12.1 Dynamics of a particle in space

Time derivative of a vector: position, velocity and acceleration

From here to the end of the book most of our calculations will involve vector-valued functions of time. For example, the vectors linear momentum $\vec{\boldsymbol{L}}$ and angular momentum $\vec{\boldsymbol{H}}$ have a central place in mechanics. Evaluating them depends, in turn, on understanding the relation between position $\vec{\boldsymbol{r}}$, and its rate of change, called velocity $\vec{\boldsymbol{v}}$. We also need to know the relation between velocity $\vec{\boldsymbol{v}}$ and its rate of change, the acceleration $\vec{\boldsymbol{a}}$.

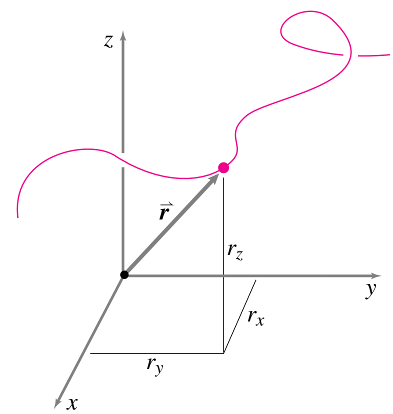

Figure 12.5: The path of a particle and its position at one time projected onto Cartesian coordinates. The path, or trajectory, of the particle is the sequence of locations of the tip of the $\vec{\boldsymbol{r}}$ vector, if the vector is drawn with its tail at the origin.

What do we mean by the rate of change of a vector? The rate of change of any quantity, including a vector, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ①

$$\text{rate of change of any } (\textit{thing}) \equiv \frac{\text{amount } \textit{thing} \text{ changes}}{\text{amount of time for that change}}$$

① As you should remember from Calculus, these words really describe the average rate of change over the time interval. Only in the mathematical limit, as the time interval approaches zero, is the ratio of "amount of change over the time interval" not just approximately, but exactly, the instantaneous rate of change.

The notation for the rate of change of a vector $\vec{\boldsymbol{r}}$ is

$$\frac{d\vec{\boldsymbol{r}}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}}$. The definition of the derivative $\frac{d\vec{\boldsymbol{r}}}{dt}$ or $\dot{\vec{\boldsymbol{r}}}$ is the same as for any*thing* else,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta\vec{\boldsymbol{r}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{r}}(t+\Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t}$$

where the top of the fraction is the change in $\vec{\boldsymbol{r}}$ and the bottom is the change in t. Underlying most dynamics calculations are derivatives of $\vec{\boldsymbol{r}}(t)$. But we also sometimes need to take derivatives of linear momentum $\vec{\boldsymbol{L}}$, angular momentum $\vec{\boldsymbol{H}}$ and some other quantities (e.g., the angular velocity $\vec{\boldsymbol{\omega}}$ of a rigid object). But all of these quantities somehow depend on the derivatives of $\vec{\boldsymbol{r}}(t)$.

Cartesian coordinates

A simple way to think about vector derivatives is with cartesian coordinates. A moving point has a location $\vec{\boldsymbol{r}}$, relative to the origin of a 'good' (i.e., $\mathcal{N}$ewtonian) reference frame as shown in fig. 12.5, which can be written as:

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} \qquad \text{or} \qquad \vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

So velocity is the derivative of $\vec{\boldsymbol{r}}$. Since the base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}.$$

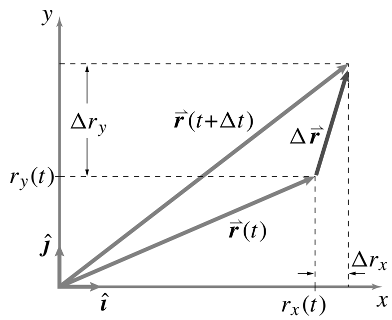

Figure 12.6: Change of position in Δt broken into components in 2-D . $\Delta\vec{r}$ is $\vec{r}(t+\Delta t)-\vec{r}(t)$. $\Delta\vec{r}$ has components Δr_x and Δr_y. So $\Delta\vec{r} = \Delta r_x\hat{\imath} + \Delta r_y\hat{\jmath}$. In the limit as Δt goes to zero, $\dot{\vec{r}}$ is the ratio of $\Delta\vec{r}$ to Δt.

