12.1 Dynamics of a particle in space

Time derivative of a vector: position, velocity and acceleration

From here to the end of the book most of our calculations will involve vector-valued functions of time. For example, the vectors linear momentum $\vec{\boldsymbol{L}}$ and angular momentum $\vec{\boldsymbol{H}}$ have a central place in mechanics. Evaluating them depends, in turn, on understanding the relation between position $\vec{\boldsymbol{r}}$, and its rate of change, called velocity $\vec{\boldsymbol{v}}$. We also need to know the relation between velocity $\vec{\boldsymbol{v}}$ and its rate of change, the acceleration $\vec{\boldsymbol{a}}$.

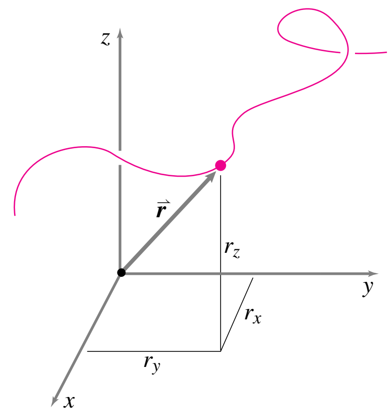


Figure 12.5: The path of a particle and its position at one time projected onto Cartesian coordinates. The path, or trajectory, of the particle is the sequence of locations of the tip of the $\vec{\boldsymbol{r}}$ vector, if the vector is drawn with its tail at the origin.

What do we mean by the rate of change of a vector? The rate of change of any quantity, including a vector, is the ratio of the change of that quantity to the amount of time that passes, for very small amounts of time. ①

$$\text{rate of change of any } (thing) \equiv \frac{\text{amount } thing \text{ changes}}{\text{amount of time for that change}}$$

①As you should remember from Calculus, these words really describe the average rate of change over the time interval. Only in the mathematical limit, as the time interval approaches zero, is the ratio of "amount of change over the time interval" not just approximately, but exactly, the instantaneous rate of change.

The notation for the rate of change of a vector $\vec{\boldsymbol{r}}$ is

$$\frac{d\vec{\boldsymbol{r}}}{dt}.$$

Or, in the short hand 'dot' notation invented by Newton for just this purpose, $\vec{\boldsymbol{v}} = \dot{\vec{\boldsymbol{r}}}$. The definition of the derivative $\frac{d\vec{\boldsymbol{r}}}{dt}$ or $\dot{\vec{\boldsymbol{r}}}$ is the same as for any*thing* else,

$$\frac{d\vec{\boldsymbol{r}}}{dt} = \lim_{\Delta t \to 0} \frac{\Delta \vec{\boldsymbol{r}}}{\Delta t} = \lim_{\Delta t \to 0} \frac{\vec{\boldsymbol{r}}(t + \Delta t) - \vec{\boldsymbol{r}}(t)}{\Delta t}$$

where the top of the fraction is the change in $\vec{\boldsymbol{r}}$ and the bottom is the change in t. Underlying most dynamics calculations are derivatives of $\vec{\boldsymbol{r}}(t)$. But we also sometimes need to take derivatives of linear momentum $\vec{\boldsymbol{L}}$, angular momentum $\vec{\boldsymbol{H}}$ and some other quantities (e.g., the angular velocity $\vec{\boldsymbol{\omega}}$ of a rigid object). But all of these quantities somehow depend on the derivatives of $\vec{\boldsymbol{r}}(t)$.

Cartesian coordinates

A simple way to think about vector derivatives is with cartesian coordinates. A moving point has a location $\vec{\boldsymbol{r}}$, relative to the origin of a 'good' (i.e., $\mathcal{N}$ewtonian) reference frame as shown in fig. 12.5, which can be written as:

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} + r_z\hat{\boldsymbol{k}} \qquad \text{or} \qquad \vec{\boldsymbol{r}} = x\hat{\boldsymbol{\imath}} + y\hat{\boldsymbol{\jmath}} + z\hat{\boldsymbol{k}}.$$

So velocity is the derivative of $\vec{\boldsymbol{r}}$. Since the base vectors $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$, and $\hat{\boldsymbol{k}}$ are constant, differentiation to get velocity and acceleration is simple:

$$\vec{\boldsymbol{v}} = \dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}} + \dot{z}\hat{\boldsymbol{k}} \qquad \text{and} \qquad \vec{\boldsymbol{a}} = \ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}} + \ddot{z}\hat{\boldsymbol{k}}.$$

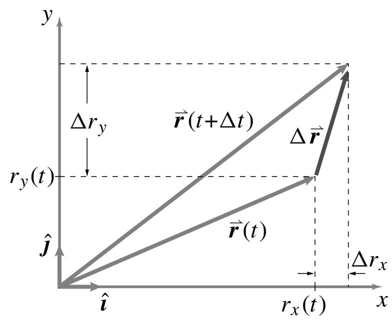


Figure 12.6: Change of position in Δt broken into components in 2-D . $\Delta\vec{\boldsymbol{r}}$ is $\vec{\boldsymbol{r}}(t+\Delta t)-\vec{\boldsymbol{r}}(t)$. $\Delta\vec{\boldsymbol{r}}$ has components Δr_x and Δr_y. So $\Delta\vec{\boldsymbol{r}} = \Delta r_x\hat{\boldsymbol{\imath}} + \Delta r_y\hat{\boldsymbol{\jmath}}$. In the limit as Δt goes to zero, $\dot{\vec{\boldsymbol{r}}}$ is the ratio of $\Delta\vec{\boldsymbol{r}}$ to Δt.

The idea is illustrated in fig. 12.6. Let's take

$$\vec{\boldsymbol{r}} = r_x\hat{\boldsymbol{\imath}} + r_y\hat{\boldsymbol{\jmath}} \quad \text{or} \quad \vec{\boldsymbol{r}}(t) = r_x(t)\hat{\boldsymbol{\imath}} + r_y(t)\hat{\boldsymbol{\jmath}}.$$

We can apply the definition of derivative and find

$$\begin{aligned}
\dot{\vec{\boldsymbol{r}}}(t) &= \lim_{\Delta t\to 0}\frac{\vec{\boldsymbol{r}}(t+\Delta t)-\vec{\boldsymbol{r}}(t)}{\Delta t}\\
&= \frac{\overbrace{\Big(r_x(t+\Delta t)\hat{\boldsymbol{\imath}} + r_y(t+\Delta t)\hat{\boldsymbol{\jmath}}\Big)}^{\vec{\boldsymbol{r}}(t+\Delta t)} - \overbrace{\Big(r_x(t)\hat{\boldsymbol{\imath}} + r_y(t)\hat{\boldsymbol{\jmath}}\Big)}^{\vec{\boldsymbol{r}}(t)}}{\Delta t}\\
&= \frac{r_x(t+\Delta t)-r_x(t)}{\Delta t}\hat{\boldsymbol{\imath}} + \frac{r_y(t+\Delta t)-r_y(t)}{\Delta t}\hat{\boldsymbol{\jmath}}\\
&= \dot{r}_x(t)\hat{\boldsymbol{\imath}} + \dot{r}_y(t)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

thus showing the palatable result that

> the components of the velocity vector are the time derivatives of the components of the position vector.

