12.1.1 Given $\vec{\boldsymbol{r}}(t) = A\sin(\omega t)\hat{\boldsymbol{\imath}} + Bt\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}$, find

1. $\vec{\boldsymbol{v}}(t)$
2. $\vec{\boldsymbol{a}}(t)$
3. $\vec{\boldsymbol{r}}(t) \times \vec{\boldsymbol{a}}(t)$.

$$\vec{r}(t) = A\sin\omega t\,\hat{i} + Bt\,\hat{j} + C\,\hat{k}$$

1) $\vec{V}(t) = \dot{\vec{r}}(t) = A\omega\cos\omega t\,\hat{i} + B\,\hat{j}$

2) $\vec{a}(t) = \ddot{\vec{r}}(t) = -A\omega^2\sin\omega t\,\hat{i}$

3) $\vec{r}(t)\times\vec{a}(t) = AB\omega^2 t\sin\omega t\,\hat{k} - AC\omega^2\sin\omega t\,\hat{j}$

$$= A\omega^2\sin\omega t\,[\,Bt\,\hat{k} - C\,\hat{j}\,]$$

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12.1.2 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\imath} + (\cos\frac{t}{\mathrm{s}})\,\mathrm{m}\hat{\jmath} + te^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s}\hat{k}$. At $t = 3\,\mathrm{s}$, find the particle's

a) velocity and

b) acceleration.

$$\vec{r} = \sin\left(\frac{t}{s}\right) m\hat{i} + \cos\left(\frac{t}{s}\right) m\hat{j} + te^{t/s}\frac{m}{s}\hat{k}$$

a) $$\vec{v} = \dot{\vec{r}}\Big|_{t=3s} = \cos\left(\frac{t}{s}\right)\frac{m}{s}\hat{i} - \sin\left(\frac{t}{s}\right)\frac{m}{s}\hat{j} + \left[e^{t/s}\frac{m}{s} + te^{t/s}\frac{m}{s^2}\right]\hat{k}$$

$$= \left[-0.989\,\hat{i} - 0.141\,\hat{j} + 80.342\,\hat{k}\right]$$

b) $$\vec{a} = \ddot{\vec{r}}\Big|_{t=3s} = -\sin\left(\frac{t}{s}\right)\frac{m}{s^2}\hat{i} - \cos\left(\frac{t}{s}\right)\frac{m}{s^2}\hat{j} + \left[2e^{t/s}\frac{m}{s^2} + te^{t/s}\frac{m}{s^3}\right]\hat{k}$$

$$= \left[-0.141\,\hat{i} + 0.989\,\hat{j} + 100.42\,\hat{k}\right]$$

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12.1.3 A particle of mass $m = 2\,\text{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{\boldsymbol{r}} = 3t^2\,\text{m/s}^2\hat{\boldsymbol{\imath}} + 4t^3\,\frac{\text{m}}{s^3}\hat{\boldsymbol{\jmath}}$. At $t = 5\,\text{s}$, find the particle's

a) velocity and

b) acceleration, and

c) draw the three vectors.

Answer:
(a) $\vec{\boldsymbol{v}}(5\,\text{s}) = (30\hat{\boldsymbol{\imath}} + 300\hat{\boldsymbol{\jmath}})\,\text{m/s}$.
(b) $\vec{\boldsymbol{a}}(5\,\text{s}) = (6\hat{\boldsymbol{\imath}} + 120\hat{\boldsymbol{\jmath}})\,\text{m/s}^2$.

$$\vec{r} = 3t^2\,\hat{i}\,\frac{m}{s^2} + 4t^3\,\hat{j}\,\frac{m}{s^3}$$

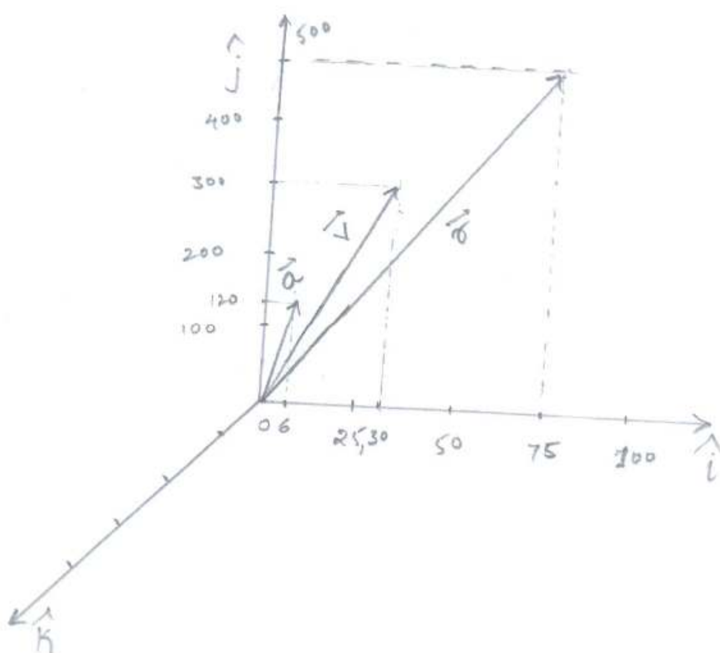
a) $$\vec{V} = \dot{\vec{r}}\Big|_{t=5} = 6t\,\hat{i}\,\frac{m}{s} + 12t^2\,\hat{j}\,\frac{m}{s}$$

$$= \left[30\,\hat{i} + 300\,\hat{j}\right]\frac{m}{s}$$

b) $$\vec{a} = \ddot{\vec{r}}\Big|_{t=5} = \left[6\,\hat{i} + 24t\,\hat{j}\right]\frac{m}{s^2}$$

$$= \left[6\,\hat{i} + 120\,\hat{j}\right]\frac{m}{s^2}$$

c)



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12.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\text{m/s})\hat{\imath} - (1.5\,\text{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

11.1.4

a) $|\Delta\vec{r}| = |2(0.5\hat{i} - 1.5\hat{j})| = |\hat{i} - 3\hat{j}| = \sqrt{10}\,\text{m}$

b) Direction vector $= \dfrac{\hat{i}}{\sqrt{10}} - \dfrac{3\hat{j}}{\sqrt{10}}$

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12.1.5 If $\dot{\vec{r}} = (u_0 \sin \Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$.

Answer:
$\vec{r}(t) = \left(x_0 + \frac{u_0}{\Omega} - \frac{u_0}{\Omega}\cos(\Omega t)\right)\hat{\imath} + \left(y_0 + v_0 t\right)\hat{\jmath}$.