The idea is illustrated in fig. 12.6. Let's take

$$\vec{r} = r_x\hat{\imath} + r_y\hat{\jmath} \quad \text{or} \quad \vec{r}(t) = r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}.$$

We can apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{r}}(t) &= \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} \\
&= \frac{\overbrace{\left(r_x(t+\Delta t)\hat{\imath} + r_y(t+\Delta t)\hat{\jmath}\right)}^{\vec{r}(t+\Delta t)} - \overbrace{\left(r_x(t)\hat{\imath} + r_y(t)\hat{\jmath}\right)}^{\vec{r}(t)}}{\Delta t} \\
&= \frac{r_x(t+\Delta t)-r_x(t)}{\Delta t}\hat{\imath} + \frac{r_y(t+\Delta t)-r_y(t)}{\Delta t}\hat{\jmath} \\
&= \dot{r}_x(t)\hat{\imath} + \dot{r}_y(t)\hat{\jmath}
\end{aligned}$$

thus showing the palatable result that

> the components of the velocity vector are the time derivatives of the components of the position vector.

(**Beware.** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components. See box 12.2.)

Figure 12.7 shows a particle P's path, its position at a sequence of times. The position vector $\vec{r}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{v}$ at time t is the rate of change of position at that time, $\vec{v} \equiv \dot{\vec{r}}$.

Example: Given position as a function of time, find the velocity.
Given that the position of a point is:

$$\vec{r}(t) = C_1\cos(\omega\, t)\hat{\imath} + C_2\sin(\omega\, t)\hat{\jmath}$$

with C_1, C_2 and ω given constants what is the velocity (a vector) at a given time t?

First we note that the components of $\vec{r}(t)$ have been given implicitly as

$$r_x(t) = C_1\cos(\omega\, t) \qquad \text{and} \qquad r_y(t) = C_2\sin(\omega\, t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{v}(t) = \dot{\vec{r}} = -C_1\omega\sin(\omega\, t)\hat{\imath} + C_2\omega\cos(\omega\, t)\hat{\jmath}$$

Now we evaluate this expression with the given values of C_1, C_2, ω and t .

Are position, velocity and acceleration all parallel? Sometimes this is a right intuition. For example, after some time has passed the change in position is exactly the average velocity times the elapsed time. And the change in velocity is exactly the average acceleration times the elapsed time. So in the long run, if something accelerates in some more-or-less constant direction then the position will change in that same direction. But actually, at any instant in time, position, velocity and acceleration are basically unrelated.

Example: Position, velocity and acceleration can be mutually orthogonal.
Here is a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal as in fig. 12.8. For example, look at the path in fig. 12.9. At the point where the path intersects the y axis the position relative to the origin is in the $\hat{\boldsymbol{\jmath}}$ direction, the velocity is tangent to the path in the $\hat{\boldsymbol{\imath}}$ direction and the acceleration is at least partially up, in the $\hat{\boldsymbol{k}}$ direction. Working this out with equations, if we take the position as a function of time to be

$$\vec{\boldsymbol{r}}(t) = A\hat{\boldsymbol{\jmath}} - Bt\hat{\boldsymbol{\imath}} + Ct^2\hat{\boldsymbol{k}}$$

we can calculate the velocity and acceleration by differentiation as

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = -B\hat{\boldsymbol{\imath}} + 2Ct\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = 2C\hat{\boldsymbol{k}}.$$

So, at $t = 0$,

$$\vec{\boldsymbol{r}} = A\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{v}} = -B\hat{\boldsymbol{\imath}}, \quad \text{and} \quad \vec{\boldsymbol{a}} = 2C\hat{\boldsymbol{k}}.$$

The dot products between $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ are: $\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{v}} = 0$, $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}} = 0$, and $\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{a}} = 0$, so these vectors are mutually orthogonal at the instant marked. (**Aside:** Why is there a $-B$ in this example? Answer: no reason, we could have used $+B$ just as well.)

In constant rate circular motion position (relative to the circle's center) and velocity remain perpendicular for all time, and so do velocity and acceleration. However, the directions of position, velocity and acceleration are not arbitrary. For example, there is no motion where position, velocity and acceleration are exactly mutually orthogonal for an extended time. Imagine a slender circular cone. If position is measured relative to the apex of the cone then constant-rate circular motion about the base of the cone *almost* has position, velocity and acceleration mutually orthogonal for all time. But position and acceleration are only exactly orthogonal in the limit as the cone becomes infinitely slender.