(**Beware.** Later in the book we will use base vectors that change in time, such as polar coordinate base vectors, path basis vectors, or basis vectors attached to a rotating frame. For these vectors the components of the vector's derivative will *not* be the derivatives of its components. See box 12.2.)

Figure 12.7 shows a particle P's path, its position at a sequence of times. The position vector $\vec{\boldsymbol{r}}_{P/O}$ is the arrow from the origin to a point on the curve, a different point on the curve at each instant of time. The velocity $\vec{\boldsymbol{v}}$ at time t is the rate of change of position at that time, $\vec{\boldsymbol{v}} \equiv \dot{\vec{\boldsymbol{r}}}$.

Example: Given position as a function of time, find the velocity.
Given that the position of a point is:

$$\vec{\boldsymbol{r}}(t) = C_1\cos(\omega\, t)\hat{\boldsymbol{\imath}} + C_2\sin(\omega\, t)\hat{\boldsymbol{\jmath}}$$

with C_1, C_2 and ω given constants what is the velocity (a vector) at a given time t?

First we note that the components of $\vec{\boldsymbol{r}}(t)$ have been given implicitly as

$$r_x(t) = C_1\cos(\omega\, t) \qquad \text{and} \qquad r_y(t) = C_2\sin(\omega\, t).$$

Then we find the velocity by differentiating each of the components with respect to time and re-assembling as a vector to get

$$\vec{\boldsymbol{v}}(t) = \dot{\vec{\boldsymbol{r}}} = -C_1\omega\sin(\omega\, t)\hat{\boldsymbol{\imath}} + C_2\omega\cos(\omega\, t)\hat{\boldsymbol{\jmath}}$$

Now we evaluate this expression with the given values of C_1, C_2, ω and t .

Are position, velocity and acceleration all parallel? Sometimes this is a right intuition. For example, after some time has passed the change in position is exactly the average velocity times the elapsed time. And the change in velocity is exactly the average acceleration times the elapsed time. So in the long run, if something accelerates in some more-or-less constant direction then the position will change in that same direction. But actually, at any instant in time, position, velocity and acceleration are basically unrelated.

Example: Position, velocity and acceleration can be mutually orthogonal.
Here is a motion where, at least at one instant in time, the position, velocity, and acceleration are mutually orthogonal as in fig. 12.8. For example, look at the path in fig. 12.9. At the point where the path intersects the y axis the position relative to the origin is in the $\hat{\boldsymbol{\jmath}}$ direction, the velocity is tangent to the path in the $\hat{\boldsymbol{\imath}}$ direction and the acceleration is at least partially up, in the $\hat{\boldsymbol{k}}$ direction. Working this out with equations, if we take the position as a function of time to be

$$\vec{\boldsymbol{r}}(t) = A\hat{\boldsymbol{\jmath}} - Bt\hat{\boldsymbol{\imath}} + Ct^2\hat{\boldsymbol{k}}$$

we can calculate the velocity and acceleration by differentiation as

$$\vec{\boldsymbol{v}} = \frac{d\vec{\boldsymbol{r}}}{dt} = -B\hat{\boldsymbol{\imath}} + 2Ct\hat{\boldsymbol{k}}, \quad \vec{\boldsymbol{a}} = \frac{d\vec{\boldsymbol{v}}}{dt} = 2C\hat{\boldsymbol{k}}.$$

So, at $t = 0$,

$$\vec{\boldsymbol{r}} = A\hat{\boldsymbol{\jmath}}, \quad \vec{\boldsymbol{v}} = -B\hat{\boldsymbol{\imath}}, \quad \text{and} \quad \vec{\boldsymbol{a}} = 2C\hat{\boldsymbol{k}}.$$

The dot products between $\vec{\boldsymbol{r}}$, $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ are: $\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{v}} = 0$, $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = 0$, and $\vec{\boldsymbol{r}}\cdot\vec{\boldsymbol{a}} = 0$, so these vectors are mutually orthogonal at the instant marked. (**Aside:** Why is there a $-B$ in this example? Answer: no reason, we could have used $+B$ just as well.)

In constant rate circular motion position (relative to the circle's center) and velocity remain perpendicular for all time, and so do velocity and acceleration. However, the directions of position, velocity and acceleration are not arbitrary. For example, there is no motion where position, velocity and acceleration are exactly mutually orthogonal for an extended time. Imagine a slender circular cone. If position is measured relative to the apex of the cone then constant-rate circular motion about the base of the cone *almost* has position, velocity and acceleration mutually orthogonal for all time. But position and acceleration are only exactly orthogonal in the limit as the cone becomes infinitely slender.

The product rule of differentiation

We know three ways to multiply vectors: multiplying a vector by a scalar, taking the dot product of two vectors, and taking the cross product of two vectors (please review Chapter 2). Because all of these quantities might be functions of time we need to know how to differentiate products. It's simple. All three kinds of vector multiplication obey 'the product rule'

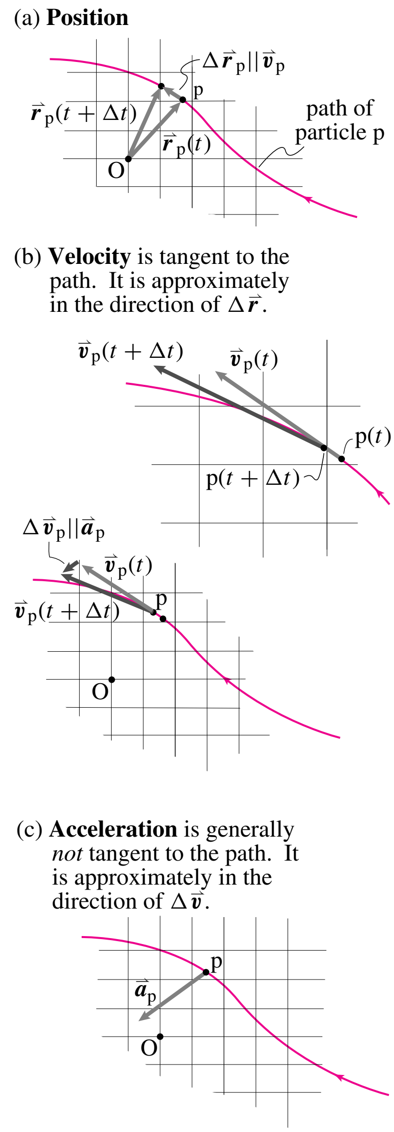


Figure 12.7: A particle moving on a curve. (a) shows the position vector is an arrow from the origin to the point on the curve. On the position curve the particle is shown at two times: t and $t + \Delta t$. The velocity at time t is roughly parallel to the difference between these two positions. The velocity is then shown at these two times in (b). The acceleration is roughly parallel to the difference between these two velocities. In (c) the acceleration is drawn on the path roughly parallel to the difference in velocities.