11.1.5

$$\dot{\vec{r}} = (u_0 \sin \Omega t)\hat{i} + v_0 \hat{j}$$

$$\vec{r}(0) = x_0 \hat{i} + y_0 \hat{j}$$

integrate $\dot{\vec{r}}$

$$\int \dot{\vec{r}} = \vec{r} = \int (u_0 \sin \Omega t)\hat{i} + v_0 \hat{j} \quad + C_1$$

$$\vec{r} = -\frac{u_0}{\Omega} \cos \Omega t \; \hat{i} + v_0 t \hat{j} \quad + C_1$$

$$\vec{r}(0) = x_0 \hat{i} + y_0 \hat{j} = -\frac{u_0}{\Omega} \hat{i} + C_1$$

$$C_1 = \left(x_0 + \frac{u_0}{\Omega}\right)\hat{i} + y_0 \hat{j}$$

$$\vec{r} = \left[x_0 + \frac{u_0}{\Omega}(1 - \cos \Omega t)\right]\hat{i} + (y_0 + v_0 t)\hat{j}$$

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12.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\mathrm{m/s^2}\hat{\imath} - (3\,\mathrm{m/s^2} - 1\,\mathrm{m/s^3}\,t)\hat{\jmath} - 5\,\mathrm{m/s^4}\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.,* find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

11.1.6

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\vec{a} = 2\hat{i} - (3-t)\hat{j} - 5t^2\hat{k}$$

$$\vec{V}(t) = \int \vec{a}\,dt + c$$

$$\vec{V}(t) = \left[2t\,\hat{i} - \left(3t - \frac{t^2}{2}\right)\hat{j} - \frac{5t^3}{3}\hat{k}\right]$$

The scalar equations are.

$$V_x = 2t$$

$$V_y = -\left(3t - \frac{t^2}{2}\right)$$

$$V_z = -\frac{5t^3}{3}$$

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12.1.7 Find $\vec{r}(5\,\mathrm{s})$ given that $\dot{\vec{r}} = v_1 \sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\mathrm{m}\hat{\imath} + 3\,\mathrm{m}\,\hat{\jmath}$, and that v_1 is a constant $4\,\mathrm{m/s}$, v_2 is a constant $5\,\mathrm{m/s}$, and c is a constant $4\,\mathrm{s}^{-1}$.

<u>12.1.7</u>

$$\vec{r}(t) = \int \dot{\vec{r}}(t)\,dt + C_1$$

$$= \int (V_1 \sin ct\ \hat{\imath} + V_2\hat{\jmath})\,dt + C_1$$

$$\vec{r}(t) = -\frac{V_1}{c}\cos ct\ \hat{\imath} + V_2 t\ \hat{\jmath} + C$$

$$\vec{r}(t)\Big|_{t=0} = -\frac{V_1}{c}\hat{\imath} + C_1 = 2\hat{\imath} + 3\hat{\jmath}$$

$$C_1 = \left(2 + \frac{V_1}{c}\right)\hat{\imath} + 3\hat{\jmath}$$

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12.1.8 Let $\dot{\vec{\boldsymbol{r}}} = v_0 \cos\alpha\hat{\boldsymbol{\imath}} + v_0 \sin\alpha\,\hat{\boldsymbol{\jmath}} + (v_0 \tan\theta - gt)\hat{\boldsymbol{k}}$, where v_0, α, θ, and g are constants. If $\vec{\boldsymbol{r}}(0) = \vec{\boldsymbol{0}}$, find $\vec{\boldsymbol{r}}(t)$.

11.1.8

$$\vec{r}(t) = \int \dot{\vec{r}}(t)\,dt + C$$

$$= \int [V_0 \cos\alpha\,\hat{i} + V_0 \sin\alpha\,\hat{j} + (V_0 \tan\theta - gt)\hat{k}]\,dt + C_1$$

$$= t(V_0 \cos\alpha\,\hat{i} + V_0 \sin\alpha\,\hat{j} + V_0 \tan\theta\,\hat{k}) - \frac{gt^2}{2}\hat{k} + C_1$$

$$\vec{r}(t)\Big|_{t=0} = 0 \quad \text{gives } C=0$$

$$\vec{r}(t) = t(V_0 \cos\alpha\,\hat{i} + V_0 \sin\alpha\,\hat{j} + V_0 \tan\theta\,\hat{k}) - \frac{gt^2}{2}\hat{k}$$

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12.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\mathrm{s})$.

11.1.9

$$\vec{r}(t) = \int(-R\sin t\,\hat{i} + R\cos t\,\hat{j} + gt\,\hat{k})\,dt + C_1$$

$$= R\cos t\,\hat{i} + R\sin t\,\hat{j} + \frac{gt^2}{2}\hat{k} + C_1$$

$$\vec{r}(0) = R\hat{i} + C_1 = R\hat{i} \Rightarrow C_1 = 0$$

$$\vec{r}\left(\frac{\pi}{3}\right) = R\cos\frac{\pi}{3}\,\hat{i} + R\sin\frac{\pi}{3}\,\hat{j} + \frac{g\left(\frac{\pi}{3}\right)^2}{2}\hat{k}$$

$$= \frac{R}{2}\hat{i} + \frac{\sqrt{3}R}{2}\hat{j} + \frac{g\pi^2}{18}\hat{k}$$

12.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\mathrm{m}})\,\mathrm{m}$. where $x(t) = t^3(\frac{\mathrm{m}}{s^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}$ s?

11.1.10

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$y(x) = \sin^2(x)$

$x(t) = t^3$

$\vec{V}(t) = \frac{d}{dt}\left[t^3\,\hat{\imath} + \sin^2(t^3)\,\hat{\jmath}\right]$

$\vec{V}(t) = \left[3t^2\,\hat{\imath} + 2\sin(t^3)\cos(t^3)\cdot 3\cdot t^2\,\hat{\jmath}\right]$

$\vec{a}(t) = \left[6t\,\hat{\imath} + \left((3t^2)(6t^2\cos 2t^3) + 6t\sin 2t^3\right)\hat{\jmath}\right]$

$\cancel{\vec{V}(t)\Big|_{t=(\pi)^{1/3}} = 3\pi^{2/3}\,\hat{\imath}\quad 6\pi^{1/3}\left(\hat{\imath} + 3\pi\,\hat{\jmath}\right)}$

$\vec{V}(t)\Big|_{t=(\pi)^{1/3}} = 3\pi^{2/3}\,\hat{\imath}$

$\vec{a}\,(\pi^{1/3}) = 6\pi^{1/3}\left[\hat{\imath} + 3\pi\,\hat{\jmath}\right]$

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12.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m}\,\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\mathrm{s}$.

Answer:
$\vec{v} = 2t\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s}\,\hat{\jmath}$, $\vec{a} = 2\,\mathrm{m/s^2}\hat{\imath} + e^{\frac{t}{\mathrm{s}}}\,\mathrm{m/s^2}\,\hat{\jmath}$.

