12.2 Linear momentum, angular momentum, work and energy

If you know a particle's starting position and velocity, and you know the force on it as it moves, then you can use $\vec{F} = m\vec{a}$ to predict its path. That is the central idea of the previous section. We had no need for ideas related to momentum, angular momentum, work and energy. For one particle the one equation $\vec{F} = m\vec{a}$ tells the whole story. So, before we go on to discuss them further, let us be clear:

> The concepts of linear and angular momentum, work and energy are not needed to study particle mechanics. $\vec{F} = m\vec{a}$ is enough.

So, why do we bother to devote a section to these topics? Because

- These concepts will sometimes be needed when we discuss more complex systems;
- These concepts sometimes provide a shorter route for answering some dynamics questions;
- The simplest place to introduce the concepts is in the context of one particle;
- The concepts give a way to check the consistency of solutions of $\vec{F} = m\vec{a}$; and
- The concepts can be an aid to physical intuition.

For more complex systems, principles of momentum and energy transcend $\vec{F} = m\vec{a}$ and can generally not be derived from $\vec{F} = m\vec{a}$. But for a single particle, all of these are derived concepts, as worked out in box 12.4 on page 9. Note that

> All of the facts and theorems below apply to any motion of a particle that is consistent with $\vec{F} = m\vec{a}$.

Example: Simple ballistics solution.
Consider a ball thrown up at 45°: $\vec{F} = -mg\hat{\jmath}$, $\vec{r}(0) = \vec{0}$ and $\vec{v}(0) = v_0\hat{\imath} + v_0\hat{\jmath}$. We claimed (page 616) that a solution is

$$\vec{r} = v_0 t\hat{\imath} + (v_0 t - gt^2/2)\hat{\jmath} \qquad \text{and} \qquad \vec{v} = v_0\hat{\imath} + (v_0 - gt)\hat{\jmath}.$$

This solution is plotted various ways in fig. 12.22. These functions of time are consistent with the initial conditions. Further they are consistent with the governing equations, the so called 'equations of motion', $\dot{\vec{v}} = \vec{F}/m$ and $\dot{\vec{r}} = \vec{v}$. All of the momentum and energy principles below must therefore apply.

Some of the ideas apply even if $\vec{F} \neq m\vec{a}$. For example, the work of a force is defined for imagined motions that might never occur.

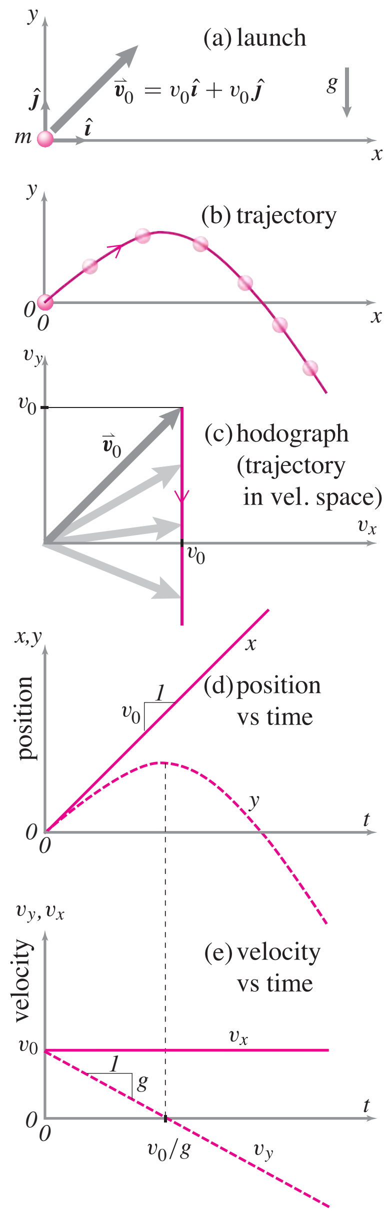

Figure 12.22: **(a) A simple ballistics problem.** (b) The well known parabolic trajectory. (c) The velocity trajectory, called a *hodograph*, shows the sequence of positions of the tip of the velocity vector. The velocity starts with $v_x = v_y$ and then v_y decreases. (d) Plot of x and y components vs time. The y component has a parabolic shape similar to the particle trajectory because x is proportional to t in this problem. (e) The velocity components versus time. Note, (b) is a 'cross plot' of the two plots in (d), and (c) is a cross plot of (e).