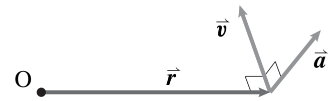


Figure 12.8: It is possible for position, velocity and acceleration to be mutually orthogonal.

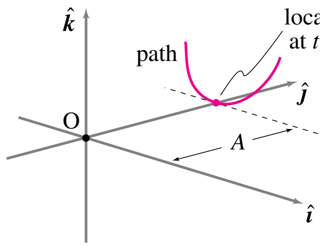


Figure 12.9: A path where, at the marked dot, the position (relative to the origin), velocity and acceleration can be mutually orthogonal: Position is in the $\hat{\jmath}$ direction, velocity is in the $\hat{\imath}$ direction and (if the particle moves at constant speed), the acceleration is in the $\hat{k}$ direction.

that you learned in freshman calculus.

$$\begin{aligned}
\frac{d}{dt}(a\vec{A}) &= \dot{a}\vec{A} + a\dot{\vec{A}} \\
\frac{d}{dt}(\vec{A}\cdot\vec{B}) &= \dot{\vec{A}}\cdot\vec{B} + \vec{A}\cdot\dot{\vec{B}} \\
\frac{d}{dt}(\vec{A}\times\vec{B}) &= \dot{\vec{A}}\times\vec{B} + \vec{A}\times\dot{\vec{B}}.
\end{aligned}$$

The proofs of these identities are the same as the proof used for scalar multiplication, it follows from the definition of derivative (above) and the elimination of terms with Δ^2 as negligible compared to terms with just Δ in the limit $\Delta \to 0$.

Example: Derivative of a vector of constant length.
Assume a vector $\vec{C}$ has constant length so

$$|\vec{C}| = \text{constant} \qquad \text{and} \qquad |\vec{C}|^2 = \text{constant}$$

so, differentiating and using the product rule, working from left to right:

$$\frac{d}{dt}|\vec{C}|^2 = \frac{d}{dt}(\vec{C}\cdot\vec{C}) = \vec{C}\cdot\dot{\vec{C}} + \dot{\vec{C}}\cdot\vec{C} = 2\vec{C}\cdot\dot{\vec{C}} = 0.$$

Because $\vec{C}\cdot\dot{\vec{C}} = 0$ we then know that $\vec{C} \perp \dot{\vec{C}}$. That is, for any vector $\vec{C}$ that has constant length, its rate of change is perpendicular to itself. This is a useful fact to remember about time-varying constant-magnitude vectors, especially time-varying unit vectors.

To make this more intuitive, imagine a dog on a taut fixed-length leash anchored to the ground. The length of the leash is the magnitude $|\vec{C}|$ of the position vector $\vec{C}$, from ground-to-neck, and is constant. So our result is obvious, the neck can only move with a velocity $\dot{\vec{C}}$ that is tangent to the circle that the neck moves on because the tangent of a circle is orthogonal to the radius.

In 3D, space-dogs on taut leashes can only move tangent to the sphere they are stuck on $|\vec{C}| = \text{constant} \quad \Rightarrow \quad \vec{C}\cdot\dot{\vec{C}} = 0 \quad \Rightarrow \quad \vec{C} \perp \dot{\vec{C}}$. And, intuitively again, all tangents to the surface of a sphere are orthogonal to the radius of the sphere at that point.

Dynamics in space

Isaac Newton wondered how the planets move around the sun. By applying his equation $\vec{F} = m\vec{a}$, his law of gravitation, his calculus, and his inimitable geometric reasoning, he learned a lot about the moon and the planets. After you learn the material in this section you will know enough to reproduce many of Newton's calculations. You won't need to be a Newton-like genius to solve Newton's differential equations. You can solve them on a computer. And you can use the same computer approach to find motions that Newton could never find, say the trajectory of a projectile with a realistic model of air friction. In this chapter, the basic recipe is this: ①

① Eventually you may gain the math skills to shortcut this brute-force numerical approach, at least for some simple problems. But for most problems, even math geniuses use the numerical approach here.

Write $\vec{F} = m\vec{a}$ and solve the equations.

In some sense it's that simple.

A sure-fire recipe. Here's how to find the motion of a particle:

1. Draw a free-body diagram of the particle,
2. Find the forces on the particle in terms of its position, velocity and time. External forces (external forces might come, for example, from a spring, dashpot, gravity, or air friction),
3. Write the linear momentum balance equation for the particle (translation: write $\vec{F} = m\vec{a}$).
4. Break the vector equation into components to make 2 or 3 2nd order scalar ODEs, in 2 or 3 dimensions, respectively.
5. Write the 2 or 3 2nd order ODEs in first order form. You now have 4 or 6 first order ordinary differential equations (for a 2 or 3 dimensional problem, respectively).
6. Write these first order equations in standard form, with all the time derivatives on the left hand side.
7. Feed these equations to the computer, substituting values for the various parameters and appropriate initial conditions.
8. Plot some aspect(s) of the solution and
   a) Use the solution to help you find errors in your formulation, and
   b) Interpret the solution so that it makes sense to you and increases your understanding of the system of study.

Instantaneous dynamics. Some problems are even easier, problems of the *instantaneous dynamics* type. They use the equations of dynamics but do not track the motion over time.

Example: Knowing the forces find the acceleration.
Say you know the forces on a particle at some instant in time, say $\vec{F}_1$ and $\vec{F}_2$, and you just want to know the acceleration at that instant. The answer is given directly by linear momentum balance as

$$\sum \vec{F}_i = m\vec{a} \quad \Rightarrow \quad \vec{a} = \frac{\vec{F}_1 + \vec{F}_2}{m}$$

Sometimes this 'instantaneous' dynamics, with the motion given and the forces to be determined, is called *'inverse dynamics'*. The inside back cover of the book compares the solution methods for instantaneous dynamics to those where differential equations need be solved.

Analytic solution. Some problems involving motion are simple and you can determine almost all you want to know with pencil and paper. You can bypass the whole computer recipe above.

Example: Parabolic trajectory of a projectile
If we assume a constant gravitational field, neglect air drag, and take the y direction as up the only force acting on a projectile is $\vec{\boldsymbol{F}} = -mg\hat{\boldsymbol{\jmath}}$. Thus the "equations of motion" (linear momentum balance) are

$$-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}.$$

Taking the dot product of this equation with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ (equivalent to taking the x and y components) we get the following two differential equations,

$$\ddot{x} = 0 \quad \text{and} \quad \ddot{y} = -g$$

which are decoupled and have the general solution

$$\vec{\boldsymbol{r}} = (A + Bt)\hat{\boldsymbol{\imath}} + (C + Dt - gt^2/2)\hat{\boldsymbol{\jmath}}$$

which is a parametric description of all possible trajectories. By making plots or by using simple algebra you could convince yourself that these trajectories are parabolas for all possible A, B, C, and D. That is, neglecting air drag, the predicted trajectory of a thrown ball is a parabola.