\# 6.1

$$\underline{r}(t) = t^2\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

$$\underline{v}(t) = \frac{d\,\underline{r}(t)}{dt} = 2t\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

$$\underline{a}(t) = \frac{d\,\underline{v}(t)}{dt} = 2\left(\frac{m}{s^2}\right)\hat{\imath} + e^{t(1/s)}\,\hat{\jmath}$$

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12.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

8

11.1.12

$$y^2 = b^2\left(1 - \frac{x^2}{a^2}\right)$$

$$x = \frac{a}{2} \Rightarrow y^2 = b^2\left(1 - \frac{1}{4}\right) = \frac{3b^2}{4}$$

$$y = \pm\frac{\sqrt{3}}{2}b \Rightarrow y = +\frac{\sqrt{3}}{2}b$$

$$\text{Speed} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = V \longrightarrow (1)$$

$$2y\frac{dy}{dt} = -b^2\frac{2x}{a^2}\frac{dx}{dt}$$

$$\frac{dy}{dt} = -\left(\frac{b^2}{a^2}\frac{x}{y}\right)\frac{dx}{dt} \longrightarrow (2)$$

Putting in (1) we get

$$\frac{dx}{dt}\sqrt{1 + \left(\frac{b^2x}{a^2y}\right)^2} = V$$

$$V_x = \frac{dx}{dt} = \frac{V}{\sqrt{1 + \left(\frac{b^2}{a^2}\frac{x}{y}\right)^2}} = \frac{V}{\sqrt{1 + \frac{b^2}{3a^2}}}$$

similarly $V_y = \dfrac{dy}{dt} = \dfrac{V}{\sqrt{1 + \frac{3a^2}{b^2}}}$

$$\vec{V} = \frac{V}{\sqrt{1 + b^2/3a^2}}\,\hat{i} + \frac{V}{\sqrt{1 + 3a^2/b^2}}\,\hat{j}$$

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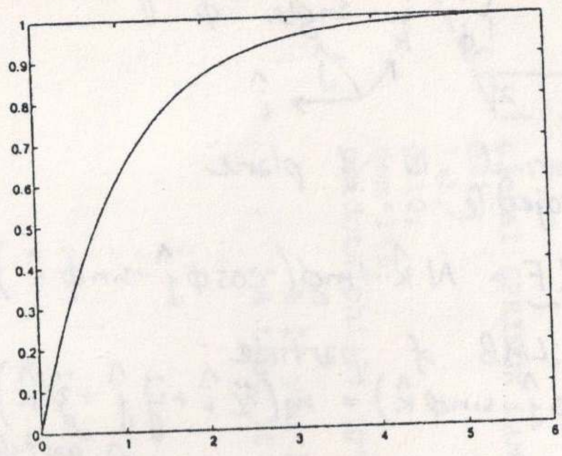
12.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\mathrm{m}}})\,\mathrm{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\mathrm{m/s^2}$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\mathrm{s}$?

6.5 p. 8

A particle travels on a path:

$$y(x) = (1 - e^{-x/m})\ m.$$

Plot:



x- coordinate: $x(t) = t^2\ [m/s^2]$

then y-coord.: $y(t) = (1 - e^{-t^2/[m]})\ [m]$ the m is a unit, not a variable

so $\underset{\sim}{r}(t) = t^2\,\hat{i} + (1 - e^{-t^2})\,\hat{j}$

then $\dfrac{d\underset{\sim}{r}(t)}{dt} = 2t\,\hat{i} + 2t\,e^{-t^2}\,\hat{j} = \underset{\sim}{v}(t)$ Drop units to avoid confusion

so $|\underset{\sim}{v}| = \sqrt{4t^2 + 4t^2 e^{-2t^2}} =$

$$\boxed{|\underset{\sim}{v}| = 2t\sqrt{1 + e^{-2t^2}}\ [m/s^2]}$$

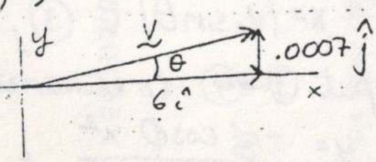
oops this is speed differentiate this wrt t to get its rate of change

b) recall $\underset{\sim}{v}(t) = 2t\,\hat{i} + 2te^{-t^2}\,\hat{j}$

put in $t = 3$ sec:

$\underset{\sim}{v}(3\text{ s.}) = (6\,\hat{i} + .0007\,\hat{j})$ m/s

$\cos\theta = \dfrac{6}{\sqrt{6^2 + .0007^2}} \Rightarrow$



$$\boxed{\theta = .007^\circ}$$

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12.1.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{s^2})t^2}$ m/s and $x(t) = t$ m/s, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What is the distance of the particle from the origin and its velocity and acceleration when $x = 3$ m?

11.1.14

9

$V(t) = \sqrt{1+4t^2}$ $\quad x > 0$

$x(t) = t$ $\quad y > 0$

$V_x = \dot{x}(t) = 1$

$V(t) = \sqrt{V_x^2 + V_y^2} = \sqrt{1 + V_y^2} = \sqrt{1+4t^2}$

$V_y = \pm 2t \implies V_y = 2t$

$\vec{r}(t) = \int \vec{V}(t)\,dt = \int (\hat{i} + 2t\,\hat{j})\,dt + C$

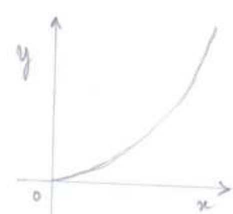
$\vec{r}(t) = t\,\hat{i} + \frac{2t^2}{2}\,\hat{j} + C$

$C = 0$

$\vec{r}(t) = t\,\hat{i} + t^2\,\hat{j}$

$\vec{V}(t) = \hat{i} + 2t\,\hat{j}$

$\vec{a}(t) = 2\,\hat{j}$



Rate of change of speed

$\dot{V} = \frac{1}{2}\frac{8t}{\sqrt{1+4t^2}} = \frac{4t}{\sqrt{1+4t^2}}$

Path of the particle is parabolic upwards constant acceleration.

At $x = 3$, $t = 3$;

$y = t^2 = 9$

distance from origin $= \sqrt{9^2+3^2} = 3\sqrt{10}$

Velocity $= (\hat{i} + 6\hat{j})$ m/s

acceleration $= 2\hat{j}$ m/s^2

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12.1.15 For a particle, $\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$. Two forces $\vec{\boldsymbol{F}}_1$ and $\vec{\boldsymbol{F}}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{\boldsymbol{a}} = -2\,\mathrm{ft/s^2}\,\hat{\boldsymbol{\imath}} + 5\,\mathrm{ft/s^2}\,\hat{\boldsymbol{\jmath}}$.

a) Write the equation

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

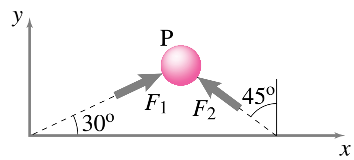
in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

c) Write the equation in matrix form.

d) Find $F_1 = |\vec{\boldsymbol{F}}_1|$ and $F_2 = |\vec{\boldsymbol{F}}_2|$ by the following methods:

1. from the scalar equations using hand algebra,
2. from the matrix equation using a computer, and
3. from the vector equation using a cross product.