Linear momentum

Linear momentum for a particle is defined as $\vec{\boldsymbol{L}} = m\vec{\boldsymbol{v}}$. The particle momentum-balance theorems (facts) are

$$\vec{\boldsymbol{F}} = \frac{d}{dt}\vec{\boldsymbol{L}} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{\boldsymbol{F}}\,dt}_{\text{Linear impulse}} = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1$$

These are so trivially related to $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ that it is hard to see any content in them. And, indeed, if we were only studying the mechanics of single particles we probably would not have introduced the concept of linear momentum. Nonetheless, the general result does apply:

> The net force $\vec{\boldsymbol{F}}$ on a particle is the rate of change of its linear momentum, $\dot{\vec{\boldsymbol{L}}}$.

A special important case is when there is no force and linear momentum is conserved (doesn't change). For a single particle momentum conservation means constant velocity motion.

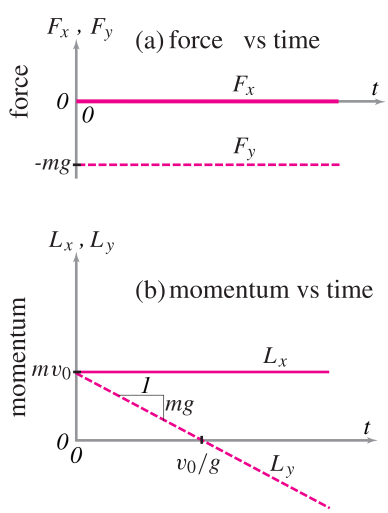

Figure 12.23: The force and linear momentum from the problem in fig. 12.22. Note that the momentum curves are the integrals of the corresponding force curves.

Example: Linear momentum check
For the simple ballistics solution above we evaluate the left side of the momentum balance equation

$$\int_0^t \vec{\boldsymbol{F}}\,dt' = -mgt\,\hat{\boldsymbol{\jmath}}.$$

Then evaluate the right side:

$$\Delta\vec{\boldsymbol{L}} = \vec{\boldsymbol{L}}_2 - \vec{\boldsymbol{L}}_1 = [m(v_0\hat{\boldsymbol{\imath}} + (v_0 - gt)\hat{\boldsymbol{\jmath}})] - [m(v_0\hat{\boldsymbol{\imath}} + v_0\hat{\boldsymbol{\jmath}})] = -mgt\,\hat{\boldsymbol{\jmath}}$$

and check for equality: $-mgt\,\hat{\boldsymbol{\jmath}} = -mgt\,\hat{\boldsymbol{\jmath}}$. This force and momentum is plotted in fig. 12.23. The solution *is* consistent with linear momentum balance. Note that in this example there is no change in the component of linear momentum in the $\hat{\boldsymbol{\imath}}$ direction; there is no force in the x direction so L_x is conserved.

Angular momentum

In dynamics angular momentum is one of the most important ideas, important theoretically and for solving problems. For one particle

> Angular momentum relative to point C is $\vec{\boldsymbol{H}}_{/\mathrm{C}} = \vec{\boldsymbol{r}}_{/\mathrm{C}} \times (m\vec{\boldsymbol{v}})$,

where $\vec{\boldsymbol{r}}_{/\mathrm{C}}$ is the position of the particle relative to fixed point C and $\vec{\boldsymbol{v}}$ is the velocity of the particle. Angular momentum can be calculated relative to any point C. Which point you pick affects the value of the angular momentum. Sometimes $\vec{\boldsymbol{H}}_{/\mathrm{C}}$ is written without the "/" as $\vec{\boldsymbol{H}}_{\mathrm{C}}$. The key angular momentum theorems (facts) are:

$$\underbrace{\vec{r}_{/\mathrm{C}} \times \vec{F}}_{\vec{M}_{\mathrm{C}}} = \dot{\vec{H}}_{/\mathrm{C}} \qquad \text{and} \qquad \underbrace{\int_{t_1}^{t_2} \vec{r}_{/\mathrm{C}} \times \vec{F}\, dt}_{\text{Angular impulse}} = (\vec{H}_{/\mathrm{C}})_2 - (\vec{H}_{/\mathrm{C}})_1.$$

The torque $\vec{M}_{\mathrm{C}}$ of *all* the external forces acting on a particle about point C is the rate of change of its angular momentum $\dot{\vec{H}}_{/\mathrm{C}}$ about point C.