Some other special problems turn out to be easy, although you might not recognize such problems at first glance.

Example: Mass tethered by a zero-length spring
Imagine a massless spring whose un-stretched length is zero (See page 382 in section 7.1 for a discussion of zero length springs). Assume one end is connected to a pivot at the origin and the other to a particle. Neglect gravity and air drag. The force on the mass is thus proportional to its distance from the pivot and the spring constant and pointed towards the origin: $\vec{\boldsymbol{F}} = -k\vec{\boldsymbol{r}}$. Thus linear momentum balance yields

$$-k\vec{\boldsymbol{r}} = m\vec{\boldsymbol{a}}.$$

Breaking into components we get

$$\ddot{x} = (-k/m)x \quad \text{and} \quad \ddot{y} = (-k/m)y.$$

Thus the motion can be thought of as two independent harmonic oscillators, one in the x direction and one in the y direction. The general solution is

$$\vec{\boldsymbol{r}} = \left(A\cos\sqrt{\frac{k}{m}}\,t + B\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\imath}} + \left(C\cos\sqrt{\frac{k}{m}}\,t + D\sin\sqrt{\frac{k}{m}}\,t\right)\hat{\boldsymbol{\jmath}}$$

which is always an ellipse (special cases of which are a circle and a straight line).

Even with gravity and springs together some (only some) problems are easy.

Example: Mass hanging from a zero-length spring
If gravity in the $-\hat{\boldsymbol{\jmath}}$ direction is included in the above problem the solution is only changed by the addition of a constant to be:

$$\vec{\boldsymbol{r}}_{\text{with gravity}} = \vec{\boldsymbol{r}}_{\text{previous example}} - (mg/k)\hat{\boldsymbol{\jmath}}$$

Analytic methods sometimes just can't do the job. Some problems are hard and can't be solved without a computer.

Example: Trajectory with quadratic air drag.
For motions of things you can see with your bare eyes moving in air, the drag force is roughly proportional to the speed squared and opposes the motion. Thus the total force on a particle is $\vec{F} = -mg\hat{\jmath} - Cv^2(\vec{v}/v)$, where $\vec{v}/v$ is a unit vector in the direction of motion. So linear momentum balance gives

$$-mg\hat{\jmath} - Cv\vec{v} = m\vec{a}.$$

If we dot this equation with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\ddot{x} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{x} \quad \text{and} \quad \ddot{y} = -(C/m)\left(\sqrt{\dot{x}^2+\dot{y}^2}\right)\dot{y} - g.$$

These are two coupled second order equations that are probably not solvable with pencil and paper. But they are easily put in the form of a set of four first order equations for direct numerical solution.

On the edge. Some problems are within the reach of advanced analytic methods, but might be easier to solve with a computer.

Example: Path of the earth around the sun.
Assume the sun is big and unmovable with mass M and the earth has mass m. Take the origin to be at the sun. The force on the earth is $\vec{F} = -(mMG/r^2)(\vec{r}/r)$ where $\vec{r}/r$ is a unit vector pointing from the sun to the earth. So linear momentum balance gives

$$-\frac{mMG\vec{r}}{r^3} = m\vec{a}.$$

This equation *can* be solved with pencil and paper, Newton did it but many of us find it too tricky. On the other hand the equations of motion for planetary trajectories are easily broken into components and then into a set of 4 ODEs which can be easily solved on the computer. Either by pencil and paper, or by investigation of numerical solutions, you will find that all solutions are conic sections (straight lines, parabolas, hyperbolas, and ellipses). The special case of circular motion is not far from what the earth does around the sun, what the moon does around the earth, and what most artificial satellites do around the earth.

Summary

If, given the time, the particle's position and the particle's velocity, you know the force on a particle, then you know $\vec{F}(t,\vec{r},\vec{v})$[2]. That means you can write $\vec{F} = m\vec{a}$ as

$$\vec{a} = \vec{F}(t,\vec{r},\vec{v})/m$$

where $\vec{F}$ is known. This can, in turn, be written as two vector first order equations

$$\begin{aligned} \dot{\vec{r}} &= \vec{v} \\ \dot{\vec{v}} &= \vec{F}(t,\vec{r},\vec{v})/m. \end{aligned} \qquad (12.2)$$

[2] Typically you would know this because an applied force would vary in time in a known way (the dependence on t), gravity and spring forces would vary with position in a known way (dependence on r), and you would know forces due to various friction (dependence on $\vec{v}$).

which are equivalent, written out long hand, to the 6 first order equations

$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{z} &= v_z \\
\dot{v}_x &= F_x(t, x, y, z, v_x, v_y, v_z)/m \\
\dot{v}_y &= F_y(t, x, y, z, v_x, v_y, v_z)/m \\
\dot{v}_z &= F_z(t, x, y, z, v_x, v_y, v_z)/m.
\end{aligned} \tag{12.3}
$$

Given the position and velocity at some starting time, these equations can be integrated, sometimes by hand but generally on the computer, to give position and velocity as a function of time.

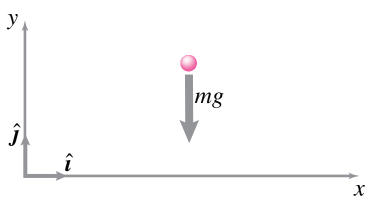


Figure 12.10: Free-body diagram of a particle in flight, neglecting air friction.

Example: Simple ballistics.
This is an example that can be solved with pencil and paper. A computer is not needed. It is the classic from high school and freshman physics. A particle has only one force on it, gravity. A free-body diagram is shown in fig. 12.10. Linear momentum balance gives

$$
\begin{aligned}
\vec{\boldsymbol{F}} &= m\vec{\boldsymbol{a}} \\
-mg\hat{\boldsymbol{j}} &= m\vec{\boldsymbol{a}} \\
\vec{\boldsymbol{a}} &= -g\hat{\boldsymbol{j}}
\end{aligned}
$$

So

$$
\begin{aligned}
\dot{x} &= v_x \\
\dot{y} &= v_y \\
\dot{v}_x &= 0 \\
\dot{v}_y &= -g
\end{aligned}
$$

Integrating the last two of these equations and plugging the result into the first two we get:

$$
\begin{aligned}
\vec{\boldsymbol{v}} &= v_{0x}\hat{\boldsymbol{\imath}} + (v_{0y} - gt)\hat{\boldsymbol{j}} \\
\vec{\boldsymbol{r}} &= (x_0 + v_{0x}t)\hat{\boldsymbol{\imath}} + (y_0 + v_{0y}t - gt^2/2)\hat{\boldsymbol{j}}
\end{aligned}
$$

This solution is plotted various ways in fig. 12.22 on page 626.

More complicated examples are given in the samples on the following pages.