The product rule of differentiation

We know three ways to multiply vectors: multiplying a vector by a scalar, taking the dot product of two vectors, and taking the cross product of two vectors (please review Chapter 2). Because all of these quantities might be functions of time we need to know how to differentiate products. It's simple. All three kinds of vector multiplication obey 'the product rule'

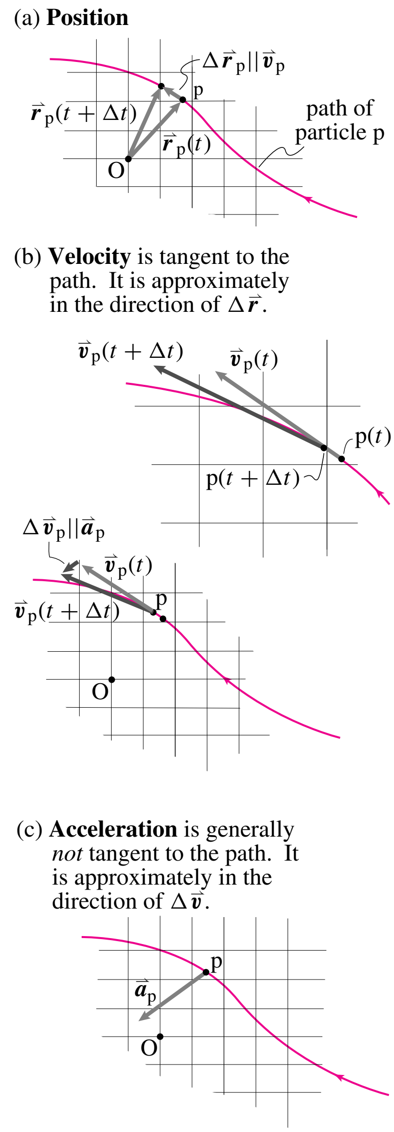

Figure 12.7: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t + \Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

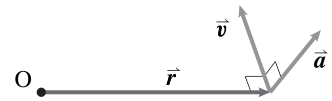

Figure 12.8: It is possible for position, velocity and acceleration to be mutually orthogonal.

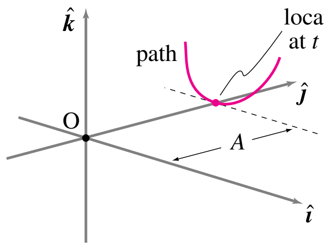

Figure 12.9: A path where, at the marked dot, the position (relative to the origin), velocity and acceleration can be mutually orthogonal: Position is in the $\hat{\jmath}$ direction, velocity is in the $\hat{\imath}$ direction and (if the particle moves at constant speed), the acceleration is in the $\hat{k}$ direction.

that you learned in freshman calculus.

$$\begin{aligned}\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}}\\ \frac{d}{dt}(\vec{A}\cdot\vec{B}) &= \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}}\\ \frac{d}{dt}(\vec{A}\times\vec{B}) &= \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}.\end{aligned}$$

The proofs of these identities are the same as the proof used for scalar multiplication, it follows from the definition of derivative (above) and the elimination of terms with Δ^2 as negligible compared to terms with just Δ in the limit $\Delta \to 0$.

Example: Derivative of a vector of constant length.
Assume a vector $\vec{C}$ has constant length so

$$|\vec{C}| = \text{constant} \qquad \text{and} \qquad |\vec{C}|^2 = \text{constant}$$

so, differentiating and using the product rule, working from left to right:

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C}\cdot\vec{C}) = \vec{C}\cdot\dot{\vec{C}} + \dot{\vec{C}}\cdot\vec{C} = 2\vec{C}\cdot\dot{\vec{C}} = 0.$$

Because $\vec{C}\cdot\dot{\vec{C}} = 0$ we then know that $\vec{C} \perp \dot{\vec{C}}$. That is, for any vector $\vec{C}$ that has constant length, its rate of change is perpendicular to itself. This is a useful fact to remember about time-varying constant-magnitude vectors, especially time-varying unit vectors.