The intuitive notion is that angular momentum represents how much a particle is 'going around' point C. A particle gets more credit for going faster, for being more massive, and for being farther away①.

If the force on the particle is zero or passes through the point C, the torque (moment) of the force is zero and its angular momentum is conserved.

Example: Angular momentum check.
Using the same ballistics example we check the solution for consistency with angular momentum balance. For no good reason let's use the origin for the angular momentum reference point. We could use any point. Again we compare the left and right sides and check for equality.

$$\underbrace{\int_0^t \vec{r}_{/0} \times \vec{F}\, dt}_{\text{Left side}} = \int_0^t \left(v_0 t' \hat{\imath} + (v_0 - g t'^2/2)\hat{\jmath}\right) \times (-mg\hat{\jmath})\, dt'$$

$$= \int_0^t -v_0 m g t' \hat{k}\, dt' = -v_0 m g \hat{k} t^2/2$$

$$\underbrace{\vec{H}_{/02} - \vec{H}_{/01}}_{\text{Right side}} = \vec{r}(t) \times (m\vec{v}(t)) - \vec{r}(0) \times (m\vec{v}(0))$$

$$= m\left(v_0 t\hat{\imath} + (v_0 t - g t^2/2)\hat{\jmath}\right) \times \left(v_0\hat{\imath} + (v_0 - g t)\hat{\jmath}\right) - m\left(\vec{0} \times \vec{v}(0)\right)$$

$$= \left(v_0 t - g t^2 - v_0 t + g t^2/2\right) m v_0 \hat{k} = -v_0 m g t^2 \hat{k}/2.$$

This torque and angular momentum are plotted in fig. 12.24. The 'left side' agrees with the 'right side' and angular momentum balance is satisfied, as it must be. In 2D problems like this there is only one non-trivial component to angular momentum, that in the out-of-plane (z) direction. In this case the angular momentum H_z is not conserved because the gravity force does have a torque about the z axis.

Power balance

The power P of a force $\vec{F}$ on a particle with velocity $\vec{v}$ is $P = \vec{F} \cdot \vec{v}$. The kinetic energy of a particle is $E_{\mathrm{K}} = mv^2/2$. The power balance equations are

$$P = \dot{E}_{\mathrm{K}} \qquad \text{and} \qquad \int_{t_1}^{t_2} P\, dt = E_{\mathrm{K}2} - E_{\mathrm{K}1}.$$

①**Angular momentum contradicts common intuition.** The notion that angular momentum is bigger if a given mass is further away is counterintuitive to most of us. Imagine a one kilogram mass is going around your head once per second at a distance of a meter. Now imagine a second equal mass going around at a radius of 10 meters and only going around once every 100 seconds. The first mass whirls around your head 100 times for each slow orbit of the second mass. Even though the second mass certainly doesn't make its rotation feel so present its angular momentum is as big. Intuitive or not, this is how angular momentum is defined. It's a useful concept so it's worth adjusting your intuition to match it.

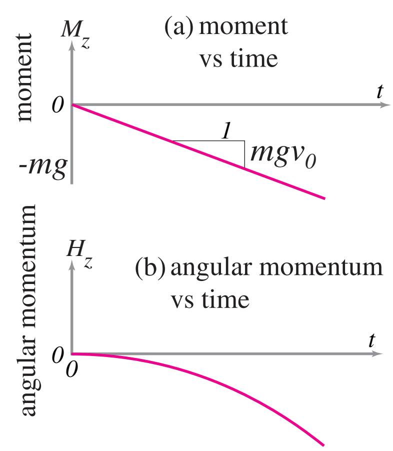

Figure 12.24: The torque and angular momentum about the origin for the problem in fig. 12.22. For this 2D problem only the z component is non-trivial. Note that the angular momentum is the integral of the torque.

The power P of *all* the external forces acting on a particle is the rate of change of its kinetic energy $\dot{E}_{\mathrm{K}}$. Or, integrating in time, the power added up over time is the net change in kinetic energy.