Problem 12.15

10

11.1.15

$\vec{a} = -2\hat{i} + 5\hat{j}$

$m = 2$

a) $\sum \vec{F} = m\vec{a}$

$\vec{F}_1 + \vec{F}_2 = m(-2\hat{i} + 5\hat{j})$

$= -2m\hat{i} + 5m\hat{j}$

$\vec{F}_2 = |\vec{F}_2|\left(-\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j}\right)$,

$\vec{F}_1 = |\vec{F}_1|\left(\frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j}\right)$

$$\left(\frac{\sqrt{3}}{2}|\vec{F}_1| - \frac{|\vec{F}_2|}{\sqrt{2}}\right)\hat{i} + \left(\frac{|\vec{F}_1|}{2} + \frac{|\vec{F}_2|}{\sqrt{2}}\right)\hat{j} = -2m\hat{i} + 5m\hat{j}$$

b) Scalar form

$$\frac{\sqrt{3}}{2}|\vec{F}_1| - \frac{|\vec{F}_2|}{\sqrt{2}} = -2m \quad \text{--- (i)}$$

$$\frac{|\vec{F}_1|}{2} + \frac{|\vec{F}_2|}{\sqrt{2}} = 5m \quad \text{--- (ii)}$$

c) Matrix form

$$\begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} |\vec{F}_1| \\ |\vec{F}_2| \end{bmatrix} = \begin{bmatrix} -2m \\ 5m \end{bmatrix}$$

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d) Solution : Hand Algebra (11)

$$\frac{\sqrt{3}+1}{2} |\vec{F}_1| = 3m \Rightarrow |\vec{F}_1| = \frac{6m}{\sqrt{3}+1}$$

$$|\vec{F}_2| = \frac{\sqrt{2}(5\sqrt{3}+2)m}{\sqrt{3}+1}$$

Matrix Method

$$|\vec{F}_1| = \frac{\begin{vmatrix} -2m & -\frac{1}{\sqrt{2}} \\ 5m & \frac{1}{\sqrt{2}} \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}}$$

$$|\vec{F}_2| = \frac{\begin{vmatrix} \sqrt{3}/2 & -2m \\ 1/2 & 5m \end{vmatrix}}{\begin{vmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{\sqrt{2}} \\ \frac{1}{2} & \frac{1}{\sqrt{2}} \end{vmatrix}}$$

Solution from vector method.

unit vector along $\vec{F}_1 \Rightarrow \frac{\sqrt{3}}{2}\hat{i} + \frac{1}{2}\hat{j} = \hat{e}_1$

Along $\vec{F}_2 = -\frac{1}{\sqrt{2}}\hat{i} + \frac{1}{\sqrt{2}}\hat{j} = \hat{e}_2$

$$\vec{F}_1 + \vec{F}_2 = m(-2\hat{i} + 5\hat{j})$$

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12

$$\hat{e}_2 \times (\vec{F}_1+\vec{F}_2) = \hat{e}_2 \times [m(-2\hat{i}+5\hat{j})]$$

$$|\vec{F}_1| \left(\frac{1}{2\sqrt{2}} - \frac{\sqrt{3}}{2\sqrt{2}}\right)\hat{k} = m\left(\frac{-5}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right)\hat{k}$$

$$|\vec{F}_1| = \frac{6m}{1+\sqrt{3}}$$

IIly

$$\hat{e}_1 \times (\vec{F}_1+\vec{F}_2) = \hat{e}_1 \times m(-2\hat{i}+5\hat{j})$$

$$|\vec{F}_2| \left(\frac{\sqrt{3}}{2\sqrt{2}} + \frac{1}{2\sqrt{2}}\right) = \left(\frac{5\sqrt{3}}{2}+1\right)m$$

$$|\vec{F}_2| = \frac{\sqrt{2}\,(5\sqrt{3}+2)\,m}{1+\sqrt{3}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.16 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{k}$. Write the equation $\sum\vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

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11. 1.16

$$\vec{F}_1 + \vec{F}_2 + \vec{F}_3 = m\vec{a}$$

$$\Rightarrow [20\hat{i} - 5\hat{j}] + [F_{2x}\hat{i} + F_{2z}\hat{k}] + \left[\frac{F_3}{2}\hat{i} + F_3\frac{\sqrt{3}}{2}\hat{j}\right] = 2[-0.2\hat{i} + 2.2\hat{j} + 1.7\hat{k}]$$

$$\left[20 + F_{2x} + \frac{F_3}{2}\right]\hat{i} + \left[-5 + \frac{F_3\sqrt{2}}{2}\right]\hat{j} + F_{2z}\hat{k} = -0.4\hat{i} + 4.4\hat{j} + 3.4\hat{k}$$