To make this more intuitive, imagine a dog on a taut fixed-length leash anchored to the ground. The length of the leash is the magnitude $|\vec{C}|$ of the position vector $\vec{C}$, from ground-to-neck, and is constant. So our result is obvious, the neck can only move with a velocity $\dot{\vec{C}}$ that is tangent to the circle that the neck moves on because the tangent of a circle is orthogonal to the radius.

In 3D, space-dogs on taut leashes can only move tangent to the sphere they are stuck on $|\vec{C}| = \text{constant} \quad \Rightarrow \quad \vec{C}\cdot\dot{\vec{C}} = 0 \quad \Rightarrow \quad \vec{C} \perp \dot{\vec{C}}$. And, intuitively again, all tangents to the surface of a sphere are orthogonal to the radius of the sphere at that point.

Dynamics in space

Isaac Newton wondered how the planets move around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning, he learned a lot about the moon and the planets. After you learn the material in this section you will know enough to reproduce many of Newton's calculations. You won't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same computer approach to find motions that Newton could never find, say the trajectory of a projectile with a realistic model of air friction. In this chapter, the basic recipe is this: ①

① Eventually you may gain the math skills to shortcut this brute-force numerical approach, at least for some simple problems. But for most problems, even math geniuses use the numerical approach here.

Write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ and solve the equations.

In some sense it's that simple.

A sure-fire recipe. Here's how to find the motion of a particle:

1. Draw a free-body diagram of the particle,
2. Find the forces on the particle in terms of its position, velocity and time. External forces (external forces might come, for example, from a spring, dashpot, gravity, or air friction),
3. Write the linear momentum balance equation for the particle (translation: write $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$).
4. Break the vector equation into components to make 2 or 3 2nd order scalar ODEs, in 2 or 3 dimensions, respectively.
5. Write the 2 or 3 2nd order ODEs in first order form. You now have 4 or 6 first order ordinary differential equations (for a 2 or 3 dimensional problem, respectively).
6. Write these first order equations in standard form, with all the time derivatives on the left hand side.
7. Feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
8. Plot some aspect(s) of the solution and
   a) Use the solution to help you find errors in your formulation, and
   b) Interpret the solution so that it makes sense to you and increases your understanding of the system of study.

Instantaneous dynamics. Some problems are even easier, problems of the *instantaneous dynamics* type. They use the equations of dynamics but do not track the motion over time.

Example: Knowing the forces find the acceleration.
Say you know the forces on a particle at some instant in time, say $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{\boldsymbol{F}}_i = m\vec{\boldsymbol{a}} \quad \Rightarrow \quad \vec{\boldsymbol{a}} = \frac{\vec{\boldsymbol{F}}_1 + \vec{\boldsymbol{F}}_2}{m}$$

Sometimes this 'instantaneous' dynamics, with the motion given and the forces to be determined, is called *'inverse dynamics'*. The inside back cover of the book compares the solution methods for instantaneous dynamics to those where differential equations need be solved.

Analytic solution. Some problems involving motion are simple and you can determine almost all you want to know with pencil and paper. You can bypass the whole computer recipe above.

Example: Parabolic trajectory of a projectile
If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{\jmath}}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}.$$

Taking the dot product of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (equivalent to taking the x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{\boldsymbol{r}} = (A + Bt)\hat{\boldsymbol{\imath}} + (C + Dt - gt^2/2)\hat{\boldsymbol{\jmath}}$$

which is a parametric description of all possible trajectories. By making plots or by using simple algebra you could convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.

Some other special problems turn out to be easy, although you might not recognize such problems at first glance.

Example: Mass tethered by a zero-length spring
Imagine a massless spring whose un-stretched length is zero (See page 382 in section 7.1 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}}$. Thus linear momentum balance yields

$$-k\vec{\boldsymbol{r}} = m\vec{\boldsymbol{a}}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{\boldsymbol{r}} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\imath}} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\jmath}}$$

which is always an ellipse (special cases of which are a circle and a straight line).

Even with gravity and springs together some (only some) problems are easy.

Example: Mass hanging from a zero-length spring
If gravity in the $-\hat{\boldsymbol{\jmath}}$ direction is included in the above problem the solution is only changed by the addition of a constant to be:

$$\vec{\boldsymbol{r}}_{\text{with gravity}} = \vec{\boldsymbol{r}}_{\text{previous example}} - (mg/k)\hat{\boldsymbol{\jmath}}$$

Analytic methods sometimes just can't do the job. Some problems are hard and can't be solved without a computer.