The situation is similar to that for 1D motion (section 10.2).

Power and work

The integral of power with respect to time can be replaced with a path integral for the work of a force. The key idea is in the differential expressions for an increment of work:

$$dW = P dt = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}} dt = \vec{\boldsymbol{F}} \cdot \frac{d\vec{\boldsymbol{r}}}{dt} dt = \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}}.$$

So

$$\begin{aligned} W_{12} &= W_{12} \\ \text{Power integrated in time} &= \text{Sum of work increments} \\ \int_{t_1}^{t_2} P dt &= \int_1^2 dW \\ \int_{t_1}^{t_2} \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\, dt &= \int_{\vec{\boldsymbol{r}}_1}^{\vec{\boldsymbol{r}}_2} \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{r}} \qquad (12.8) \end{aligned}$$

Thus the power balance equation, integrated in time is equivalent to the "work energy" equation:

$$\begin{aligned} \text{Work on particle} &= \text{Change in its kinetic energy} \\ W_{12} &= \Delta E_{\mathrm{K}} \qquad (12.9) \\ &= E_{\mathrm{K}2} - E_{\mathrm{K}1} \end{aligned}$$

Example: Energy check.
We can check the simple ballistics solution for consistency with energy balance.

First let's compare the work to the change of kinetic energy.

$$\underbrace{\int_0^t \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}\, dt}_{\text{Work}} = \int_0^t (-mg\hat{\boldsymbol{\jmath}}) \cdot \left(v_0\hat{\boldsymbol{\imath}} + (v_0 - gt')\hat{\boldsymbol{\jmath}}\right)\, dt'$$

$$= \int_0^t -mg(v_0 - gt')\, dt' = -mgv_0t + mg^2t^2/2$$

$$\underbrace{E_{K2} - E_{K1}}_{\text{Change in kinetic energy}} = m(|\vec{\boldsymbol{v}}|^2 - |\vec{\boldsymbol{v}}_0|^2)/2$$

$$= m\left((v_0^2 + (v_0 - gt)^2 - 2v_0^2\right)/2 = \underbrace{-mgv_0t + mg^2t^2/2}_{\text{agrees}}$$

This power and kinetic energy are plotted in fig. 12.25. In this check we have not taken advantage of the fact that this particular force is conservative.

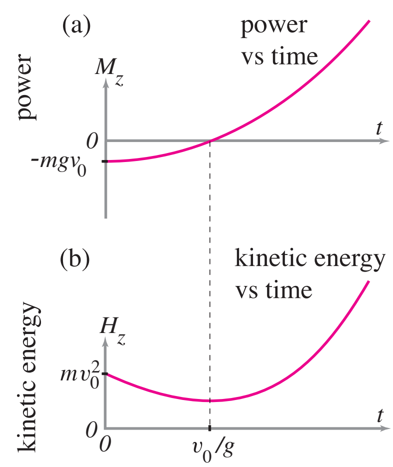

Figure 12.25: The power of the gravitational force and the total kinetic energy are plotted vs time for the ballistics problem of fig. 12.22. Note that the kinetic energy is the integral of the power. The gravitational power starts out negative, goes to zero when the particle reaches the apex of its trajectory and then becomes positive evermore as the downwards gravitational force acts on a downwards moving particle. The kinetic energy starts at mv_0^2 and drops to $mv_0^2/2$ at the particle apex when v_y goes to zero. Then the kinetic energy increases forever more as the gravitational force does more and more work.

The work of a force $\vec{F}$: W_{12}

Previously in Physics, and more recently in one dimensional dynamics here, you learned that

Work is force times distance.

This is actually a special case of the formula

$$P = \vec{\boldsymbol{F}} \cdot \vec{\boldsymbol{v}}.$$

How is that? If $\vec{\boldsymbol{F}}$ is constant and parallel to the displacement $\Delta\vec{\boldsymbol{x}}$, then

$$W_{12} = \int \dot{W}\, dt = \int P dt = \int \vec{\boldsymbol{F}} \cdot \underbrace{\vec{\boldsymbol{v}}\, dt}_{d\vec{\boldsymbol{x}}} = \int \vec{\boldsymbol{F}} \cdot d\vec{\boldsymbol{x}} = \vec{\boldsymbol{F}} \cdot \int d\vec{\boldsymbol{x}}$$

$$= \vec{\boldsymbol{F}} \cdot \Delta\vec{\boldsymbol{x}} = F\Delta x = \text{Force} \cdot \text{distance}.$$

In 2 and 3 dimensions there are subtleties involved with the concept of work because of its dependence on which path in space the force works on. These 'path dependence' subtleties are often covered in some detail in calculus courses in the sections on vector calculus, path integrals, gradient and curl. We discuss the relevant highlights below.