12.2 The rate of change of a vector depends on reference frame

The time derivative of a vector can be found by differentiating each of its components. This calculation depended on having a reference frame, an imaginary piece of big graph paper, and a corresponding set of base (or basis) vectors, say $\hat{\boldsymbol{\imath}}$, $\hat{\boldsymbol{\jmath}}$ and $\hat{\boldsymbol{k}}$. But there can be more than one piece of imaginary graph paper. You could be holding one, Jo another, and Tanya a third. Each could be moving their graph paper around and on each paper the same given vector would change in a different way.

As noted earlier, but for the special case of one frame moving at constant velocity (without rotation) with respect to another, the rate of change of a given vector is different if calculated in different reference frames.

For mechanics we have to differentiate vectors with respect to a Newtonian frame.

Because most often we use the "fixed" ground under us as a practical approximation of a Newtonian frame, we label a Newtonian frame with a curly script $\mathcal{F}$, for fixed. So, when being fussy about notation we will sometimes write

$$\overset{\mathcal{F}}{\dot{\vec{\boldsymbol{r}}}}_{B/O} = \text{The velocity of point B as calculated in frame } \mathcal{F}.$$

Non-Newtonian frames

Even though the laws of mechanics are not valid in non-Newtonian frames, non-Newtonian frames are a useful aid with the understanding of the motion of and forces on systems composed of objects with complex relative motion. So eventually we need to understand frames that accelerate and rotate with respect to each other and with reference to Newtonian frames. Such non-Newtonian frames will be discussed in later chapters.

SAMPLE 12.1 Velocity and acceleration from derivative of position:
The position vector of a particle is given as a function of time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}$$

where $C_1 = 1\,\text{m}, C_2 = 3\,\text{m/s}, C_3 = 1\,\text{m/s}^2,$ and $C_4 = 2\,\text{m/s}.$

1. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.
2. Find the change in the position of the particle between $t = 2\,\text{s}$ and $t = 3\,\text{s}$.

Solution We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}.$$

Therefore,

$$\begin{aligned}\vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t)\hat{\imath} + C_4\hat{\jmath}\\ \vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3\hat{\imath}.\end{aligned}$$

1. Substituting the given values of the constants and $t = 2\,\text{s}$ in the equations above we get,

$$\begin{aligned}\vec{r}(t = 2\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 2\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 4\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 2\,\text{s})\hat{\jmath}\\ &= 11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}\\ \vec{v}(t = 2\,\text{s}) &= (3\frac{\text{m}}{\text{s}} + 2\cdot 1\frac{\text{m}}{\text{s}^2}\cdot 2\,\text{s})\hat{\imath} + (2\frac{\text{m}}{\text{s}})\hat{\jmath}\\ &= 7\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath}\\ \vec{a}(t = 2\,\text{s}) &= (2\cdot 1\frac{\text{m}}{\text{s}^2})\hat{\imath} = 2\,\text{m/s}^2\hat{\imath}.\end{aligned}$$

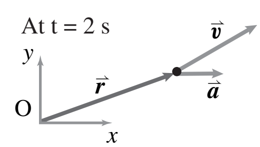


Figure 12.11

$$\boxed{\vec{r} = (11\hat{\imath} + 4\hat{\jmath})\,\text{m}, \quad \vec{v} = (7\hat{\imath} + 2\hat{\jmath})\,\text{m/s}, \quad \vec{a} = 2\,\text{m/s}^2\hat{\imath}}$$

2. The change in the position of the particle between the two time instants is,

$$\Delta\vec{r} = \vec{r}(t = 3\,\text{s}) - \vec{r}(t = 2\,\text{s}).$$

We already have $\vec{r}$ at $t = 2\,\text{s}$. We need to calculate $\vec{r}$ at $t = 3\,\text{s}$.

$$\begin{aligned}\vec{r}(t = 3\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}}\cdot 3\,\text{s} + 1\frac{\text{m}}{\text{s}^2}\cdot 9\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}}\cdot 3\,\text{s})\hat{\jmath}\\ &= 19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}.\end{aligned}$$

Therefore,

$$\begin{aligned}\Delta\vec{r} &= (19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}) - (11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath})\\ &= 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}.\end{aligned}$$

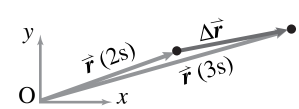


Figure 12.12

$$\boxed{\Delta\vec{r} = 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}}$$

SAMPLE 12.2 Velocity and acceleration from position on a helix. Given that the position of a particle is

$$\vec{\boldsymbol{r}} = A\cos(\lambda t)\hat{\boldsymbol{\imath}} + B\sin(\lambda t)\hat{\boldsymbol{\jmath}} + Ct\hat{\boldsymbol{k}},$$

where A, B, C, and λ are constants, find

1. the velocity as a function of time,
2. the acceleration as a function of time,
3. a condition under which the acceleration vector is normal to the velocity vector.

Solution

1. The velocity:

$$\begin{aligned}\vec{\boldsymbol{v}} &= \frac{d\vec{\boldsymbol{r}}}{dt}\\ &= \frac{d}{dt}[A\cos(\lambda t)\hat{\boldsymbol{\imath}} + B\sin(\lambda t)\hat{\boldsymbol{\jmath}} + Ct\hat{\boldsymbol{k}}]\\ &= -A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = -A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}}$$

2. The acceleration:

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{d\vec{\boldsymbol{v}}}{dt}\\ &= \frac{d}{dt}[-A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}}\\ &= -A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}}\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = -A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}}}$$

3. The velocity vector and the acceleration vector will be orthogonal to each other if $\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}} = 0$. Taking the dot product of the two vectors, we find,

$$\begin{aligned}\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}} &= (-A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}) \cdot (-A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}})\\ &= A^2\lambda^3\sin(\lambda t)\cos(\lambda t) - B^2\lambda^3\sin(\lambda t)\cos(\lambda t)\\ &= (A^2 - B^2)\lambda^3\sin(\lambda t)\cos(\lambda t).\end{aligned}$$

Now, this dot product must be zero for all t if $\vec{\boldsymbol{a}}$ is normal to $\vec{\boldsymbol{v}}$. This is indeed the case if $A = B$. Thus, the condition for orthogonality of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ is $A = B$.

$$\boxed{A = B \quad \Rightarrow \quad \vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{a}} = 0}$$

Note: The path is an elliptical helix with axis in the z direction. The z-component of velocity is constant so the acceleration is entirely in the xy plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

SAMPLE 12.3 Position from velocity. Assume the expression for velocity $\vec{v}$ of a particle is given: $\vec{v} = v_0\hat{\imath} - gt\,\hat{\jmath}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0). Plot the path of the particle taking $x_0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and $t = 1 \ldots 4\,\mathrm{s}$.

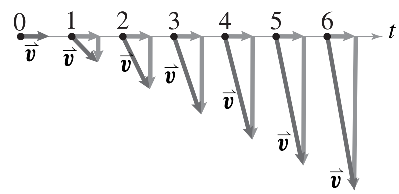


Figure 12.13: The given velocity vector at different time instants ($t = 0,\ 1\ 2$, etc.). Note that the x-component of the velocity remains constant (v_0) while the y-component grows linearly with time ($-gt$).