$$20 + F_{2x} + \frac{F_3}{2} = -0.4 \longrightarrow (i)$$

$$-5 + \frac{F_3\sqrt{3}}{2} = 4.4 \longrightarrow (ii)$$

$$F_{2z} = 3.4 \longrightarrow (iii)$$

$F_{2z} = 3.4$

$F_3 = 10.85$

$F_{2x} = -25.827$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\text{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\ \text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
     .
     .
     .
T   =
```

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11.1.17

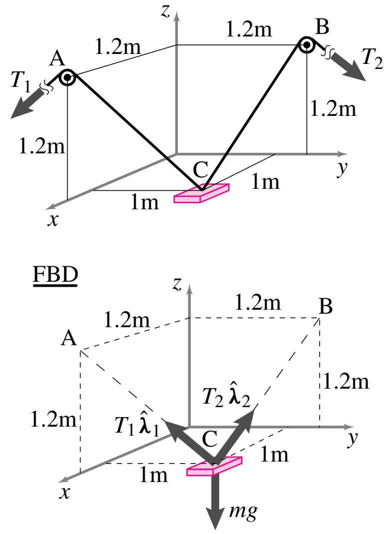
Say the tension be T_1, T_2, T_3

then

$$T_1 \frac{\vec{r}_{AB}}{|\vec{r}_{AB}|} + T_2 \frac{\vec{r}_{AC}}{|\vec{r}_{AC}|} + T_3 \frac{\vec{r}_{AD}}{|\vec{r}_{AD}|} = m\vec{a}$$

$$\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix} = \begin{bmatrix} 3.4675 \\ 0.3366 \\ 0.1083 \end{bmatrix}$$

12.1.18 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\mathrm{N}$ and $T_2 = 1500\,\mathrm{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]

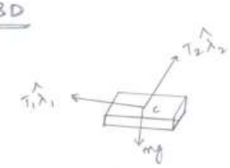


Problem 12.18

16

11.1.19

FBD



$$\hat{\lambda}_1 = \frac{\vec{r}_{CA}}{|\vec{r}_{CA}|}$$

$$\hat{\lambda}_1 = \frac{(\hat{i}+\hat{j}) - (1.2\hat{i} + 1.2\hat{k})}{|\vec{r}_{CA}|}$$

$$= -\frac{1}{\sqrt{1.08}}\left[-0.2\hat{i} + \hat{j} - 0.2\hat{k}\right]$$

Illy.

$$\hat{\lambda}_2 = -\frac{[\hat{i} - 0.2\hat{j} - 0.2\hat{k}]}{\sqrt{1.08}}$$

using $\Sigma\vec{F} = m\vec{a}$

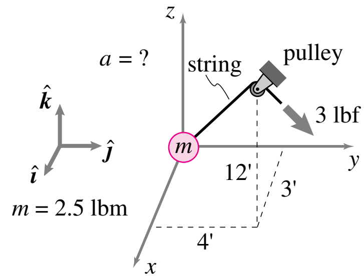
we get $m\vec{a} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 - mg\hat{k}$

$$m\vec{a} = \frac{T_1}{\sqrt{1.08}}\begin{bmatrix}0.2\\-1\\0.2\end{bmatrix} + \frac{T_2}{\sqrt{1.08}}\begin{bmatrix}-1\\0.2\\0.2\end{bmatrix} - mg\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$\vec{a} = 6.2\hat{i} - 25.98\hat{j} - 4.8038\hat{k}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.19 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and lbf / lbm = g. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.



Problem 12.19

11.1.20

$$\vec{F} = m\vec{a}$$

$$3g\,\frac{[3\hat{i} + 4\hat{j} + 12\hat{k}]}{13} = m\vec{a}$$

$$\cancel{\vec{a} = \frac{3}{32} \times (32)}$$

$$\vec{a} = \frac{3}{2.5} \times 32 \left[\frac{3\hat{i} + 4\hat{j} + 12\hat{k}}{13}\right]$$

$$\vec{a} = 5.9\,\hat{i} + 7.87\,\hat{j} + 23.63\,\hat{k}$$

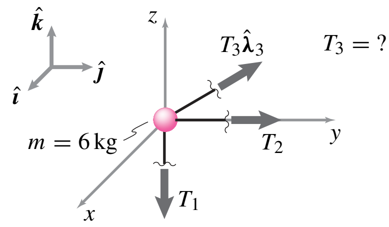
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.20 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{\boldsymbol{a}} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free-body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3.

$\hat{\boldsymbol{\lambda}}_3 = \frac{1}{13}(3\hat{\imath} + 12\hat{\jmath} + 4\hat{k})$



$\vec{\boldsymbol{a}} = (\text{-}.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\frac{m}{s^2}$

Problem 12.20

Answer:
$T_3 = 13\,\mathrm{N}$

11.1.21

$\sum_i \vec{F} = m\vec{a}$

a) $-T_1\hat{k} + T_2\hat{j} + T_3\frac{1}{13}(3\hat{i} + 12\hat{j} + 4\hat{k}) = [-0.5\hat{i} + 2.5\hat{j} + \frac{1}{3}\hat{k}]\,6$

$\Rightarrow \frac{3T_3\hat{i}}{13} = -3\hat{i}$

$T_3 = -13N$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

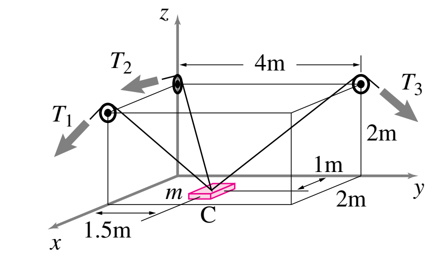
12.1.21

An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = (-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k})\,\mathrm{m/s^2}$.

a) Draw a free-body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use λ's as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.

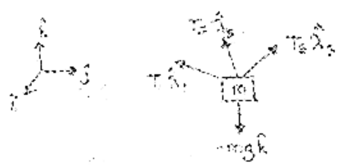


Problem 12.21

10.22

$\vec{a} = -0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}$ [m/s²], $m = 2$ kg

a) Free body diagram:



$\vec{r}_m = 1.0\hat{\imath} + 1.5\hat{\jmath}$ [m]

$\vec{r}_1 = 2.0\hat{\imath} + 2.0\hat{k}$ [m]

$\vec{r}_2 = 2.0\hat{k}$ [m]

$\vec{r}_3 = 4.0\hat{\jmath} + 2.0\hat{k}$ [m]

b) $\vec{r}_{m1} = \vec{r}_1 - \vec{r}_m = 1.0\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m1}| = \frac{1}{2}\sqrt{29}$

$\vec{r}_{m2} = \vec{r}_2 - \vec{r}_m = -1.0\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m2}| = \frac{1}{2}\sqrt{29}$

$\vec{r}_{m3} = \vec{r}_3 - \vec{r}_m = -1.0\hat{\imath} + 2.5\hat{\jmath} + 2.0\hat{k}$ $\quad |\vec{r}_{m3}| = \frac{3}{2}\sqrt{5}$

$$\sum \vec{F} = m\vec{a} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 - T_3\hat{\lambda}_3 - mg\hat{k}$$

or $2\,\mathrm{kg}(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}) = T_1\left(\frac{2\sqrt{29}}{29}\right)(\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k})$

$+ T_2\left(\frac{2\sqrt{29}}{29}\right)(-\hat{\imath} - 1.5\hat{\jmath} + 2.0\hat{k}) + T_3\left(\frac{2\sqrt{5}}{15}\right)(-\hat{\imath} + 2.5\hat{\jmath} + 2.0\hat{k}) - mg\hat{k}$

In component form:

$\hat{\imath}\,(-1.2) = \hat{\imath}\,(0.3714T_1 - 0.3714T_2 - 0.2981T_3)$

$\hat{\jmath}\,(-0.4) = \hat{\jmath}\,(-0.5571T_1 - 0.5571T_2 + 0.7454T_3)$

$\hat{k}\,(4 + 19.62) = \hat{k}\,(0.7428T_1 + 0.7428T_2 + 0.5963T_3)$

Continued on page 24

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

10.22 continued.

In matrix form, we have:

$$\begin{bmatrix} -1.2 \\ -0.4 \\ 23.62 \end{bmatrix} = \begin{bmatrix} 0.3714 & -0.3714 & -0.2981 \\ -0.5571 & -0.5571 & 0.7454 \\ 0.7428 & 0.7428 & 0.5963 \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}$$

Solving in Matlab yields:

$$\boxed{\begin{aligned} T_1 &= 14.28\ \text{N} \\ T_2 &= 5.86\ \text{N} \\ T_3 &= 14.52\ \text{N} \end{aligned}}$$

Alternately, see Matlab code (modified from problem 10.17) on previous page.

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

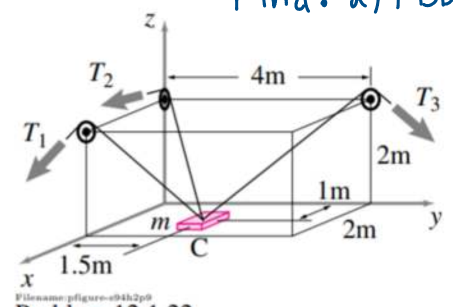
#21 Solution Michelle deSouza

Monday, March 8, 2021 2:54 PM

$\boxed{12.1.22}$

Given: accelerating mass pulled by 3 strings

Find: a) FBD b) LMB c) T_1, T_2, T_3

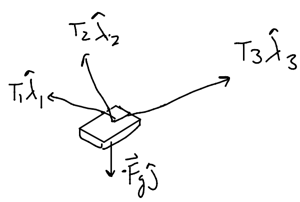


Problem 12.1.22

$m = 2\ kg$

$\vec{a} = -0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\ m/s^2$

a) **FBD**:



$\hat{\lambda}$s defined as unit vector in direction of string

Assumption: ideal pulley, tension is equal on either side

b) **LMB**: $\sum \vec{F} = T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 + T_3\hat{\lambda}_3 - mg\hat{k} = m\vec{a}$

What are $\hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3$?

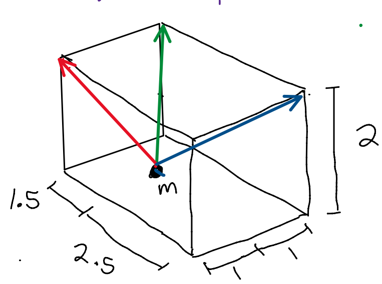
- Find vector $\vec{v}$ in direction of $\hat{\lambda}$, use components of string lengths

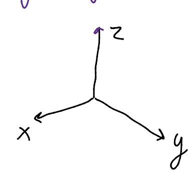
$\vec{v} = a\hat{\imath} + b\hat{\jmath} + c\hat{k}$

$\vec{v}_1 = (1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$

$\vec{v}_2 = (-1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$

$\vec{v}_3 = (-1\hat{\imath} + 2.5\hat{\jmath} + 2\hat{k})\ m$





pick mass as origin

- Convert to unit vector

$v = \sqrt{a^2+b^2+c^2}$ (magnitude!)

$\hat{\lambda} = \frac{\vec{v}}{v}$ → total vector has magnitude 1 ✓

$$V_1 = \sqrt{(1m)^2+(-1.5m)^2+(2m)^2} = 2.69\ m \qquad \hat{\lambda}_1 = \frac{\vec{V}_1}{V_1} = \frac{1}{2.69}(1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$$
$$V_2 = \sqrt{(-1m)^2+(-1.5m)^2+(2m)^2} = 2.69\ m \qquad \hat{\lambda}_2 = \frac{\vec{V}_2}{V_2} = \frac{1}{2.69}(-1\hat{\imath} - 1.5\hat{\jmath} + 2\hat{k})\ m$$
$$V_3 = \sqrt{(-1m)^2+(2.5m)^2+(2m)^2} = 3.35\ m \qquad \hat{\lambda}_3 = \frac{\vec{V}_3}{V_3} = \frac{1}{3.35}(-1\hat{\imath} + 2.5\hat{\jmath} + 2\hat{k})\ m$$

c) Solve for T_1, T_2, T_3

$$T_1\hat{\lambda}_1 + T_2\hat{\lambda}_2 + T_3\hat{\lambda}_3 = m\vec{a} + mg\hat{k}$$

split into $\hat{\imath}, \hat{\jmath}, \hat{k}$ by dotting

$$T_1(\hat{\lambda}_1\cdot\hat{\imath}) + T_2(\hat{\lambda}_2\cdot\hat{\imath}) + T_3(\hat{\lambda}_3\cdot\hat{\imath}) = m(\vec{a}\cdot\hat{\imath})$$
$$T_1(\hat{\lambda}_1\cdot\hat{\jmath}) + T_2(\hat{\lambda}_2\cdot\hat{\jmath}) + T_3(\hat{\lambda}_3\cdot\hat{\jmath}) = m(\vec{a}\cdot\hat{\jmath})$$
$$T_1(\hat{\lambda}_1\cdot\hat{k}) + T_2(\hat{\lambda}_2\cdot\hat{k}) + T_3(\hat{\lambda}_3\cdot\hat{k}) = m(\vec{a}\cdot\hat{k}) + mg$$

put into matrix form

$$\underbrace{\begin{bmatrix} \hat{\lambda}_1\cdot\hat{\imath} & \hat{\lambda}_2\cdot\hat{\imath} & \hat{\lambda}_3\cdot\hat{\imath} \\ \hat{\lambda}_1\cdot\hat{\jmath} & \hat{\lambda}_2\cdot\hat{\jmath} & \hat{\lambda}_3\cdot\hat{\jmath} \\ \hat{\lambda}_1\cdot\hat{k} & \hat{\lambda}_2\cdot\hat{k} & \hat{\lambda}_3\cdot\hat{k} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} T_1 \\ T_2 \\ T_3 \end{bmatrix}}_{T} = m \underbrace{\begin{bmatrix} \vec{a}\cdot\hat{\imath} \\ \vec{a}\cdot\hat{\jmath} \\ \vec{a}\cdot\hat{k} + g \end{bmatrix}}_{b}$$

3 equations, 3 variables — can solve using substitution, dot + cross products (shown in Matlab), etc

$$A \cdot T = b$$
$$T = A^{-1}b$$

solve in Matlab

↓

my code shown below, for another example, refer to "Matlab Code for Three Strings" on Ed

$T_1 = 14.3\ N$

$T_2 = 5.9\ N$

$T_3 = 14.5\ N$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
%MAE 2030 HW 4 SOLUTION
%Problem #21 12.1.22
%Michelle deSouza
%For another example, see 'Matlab Code for Three Strings' on Ed

%Define unit vectors
i = [1;0;0];
j = [0;1;0];
k = [0;0;1];

%Define m, a, g
m = 2; %kg
a = -0.6*i -0.2*j + 2.0*k;%m/s2
g = 9.8; %m/s^2

%%FIND LAMBDAS%%
%Define v1,2,3 vectors
v1 =  1*i - 1.5*j + 2*k;
v2 = -1*i - 1.5*j + 2*k;
v3 = -1*i + 2.5*j + 2*k;
%Find v1,2,3 magnitude
v1mag = norm(v1);
v2mag = norm(v2);
v3mag = norm(v3);
%Convert to unit vectors
lam1 = v1/v1mag;
lam2 = v2/v2mag;
lam3 = v3/v3mag;

%%%%%%%%Method 1: invert a matrix%%%%%%%%
%Create A and b
A = [lam1, lam2, lam3];
b = m*a+m*g*k;
%Invert to solve for T1,2,3
T = A\b


%%%%%%%%Method 2: choose good cross product%%%%%%%%
% T1*lam1 + T2*lam2 + T3*lam3 = b
% dot both sides by a vector perpendicular to two of the terms e.g.
 lam2 and lam3
% those two terms go to 0, so -> dot(perp_vec1,lam1)*T1 =
 dot(perp_vec1,b)
% T = dot(perp_vec1,b)/dot(perp_vec1,lam1)

perp_vec1 = cross(lam2,lam3); %perpendicular to T2 and T3
perp_vec2 = cross(lam1,lam3); %perpendicular to T1 and T3
perp_vec3 = cross(lam1,lam2); %perpendicular to T1 and T2

T1 = dot(perp_vec1, b)/dot(perp_vec1, lam1);
T2 = dot(perp_vec2, b)/dot(perp_vec2, lam2);
T3 = dot(perp_vec3, b)/dot(perp_vec3, lam3);
```


Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
T = [T1 T2 T3]' %identical to Method 1 :)
```

```
T =

   14.2707
    5.8564
   14.5065


T =

   14.2707
    5.8564
   14.5065
```

Published with MATLAB® R2020b

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.22 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\vec{r} = (5\,\mathrm{m})\cos^2(t^2/\mathrm{s}^2)\hat{\imath} + (5\,\mathrm{m})\sin(t^2/\mathrm{s}^2)\cos(t^2/\mathrm{s}^2)\hat{\jmath}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

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11.1.23

$$\vec{r} = 5\cos^2(t^2)\hat{i} + 5\sin(t^2)\cos(t^2)\hat{j}$$

1. Plot trajectory

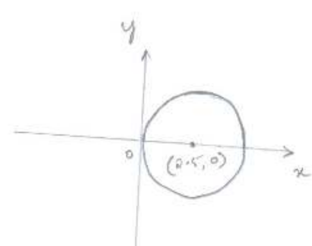
$x = 5\cos^2(t^2)$, $y = 5\sin(t^2)\cdot\cos(t^2)$

$x = 2.5(\cos 2t^2 + 1)$, $y = 2.5\sin(2t^2)$

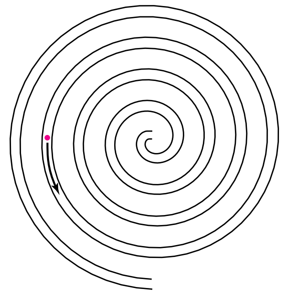
$\frac{x-2.5}{2.5} = \cos 2t^2$, $\frac{y}{2.5} = \sin 2t^2$

$$(x-2.5)^2 + y^2 = (2.5)^2$$

Trajectory is a circle with radius 2.5 and the centre $x = 2.5$, $y = 0$

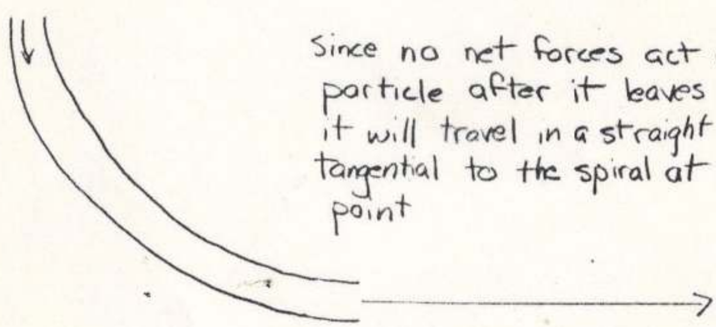


12.1.24 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



Problem 12.24

6.1 Determine the path of a particle after it is blown out of a spiral tube which rests on a frictionless table.



Since no net forces act on the particle after it leaves the tube, it will travel in a straight line tangential to the spiral at its end point

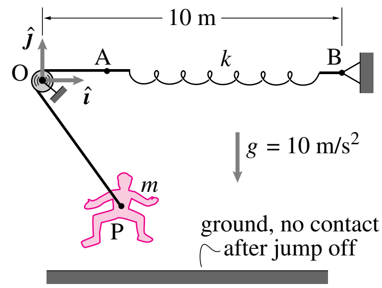
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.25 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\mathrm{m}$ and a stiffness of $k = 200\,\mathrm{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\mathrm{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

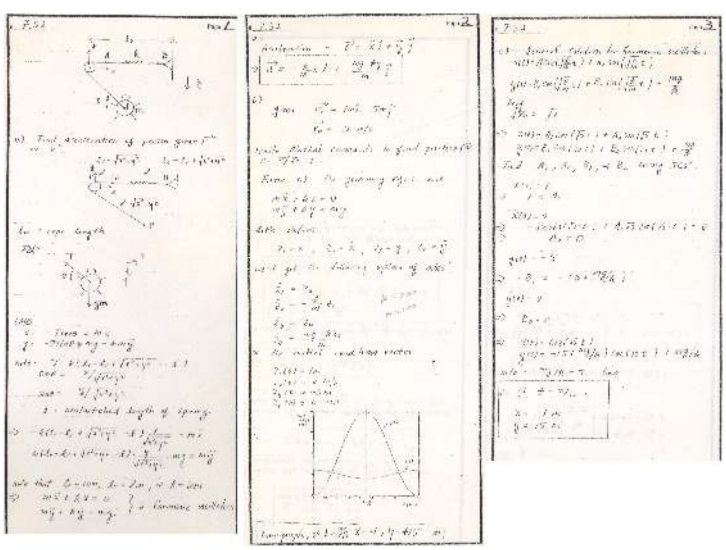
a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\mathrm{m}\hat{\imath} - 5\,\mathrm{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \boldsymbol{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\mathrm{s}$.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



Problem 12.25: Conceptual setup for a bungy jumping system.



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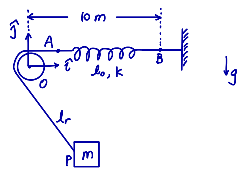
HW5 P22 12.1.26 Teaya Yang



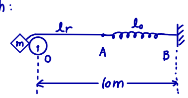
Assume: - pulley has negligible size, mass and friction
- rope is massless and inextensible

Given: $l_o = 2m$ $k = 200 N/m$
$l_r = 8m$ $m = 100\ kg$
$g = 10 m/s^2$

a) Given $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, find $\vec{a}$.

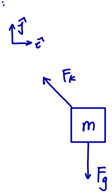
Note that $l_o + l_r = 10m$.

At rest length:



* This means the person is at origin when the spring is at rest length so the spring force is proportional to the position vector.

FBD:



$$\vec{F}_g = -mg\,\hat{\jmath}$$

$$\vec{F}_k = kr\left(-\frac{\vec{r}}{r}\right) = -k\vec{r}$$

LMB: $\Sigma \vec{F} = m\vec{a}$

$$-mg\,\hat{\jmath} - k(x\hat{\imath} + y\hat{\jmath}) = m\vec{a}$$

$$m\vec{a} = -mg\,\hat{\jmath} - kx\,\hat{\imath} - ky\,\hat{\jmath}$$

$$\vec{a} = -\frac{k}{m}x\,\hat{\imath} - \left(\frac{k}{m}y + g\right)\hat{\jmath}$$

$$\vec{a} = -\frac{200}{100}\left(\frac{N/m}{kg}\right)x\,\hat{\imath} - \left[\frac{200\ N/m}{100\ kg}\,y + 10\ m/s^2\right]\hat{\jmath}$$

$$= -2\left(\frac{\cancel{N}}{\cancel{kgm}}\right)\left(\frac{\cancel{kgm}/s^2}{\cancel{N}}\right) - \left[2\left(\frac{\cancel{N}}{\cancel{kgm}}\right)\left(\frac{\cancel{kgm}/s^2}{\cancel{N}}\right) + 10\ m/s^2\right]\hat{\jmath}$$

$$\boxed{\vec{a} = -(2\,s^{-2})\,x\,\hat{\imath} - \left[(2\,s^{-2})y + 10\,ms^{-2}\right]\hat{\jmath}}$$

b) Given ICs, find $\vec{r}\left(t = \frac{\pi}{\sqrt{2}}s\right)$ numerically.

ICs: $\vec{r}_o = 1m\,\hat{\imath} - 5m\,\hat{\jmath}$

$\vec{V}_o = \vec{0}$

$$z = \begin{bmatrix} r_x \\ r_y \\ V_x \\ V_y \end{bmatrix} \quad \dot{z} = \begin{bmatrix} V_x \\ V_y \\ a_x \\ a_y \end{bmatrix} \quad z_o = \begin{bmatrix} 1 \\ -5 \\ 0 \\ 0 \end{bmatrix}$$

$$[r] = \begin{bmatrix} r_x \\ r_y \end{bmatrix} = z(1:2)$$

$$[V] = \begin{bmatrix} V_x \\ V_y \end{bmatrix} = z(3:4)$$

In rhs:

$$[F_{tot}] = -mg[\hat{\jmath}] - k[r] \quad \text{where} \quad [\hat{\jmath}] = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$[a] = [F_{tot}]/m$$

$$\dot{z} = \begin{bmatrix} [V] \\ [a] \end{bmatrix}$$

This is integrated with ode45 until tend $= \frac{\pi}{\sqrt{2}}$ s.
Then the last data point is printed.

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b) cont.

solution : $\boxed{\vec{r}\left(t=\frac{\pi}{\sqrt{2}}s\right) = -1m\,\hat{\imath} - 5m\,\hat{\jmath}}$

what the trajectory looks like :

See next page for Matlab file, part c) comes after.

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Contents