Potential energy of a force

Some forces (read force fields $\vec{\boldsymbol{F}} = \vec{\boldsymbol{F}}(\vec{\boldsymbol{r}})$) have the property that the work they do is independent of the path followed by the material point as the force acts. If the work of a force is path independent in this way (see box 12.3 on page 7), then a potential energy can be defined so that the work done by the force is the decrease in the Potential Energy

$$-\Delta E_{\mathrm{P}} = W_{12} = E_{\mathrm{P}_1} - E_{\mathrm{P}_2}$$

The common examples are listed below:

- **linear spring:** $E_{\mathrm{P}} = (1/2)k(\text{stretch})^2$.
- **gravity near earth's surface:** $E_{\mathrm{P}} = mgh$
- **gravity between spheres or points:** $E_{\mathrm{P}} = -MmG/r$
- **constant force $\vec{F}$ acting on a point:** $E_{\mathrm{P}} = -\vec{F} \cdot \vec{r}$

In all cases a constant could be added to the potential energy and it would still be a legitimate potential energy for the force.

In the cases of the spring and gravity between spheres, the change in potential energy is the net work done by the spring or gravity on the pair of objects between which the force acts. If both ends of a spring are moving, the net work of the spring on the two objects to which it is connected is the decrease in potential energy of the spring.

There is a possible source of confusion in our using the same symbol E_{P} to represent the potential work of an external force and for internal potential energy. In practice, however, they are used identically, so we use the same symbol for both. The potential energy in a stretched spring is the same whether it is the cause of force on a system or it is internal to the system.

Example: Checking conservation of energy
Because the gravity force is conservative we can also check our simple ballistics solution for consistency with conservation of energy. Taking the potential energy as $E_{\mathrm{P}} = mgh = mgy$ we find, as expected, that the solution does have the property that

$$\begin{aligned}
E_{tot1} &\stackrel{?}{=} E_{tot2} \\
E_{\mathrm{K}_1} + E_{\mathrm{P}_1} &\stackrel{?}{=} E_{\mathrm{K}_2} + E_{\mathrm{P}_2} \\
mv_0^2 + 0 &\stackrel{?}{=} \left(v_0^2 + (v_0 - gt)^2\right) m/2 + mg\overbrace{(v_0 t - gt^2/2)}^{y} \\
mv_0^2 &\stackrel{\surd}{=} mv_0^2 \quad \text{(Checks)}
\end{aligned}$$

Using momentum and energy as a check of a numerical solution

You obtain a numerical solution to $\vec{F} = m\vec{a}$ by setting up the set of first order differential equations 12.2 on page 615. In turn, these can be written in explicit scalar form as eqn. (12.3).

While you solve these equations you can add further first order equations that you can use in your energy and momentum checks. These evaluate the integrals for linear impulse, angular impulse and work.

$$\begin{aligned}\frac{d}{dt}(\text{linear impulse}) &= \vec{F},\\ \frac{d}{dt}(\text{angular impulse}) &= \vec{r}_{/C}\times\vec{F}, \text{and}\\ \dot{W} &= \vec{F}\cdot\vec{v}.\end{aligned}$$

The first two equations are short hand for 2 (or 3) first order scalar equations for motion in 2 (or 3) spatial dimensions. If these are added to the system of ordinary differential equations that you solve, they can be used to check the solution.

Summary on using energy and momentum to check a solution

Because the momentum and energy facts and theorems apply to any motion consistent with $\vec{F} = m\vec{a}$ they can be used as a consistency check on any solutions you find to the differential equations of motion ($\vec{F} = m\vec{a}$).