Solution The position vector of the particle at any time t is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}.$$

We are given that

$$\vec{r}(t=0) = x_0\hat{\imath} + y_0\hat{\jmath}.$$

Now

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= v_0\hat{\imath} - gt\,\hat{\jmath} \\ \text{or} \qquad dx\hat{\imath} + dy\,\hat{\jmath} &= (v_0\hat{\imath} - gt\,\hat{\jmath})\,dt. \end{aligned}$$

Integrating both sides of the equation with appropriate limits, we get

$$\begin{aligned} \int_{x_0\hat{\imath}+y_0\hat{\jmath}}^{x\hat{\imath}+y\hat{\jmath}} (dx\hat{\imath} + dy\,\hat{\jmath}) &= \int_0^t (v_0\hat{\imath} - gt\,\hat{\jmath})dt \\ \int_{x_0}^{x} dx\hat{\imath} + \int_{y_0}^{y} dy\,\hat{\jmath} &= v_0\hat{\imath}\int_0^t dt - g\hat{\jmath}\int_0^t t\,dt \\ (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} &= v_0 t\hat{\imath} - \frac{1}{2}gt^2\,\hat{\jmath} \\ x\hat{\imath} + y\,\hat{\jmath} &= (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0 t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0 t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0 t,\ y_0 - \tfrac{1}{2}gt^2)}$$

Plugging in $x0 = 0$, $y_0 = 80\,\mathrm{m}$, $v_0 = 2\,\mathrm{m/s}$, $g = 10\,\mathrm{m/s^2}$, and taking 20 points between $t = 0$ to $t = 4$, we compute the values of x and y and plot them to get the path of the particle. The plot is shown in fig. 12.14 with a few intermediate positions marked on the path.

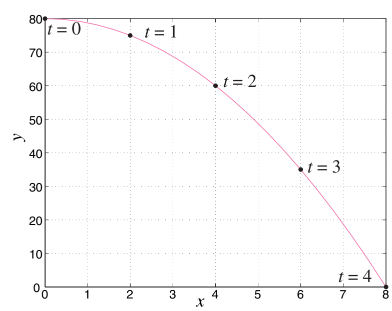


Figure 12.14: Path of the particle plotted by computing the coordinates $x(t)$ and $y(t)$ at 20 equal time intervals between $t = 0$ to 4 seconds. Five positions are marked on the path corresponding to $t = 0$, 1, 2, 3, and 4 seconds.

Comments: From the x and y coordinates, it is possible to get the equation of the path of the particle by eliminating the time from the two equations. From the expression for $x(t)$, we get $t = (x - x_0)/v_0$. Substituting this expression for t in the equation for $y(t)$, we get,

$$y - y_0 = \frac{g}{2v_0^2}(x - x_0)^2$$

which is the equation of the path. From this equation it should be clear that the path is parabolic. It is easier to see this if you shift the origin to (x_0, y_0) and use the new coordinates $\hat{x} = x - x_0$ and $\hat{y} = y - y_0$. Then, in terms of the new coordinates, the path becomes,

$$\hat{y} = \frac{g}{2v_0^2}\hat{x}^2.$$

SAMPLE 12.4 **Acceleration of a point mass in 3-D.** A ball of mass $m = 13\,\text{kg}$ is being pulled by three strings as shown in Fig. 12.15. The tension in each string is $T = 13\,\text{N}$. Find the acceleration of the ball.

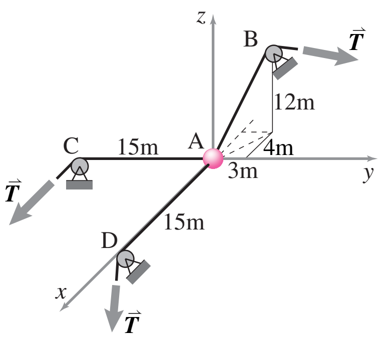


Figure 12.15: A ball in 3-D

Solution The forces acting on the body are shown in the free-body diagram in Fig.12.16. From geometry:

$$\begin{aligned}\hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2 + 3^2 + 12^2}} \\ &= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}.\end{aligned}$$

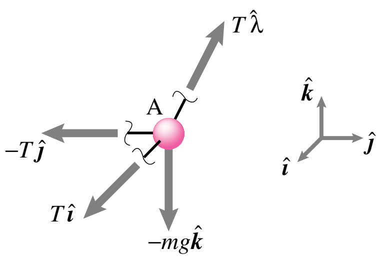


Figure 12.16: FBD of the ball

The balance of linear momentum for the ball gives

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \qquad (12.4)$$

where

$$\begin{aligned}\sum \vec{\boldsymbol{F}} &= T\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + T\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{k}} \\ &= T\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\ &= \frac{T}{13}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}.\end{aligned}$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (12.4):

$$\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.$$

Now plugging in the given values: $T = 13\,\text{N}$, $m = 13\,\text{kg}$, and $g = 10\,\text{m/s}^2$, we get

$$\begin{aligned}\vec{\boldsymbol{a}} &= \frac{\cancel{13}\,\text{N}}{\cancel{13}\cdot 13\,\text{kg}}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - 10\,\text{m/s}^2\hat{\boldsymbol{k}} \\ &= (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2.\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

SAMPLE 12.5 Projectile motion with air drag. A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

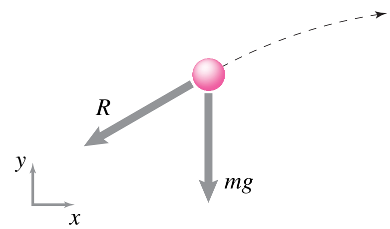


Figure 12.17: FBD of the projectile.

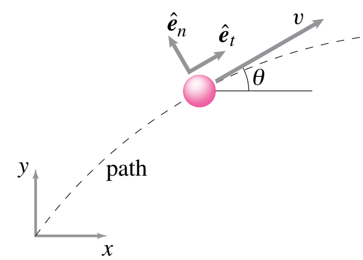


Figure 12.18

Solution The free-body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{v} = v\hat{e}_t$ where

$$\hat{e}_t = \vec{v}/|\vec{v}| = (v_x/|\vec{v}|)\hat{\imath} + (v_y/|\vec{v}|)\hat{\jmath}.$$

Then the force due to air resistance is

$$\vec{R} = -kv^2\hat{e}_t.$$

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{R} + m\vec{g} &= m\vec{a} \\ \text{or} \qquad -kv^2\hat{e}_t - mg\hat{\jmath} &= m\overbrace{(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})}^{\vec{a}}. \end{aligned} \tag{12.5}$$

Noting that $v = |\vec{v}| = |\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of eqn. (12.5) with $\hat{\imath}$ and $\hat{\jmath}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\imath}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{e}_t\cdot\hat{\jmath}) - mg &= m\ddot{y}. \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_x/|\vec{v}|) \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_y/|\vec{v}|). \end{aligned}$$

Substituting these expressions in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

$$\boxed{\ddot{x} = -\tfrac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\tfrac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

For initial conditions we could use

$$x_0 = 0, y_0 = 0 \qquad \text{and} \qquad \dot{x}_0 = v_{x0} = v_0\cos\theta_0, \qquad \text{and} \qquad \dot{y}_0 = v_{y0} = v_0\sin\theta_0.$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$. We can calculate the slope of the trajectory from

$$\tan\theta = \frac{\dot{y}}{\dot{x}}.$$

SAMPLE 12.6 Trajectory of a food-bag. In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$. Ignore air drag.