Example: Trajectory with quadratic air drag.
For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{F} = -mg\hat{\jmath} - Cv^2(\vec{v}/v)$, where $\vec{v}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{\jmath} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations for direct numerical solution.

On the edge. Some problems are within the reach of advanced analytic methods, but might be easier to solve with a computer.

Example: Path of the earth around the sun.
Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$-\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it but many of us find it too tricky. On the other hand the equations of motion for planetary trajectories are easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does around the sun, what the moon does around the earth, and what most artificial satellites do around the earth.

Summary

If, given the time, the particle's position and the particle's velocity, you know the force on a particle, then you know $\vec{F}(t, \vec{r}, \vec{v})$ ②. That means you can write $\vec{F} = m\vec{a}$ as

$$\vec{a} = \vec{F}(t, \vec{r}, \vec{v})/m$$

where $\vec{F}$ is known. This can, in turn, be written as two vector first order equations

$$\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{F}(t, \vec{r}, \vec{v})/m. \end{aligned} \qquad (12.2)$$

② Typically you would know this because an applied force would vary in time in a known way (the dependence on t), gravity and spring forces would vary with position in a known way (dependence on r), and you would know forces due to various friction (dependence on $\vec{v}$).

which are equivalent, written out long hand, to the 6 first order equations

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{z} &= v_z \\ \dot{v}_x &= F_x(t,x,y,z,v_x,v_y,v_z)/m \\ \dot{v}_y &= F_y(t,x,y,z,v_x,v_y,v_z)/m \\ \dot{v}_z &= F_z(t,x,y,z,v_x,v_y,v_z)/m.\end{aligned} \tag{12.3}$$

Given the position and velocity at some starting time, these equations can be integrated, sometimes by hand but generally on the computer, to give position and velocity as a function of time.

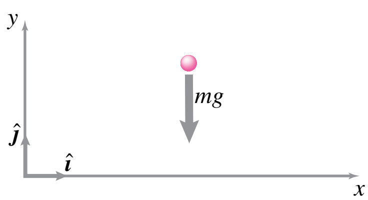

Figure 12.10: Free-body diagram of a particle in flight, neglecting air friction.

Example: Simple ballistics.
This is an example that can be solved with pencil and paper. A computer is not needed. It is the classic from high school and freshman physics. A particle has only one force on it, gravity. A free-body diagram is shown in fig. 12.10. Linear momentum balance gives

$$\begin{aligned}\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\ -mg\hat{\boldsymbol{\jmath}} &= m\vec{\boldsymbol{a}} \\ \vec{\boldsymbol{a}} &= -g\hat{\boldsymbol{\jmath}}\end{aligned}$$

So

$$\begin{aligned}\dot{x} &= v_x \\ \dot{y} &= v_y \\ \dot{v}_x &= 0 \\ \dot{v}_y &= -g\end{aligned}$$

Integrating the last two of these equations and plugging the result into the first two we get:

$$\begin{aligned}\vec{\boldsymbol{v}} &= v_{0x}\hat{\boldsymbol{\imath}} + (v_{0y} - gt)\hat{\boldsymbol{\jmath}} \\ \vec{\boldsymbol{r}} &= (x_0 + v_{0x}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{0y}t - gt^2/2)\hat{\boldsymbol{\jmath}}\end{aligned}$$

This solution is plotted various ways in fig. 12.22 on page 626.

More complicated examples are given in the samples on the following pages.

12.2 The rate of change of a vector depends on reference frame

The time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

As noted earlier, but for the special case of one frame moving at constant velocity (without rotation) with respect to another, the rate of change of a given vector is different if calculated in different reference frames.

For mechanics we have to differentiate vectors with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being fussy about notation we will sometimes write

$$\overset{\mathcal{F}}{\dot{\boldsymbol{r}}}_{\mathrm{B/O}} = \text{The velocity of point B as calculated in frame } \mathcal{F}.$$

Non-Newtonian frames

Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are a useful aid with the understanding of the motion of and forces on systems composed of objects with complex relative motion. So eventually we need to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. Such non-Newtonian frames will be discussed in later chapters.