution. Newton discovered that with $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and a central attractive force of $F = C/r^2$ that all motions were conic sections. In particular, consider this problem, all in consistent units: $m = 1, C = 1, x_0 = 1, y_0 = 0, \dot{x}_0 = 0, \dot{y}_0 = v_0$. Newton claimed that there are special values for v_0, let's call them v_0^c and v_0^p with $v_0^c < v_0^p$ so that

- for $v_0 < v_0^c$ all orbits are ellipses with maximum distance from the origin being 1;
- for $v_0 = v_0^c$ the orbit is a circle of radius 1;
- for $v_0^c < v_0 < v_0^p$ the orbit is an ellipse with minimum distance from the origin being 1;
- for $v_0 = v_0^p$ the orbit is a left-opening parabola with base at the initial condition. (to draw the complete parabola you need also to use all the same initial conditions but with $\dot{y}_0 = -v_0$).
- for $v_0^p < v_0$ the orbit is a left-opening hyperbola, asymptoting to a straight line. (again you need to use $\dot{y}_0 = -v_0$ to draw the complete hyperbola.)

a) By a sequence of more or less systematic numerical guesses find as accurately, as you can, v_0^c and v_0^p.

b) Numerically solve the 4 coupled first order ODEs for initial conditions that correspond to each of the 5 cases above.

c) Plot all 5 cases on one plot showing the 5 shapes clearly.