FBD of the bag

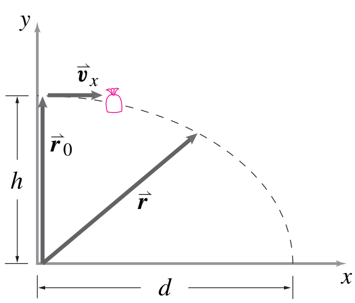


Figure 12.19: Free-body diagram of the bag and the geometry of its motion.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free-body diagram of the bag during its free flight is shown in Fig. 12.19. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{\boldsymbol{a}} = -mg\hat{\boldsymbol{\jmath}}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} = 80\,\mathrm{m}\,\hat{\boldsymbol{\jmath}}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{\boldsymbol{v}}(0) \equiv \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}} = 200\,\mathrm{km/h}\hat{\boldsymbol{\imath}}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{\boldsymbol{r}}} = -g\hat{\boldsymbol{\jmath}}$$

with two initial conditions:

$$\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned} \ddot{\vec{\boldsymbol{r}}} &\equiv \frac{d\dot{\vec{\boldsymbol{r}}}}{dt} = -g\hat{\boldsymbol{\jmath}} \\ \int d\dot{\vec{\boldsymbol{r}}} &= -\hat{\boldsymbol{\jmath}}\int g\,dt \\ \text{or} \qquad \dot{\vec{\boldsymbol{r}}} &= -gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1 \end{aligned} \tag{12.6}$$

and integrating once again, we get

$$\begin{aligned} \vec{\boldsymbol{r}} &= \int(-gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1)\,dt \\ &= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1 t + \vec{\boldsymbol{c}}_2 \end{aligned} \tag{12.7}$$

where $\vec{\boldsymbol{c}}_1$ and $\vec{\boldsymbol{c}}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eqn. (12.6) and eqn. (12.7) we get

$$\begin{aligned} \dot{\vec{\boldsymbol{r}}}(0) &= v_x\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{c}}_1, \quad \text{and} \\ \vec{\boldsymbol{r}}(0) &= h\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{c}}_2. \end{aligned}$$

Therefore, the solution is

$$\begin{aligned} \vec{\boldsymbol{r}}(t) &= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + v_x t\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} \\ &= v_x t\hat{\boldsymbol{\imath}} + (h - \frac{1}{2}gt^2)\hat{\boldsymbol{\jmath}}. \end{aligned}$$

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{\boldsymbol{r}}$, *i.e.,* $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.,* we can write $\vec{\boldsymbol{r}} = d\hat{\boldsymbol{\imath}} + 0\hat{\boldsymbol{\jmath}}$ where d is the distance we are interested in. Now equating the components of $\vec{\boldsymbol{r}}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2}gt^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too "*mathy*" or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\boldsymbol{\jmath}} = m\vec{\boldsymbol{a}}$, writing the acceleration as $\vec{\boldsymbol{a}} = a_x\hat{\boldsymbol{\imath}} + a_y\hat{\boldsymbol{\jmath}}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2}at^2$, in both x and y directions, we get

$$\begin{aligned}
d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2}\overbrace{a_x}^{0}t^2 \\
&= v_x t \\
0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0}t + \frac{1}{2}\overbrace{a_y}^{-g}t^2 \\
&= h - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \sqrt{\frac{2h}{g}}.
\end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

SAMPLE 12.7 Cartoon mechanics: The cannon. It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

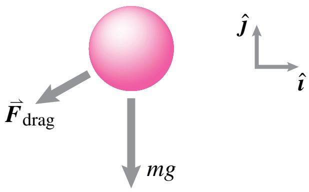


Figure 12.20

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$\begin{aligned}
\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\
&= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\
&= -c|\vec{v}|\vec{v} \\
&= -c\sqrt{\dot{x}^2+\dot{y}^2}\,(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath})
\end{aligned}$$

So the linear momentum balance gives

$$\sum \vec{F} = \dot{\vec{L}}$$
$$\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2+\dot{y}^2}(\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath}+\ddot{y}\hat{\jmath})\right\}$$

$$\begin{aligned}
\{\}\cdot\hat{\imath} \quad &\Rightarrow \quad \ddot{x} = \left[-c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{x}/m\right] \\
\{\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \ddot{y} = -c\sqrt{\dot{x}^2+\dot{y}^2}\,\dot{y}/m - g
\end{aligned}$$

Solving these equations numerically with reasonable values (1) of $\dot{x}_0$, $\dot{y}_0$, m and c gives which is closer to a cartoon's triangle than to a naive physicist's parabola.

(1) To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

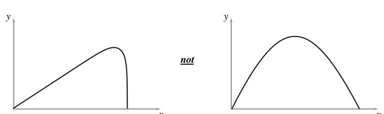


Figure 12.21

12.1 Dynamics of a particle in space

Preparatory Problems

12.1.1 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\hat{\jmath} + C\hat{k}$, find

1. $\vec{v}(t)$
2. $\vec{a}(t)$
3. $\vec{r}(t)\times\vec{a}(t)$.

12.1.2 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\imath} + (\cos\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\jmath} + te^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s}\hat{k}$. At $t = 3\,\mathrm{s}$, find the particle's

a) velocity and
b) acceleration.

12.1.3 A particle of mass $m = 2\,\mathrm{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\mathrm{m/s^2}\hat{\imath} + 4t^3\frac{\mathrm{m}}{s^3}\hat{\jmath}$. At $t = 5\,\mathrm{s}$, find the particle's

a) velocity and *
b) acceleration, and *
c) draw the three vectors.

12.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\mathrm{m/s})\hat{\imath} - (1.5\,\mathrm{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

12.1.5 If $\dot{\vec{r}} = (u_0\sin\Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. *

12.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\mathrm{m/s^2}\hat{\imath} - (3\,\mathrm{m/s^2} - 1\,\mathrm{m/s^3}\,t)\hat{\jmath} - 5\,\mathrm{m/s^4}\,t^2\hat{k}$, write the vector equation $\vec{v} = \int\vec{a}\,dt$ as three scalar equations (*i.e.*, find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

12.1.7 Find $\vec{r}(5\,\mathrm{s})$ given that $\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\hat{\jmath}$, and that v_1 is a constant $4\,\mathrm{m/s}$, v_2 is a constant $5\,\mathrm{m/s}$, and c is a constant $4\,\mathrm{s^{-1}}$.