Here is the general situation. You are given $\vec{F}(t, \vec{r}, \vec{v})$. You are given initial conditions $\vec{r}_0$ and $\vec{v}_0$ at, say, $t = 0$. Using computer integration or pencil and paper methods, you solve the differential equation $\vec{a} = \vec{F}/m$ to get $\vec{r}(t)$ and $\vec{v}(t)$. Now your solution can be checked for consistency with the energy and momentum theorems. In particular, your solution, if it is correct, must satisfy

- Linear momentum balance: $\int_0^t \vec{F}dt = \vec{L}_2 - \vec{L}_1$;
- Angular momentum balance: $\int_0^t \vec{r}\times\vec{F}\,dt = \vec{H}_2 - \vec{H}_1$;
- Work-energy: $\int_0^t \vec{F}\cdot\vec{v}\,dt = E_{K_2} - E_{K_1}$.

12.3 Conservative forces and non-conservative forces

Imagine that the force $\vec{F}$ on a particle is known to depend on the position $\vec{r}$ of a particle as it moves. This dependence of $\vec{F}$ on $\vec{r}$ is called a *force field*:

$$\vec{F} = \vec{F}(\vec{r}).$$

As the particle moves from one point $\vec{r}_1$ to another $\vec{r}_2$ we can evaluate the work of this force field as

$$W_{12} = \int_{t_1}^{t_2} P\,dt = \int_{t_1}^{t_2} \vec{F}\cdot\vec{v}\,dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F}(\vec{r})\cdot d\vec{r}.$$

But what if the particle moves between the same two points but along a different path, is the work W_{12} the same? If that is true then the work going from $\vec{r}_1$ to $\vec{r}_2$ and back would be zero. Which means the work of the force when the particle moves on any closed path would be zero. Here is an example of a force field $\vec{F}(\vec{r})$ in the xy plane where the work in going on a closed path, from $\vec{r}_1$ to $\vec{r}_1$, from home to home, is not zero:

$$\vec{F} = C[\hat{k}\times\vec{r}] = C[-y\hat{\imath} + x\hat{\jmath}]$$

where C is a constant. This force pushes the particle around in circles. So, if the particle moves on the circular path

$$\vec{r} = r_0[\cos\theta\hat{\imath} + \sin\theta\hat{\jmath}] \qquad (0 \le \theta \le 2\pi)$$

then the work is the force magnitude times the arc-length (the force is parallel to the velocity for this path) and so,

$$\int_{\theta_1}^{\theta_2} \vec{F}\cdot d\vec{r} = 2\pi C r_0^2 \neq 0.$$

This force field gives non-zero work for some closed paths, thus is path dependent for open paths and therefore is *non-conservative*. How can you tell if a force field is conservative or not. This, you learn in vector calculus, holds if the curl of $\vec{F}$ is zero, $\vec{\nabla}\times\vec{F} = \vec{0}$, everywhere.

Forces from any combination of springs and gravity are always conservative.

These have been used in the simple ballistics example above. That linear momentum balance, angular momentum balance and energy balance, all are consistent to an assumed solution lends credence to its correctness. For simple problems with such simple analytical solutions, using this consistency is not the most efficient way of checking a candidate solution's veracity. We would be better off just plugging the proposed solution back into the differential equation to see if it was satisfied. But in more complex problems and in numerical solutions, checks like those here are sometimes simpler to make. Some more comments about these checks:

- The angular momentum check can be used relative to any fixed point you choose. If you can find a point where, say, the applied force has no moment, then the change of angular momentum should be zero about that point.
- If the applied force is conservative, the work integral can be replaced by the change in potential energy and the work-energy check is a check of the conservation of energy.
- If you try to make the checks with pencil and paper the checks can sometimes be harder to implement than it was to find the original solution.
- These checks are often *very useful*, and this is perhaps an understatement, for checking the validity of numerical solutions of dynamics equations. Basically you shouldn't trust yours or any body else's code unless such checks have been made. It is hard to write correct code without making such checks. And such checks are a strong sign of code reliability because an error in computer code will usually lead to an error in momentum balance, angular momentum balance or energy balance.

12.4 Derivation of momentum, angular momentum and energy theorems for a point mass

For a point-mass particle the principles of linear momentum, angular momentum and energy are theorems that can be derived simply from

$$\vec{F} = m\vec{a}$$

as follows.