12.1.8 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

12.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\mathrm{s})$.

More-Involved Problems

12.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\mathrm{m}})\,\mathrm{m}$. where $x(t) = t^3(\frac{\mathrm{m}}{s^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}\,\mathrm{s}$?

12.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m}\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\mathrm{s}$. *

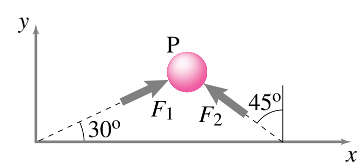
12.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

12.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\mathrm{m}}})\,\mathrm{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\mathrm{m/s^2}$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\mathrm{s}$?

12.1.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}\,\mathrm{m/s}$ and $x(t) = t\,\mathrm{m/s}$, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What is the distance of the particle from the origin and its velocity and acceleration when $x = 3\,\mathrm{m}$?

12.1.15 For a particle, $\sum\vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\mathrm{ft/s^2}\hat{\imath} + 5\,\mathrm{ft/s^2}\hat{\jmath}$.

a) Write the equation
$$\sum\vec{F} = m\vec{a}$$
in vector form (evaluating each side as much as possible).
b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).
c) Write the equation in matrix form.
d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:
   1. from the scalar equations using hand algebra,
   2. from the matrix equation using a computer, and
   3. from the vector equation using a cross product.



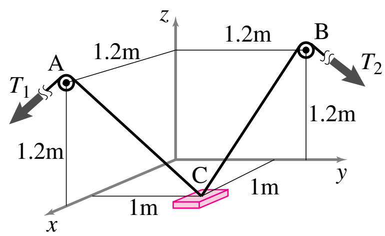
Problem 12.1.15

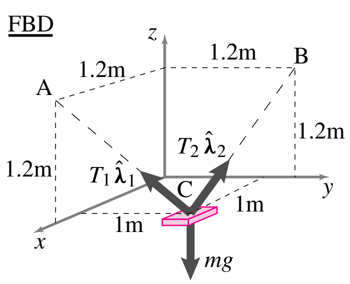
12.1.16 Three forces, $\vec{F}_1 = 20\,\mathrm{N}\hat{\imath} - 5\,\mathrm{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\mathrm{m/s^2}\hat{\imath} + 2.2\,\mathrm{m/s^2}\hat{\jmath} + 1.7\,\mathrm{m/s^2}\hat{k}$. Write the equation $\sum\vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

12.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\mathrm{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\,\mathrm{m/s^2}$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
     .
     .
     .
T   =
```

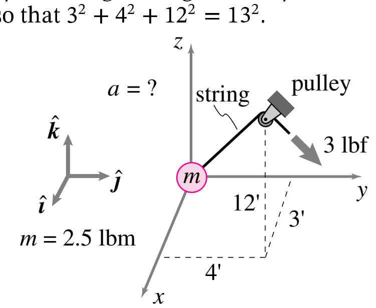
12.1.18 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\mathrm{N}$ and $T_2 = 1500\,\mathrm{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]





Problem 12.1.18

12.1.19 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and $\mathrm{lbf}/\mathrm{lbm} = g$. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



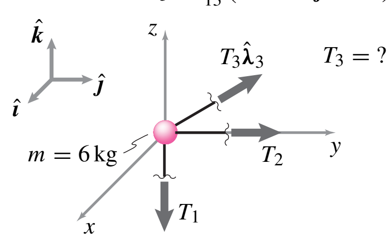
Problem 12.1.19

12.1.20 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free-body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3. *

$$\hat{\lambda}_3 = \tfrac{1}{13}(3\hat{\imath} + 12\hat{\jmath} + 4\hat{k})$$



$$\vec{a} = (\text{-}.5\hat{\imath} + 2.5\hat{\jmath} + \tfrac{1}{3}\hat{k})\,\tfrac{m}{s^2}$$

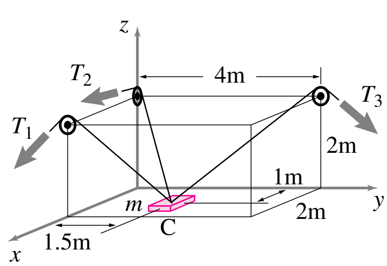
Problem 12.1.20

12.1.21 An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = (-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k})\,\mathrm{m/s^2}$.

a) Draw a free-body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\boldsymbol{\lambda}$'s as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



Problem 12.1.21

12.1.22 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\vec{r} = (5\,\mathrm{m})\cos^2(t^2/\mathrm{s}^2)\hat{\imath} + (5\,\mathrm{m})\sin(t^2/\mathrm{s}^2)\cos(t^2/\mathrm{s}^2)\hat{\jmath}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

12.1.23 *Computer question:* **What's the plot? What's the mechanics question?** Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

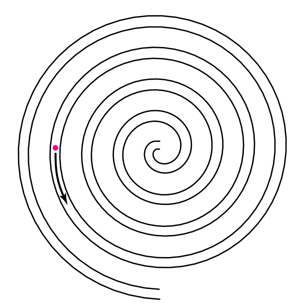
```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

plot z2 and z1 vs t on the same plot
```

Problem 12.1.23

12.1.24 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



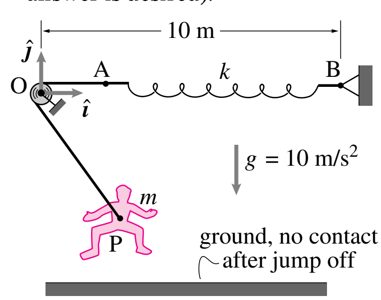
Problem 12.1.24

12.1.25 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\text{m}$ and a stiffness of $k = 200\,\text{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\text{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\text{m}\hat{\imath} - 5\,\text{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \mathbf{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\text{s}$.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



Problem 12.1.25: Conceptual setup for a bungy jumping system.

12.1.26 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

12.1.27 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\text{m/s}$, the drag proportionality constant is $C = 5\,\text{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free-body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

12.1.28 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{\boldsymbol{e}}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates. *

12.1.29 The equations of motion from problem 12.1.28 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations. *

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plot of its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for the ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

12.1.30 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of $v_{x0}, v_{y0}, m, g, c,$ and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

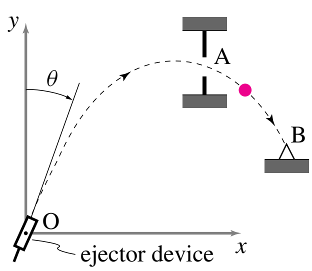
d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

12.1.31 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01\,\text{kg/m}$.
Gravity $g = 10\,\text{m/s}$.

a) Write and execute computer code to find the height at $t = 1\,\text{s}$. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1\,\text{s}$ using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

12.1.32 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that it passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 12.1.32