Linear momentum

Define linear momentum as $\vec{L} = m\vec{v}$ then differentiating we have the equation $\vec{F} = m\vec{a}$. It is not so much a derivation but a restatement to write:

$$\vec{F} = \dot{\vec{L}}.$$

Integrating both sides in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F}\, dt}_{\text{impulse}} = \underbrace{\vec{L}_2 - \vec{L}_1}_{\text{change of momentum}} .$$

This is the principle of impulse and momentum.

Angular momentum

Start with $\vec{F} = m\vec{a}$ and take the cross product of both sides with the position relative to a fixed point C and you get

$$\vec{r}_{/C} \times \vec{F} = \vec{r}_{/C} \times (m\vec{a}).$$

Now if we define $\vec{H}_{/C} = \vec{r}_{/C} \times (m\vec{v})$ we can differentiate to find that, writing out all details,

$$\begin{aligned}
\dot{\vec{H}}_{/C} &= \frac{d}{dt}\left(\vec{r}_{/C} \times (m\vec{v})\right) \\
&= \frac{d}{dt}\left((\vec{r} - \vec{r}_C) \times m\vec{v}\right) \\
&= m\left(\dot{\vec{r}} - \dot{\vec{r}}_C\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= m\left(\vec{v} - \vec{0}\right) \times \vec{v} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= m\underbrace{\vec{v} \times \vec{v}}_{\vec{0}} + m\left(\vec{r} - \vec{r}_C\right) \times \dot{\vec{v}} \\
&= \vec{r}_{/C} \times (m\dot{\vec{v}}) \\
&= \vec{r}_{/C} \times (m\vec{a})
\end{aligned}$$

Putting these together we have

$$\vec{r}_{/C} \times \vec{F} = \dot{\vec{H}}_{/C}.$$

Integrating both sides with respect to time we get that the net angular impulse is the change in angular momentum.

$$\int_{t_1}^{t_2} \vec{r}_{/C} \times \vec{F}\, dt = (\vec{H}_{/C})_2 - (\vec{H}_{/C})_1.$$

Power and kinetic energy

The power equation is found with a shade more difficulty. We take the equation $\vec{F} = m\vec{a}$ and dot both sides with the velocity $\vec{v}$ of the particle:

$$\vec{F} \cdot \vec{v} = m\vec{a} \cdot \vec{v}. \qquad (12.10)$$

Evaluating $\vec{v} \cdot \vec{a}$ is most easily done with the benefit of hindsight. So we cheat and look at the time derivative of the speed squared:

$$\begin{aligned}
\frac{d}{dt}\left(\frac{1}{2}v^2\right) &= \frac{1}{2}\frac{d}{dt}(\vec{v} \cdot \vec{v}) \\
&= \frac{1}{2}\left(\dot{\vec{v}} \cdot \vec{v} + \vec{v} \cdot \dot{\vec{v}}\right) \\
&= \vec{v} \cdot \dot{\vec{v}} \\
&= \vec{v} \cdot \vec{a}
\end{aligned}$$

Applying this result to eqn. (12.10) we get

$$\underbrace{\vec{F} \cdot \vec{v}}_{P} = \frac{d}{dt}\underbrace{\left(\frac{1}{2}mv^2\right)}_{E_K},$$

the energy (or power balance) equation for a particle.

Integrating in time we get

$$\underbrace{\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt}_{\text{Work}} = \underbrace{E_{K2} - E_{K1}}_{\text{Change in kinetic energy}} ,$$

Power and work and energy

Because $\vec{F} \cdot \vec{v}\, dt = \vec{F} \cdot d\vec{r}$ the time integral of power can be replaced with a path integral, the standard work integral:

$$\int_{t_1}^{t_2} \vec{F} \cdot \vec{v}\, dt = \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r}.$$

If $\vec{F}$ is a conservative force field, meaning a function of position, then $E_P(\vec{r})$ exists, so that

$$-\vec{\nabla} E_P = \vec{F} \quad \text{then} \quad \int_{\vec{r}_1}^{\vec{r}_2} \vec{F} \cdot d\vec{r} = E_P(\vec{r}_1) - E_P(\vec{r}_2)$$

and the work-energy equation becomes

$$E_2 = E_1$$

Where $E = E_K + E_P$ is defined as the total energy.