```
% MAE2030 HW5 P22 12.1.26
% Finding the position of bungy jumper using ODE45
```

Part b

Fining position at t = pi/sqrt(2)

```
% Parameters
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
r0 = [1;-5];
v0 = [0;0];
z0 = [r0;v0];

% Time
tend = pi/sqrt(2);
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Display Solution
fprintf('Position at t = pi/sqrt(2) is %.2f m i %.2f m j\n',zarray(end,1),zarray(end,2))


% rhs function
function zdot = rhs(t,z,p)
% Unit Vectors
i = [1;0]; j = [0;1];

% Unpack Variables
r = z(1:2);
v = z(3:4);
% Unpack Parameters
m = p.m; k = p.k; g = p.g;

% Equation
Ftot = -m*g*j-k*r;
vdot = Ftot/m;

% Return zdot
zdot = [v;vdot];
```

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```
end
```

```
Position at t = pi/sqrt(2) is -1.00 m i -5.00 m j
```

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c) Solve part b) analytically.

$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$

$\{\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = -(2s^{-2})x\hat{\imath} - [(2s^{-2})y + 10\,ms^{-2}]\hat{\jmath}\}$

$\{\ \}\cdot\hat{\imath}$: $\ddot{x} = -(2s^{-2})x$ ①

$\{\ \}\cdot\hat{\jmath}$: $\ddot{y} = -(2s^{-2})y - 10\,ms^{-2}$ ②

① $\ddot{x} + (2s^{-2})x = 0$

Sol : $x = A\cos(\sqrt{2}s^{-1}t) + B\sin(\sqrt{2}s^{-1}t)$

$\dot{x} = -\sqrt{2}s^{-1}A\sin(\sqrt{2}s^{-1}t) + \sqrt{2}s^{-1}B\cos(\sqrt{2}s^{-1}t)$

IC: $x(0) = 1\,m = A$

$V_x(0) = 0\,ms^{-1} = \sqrt{2}s^{-1}B \Rightarrow x(t) = \cos(\sqrt{2}s^{-1}t)\ m$

$B = 0\,m$

② $\ddot{y} + (2s^{-2})y = -10\,ms^{-2}$

Sol: $y_h = C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t)$

$y_p = E$

$\ddot{y}_p + (2s^{-2})y_p = -10\,ms^{-2}$

$y_p = -5\,m$

IC: $y(0) = -5\,m = C - 5\,m \rightarrow C = 0\,m$

$V_y(0) = 0\,ms^{-1} = \sqrt{2}s^{-1}D \longrightarrow D = 0\,m$

↳ $y(t) = -5\,m$

$y = y_h + y_p = C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t) - 5\,m$

$\dot{y} = -\sqrt{2}s^{-1}C\sin(\sqrt{2}s^{-1}t) + \sqrt{2}s^{-1}D\cos(\sqrt{2}s^{-1}t)$

Finding $\vec{r}(t = \frac{\pi}{\sqrt{2}}s)$: $x(t = \frac{\pi}{\sqrt{2}}s) = \cos(\pi s^{-1}t)\,m = -1\,m$

$y(t = \frac{\pi}{\sqrt{2}}s) = -5\,m$

$\boxed{\vec{r}(t = \frac{\pi}{\sqrt{2}}s) = -1\,m\,\hat{\imath} - 5\,m\,\hat{\jmath}}$

d) General solution.

$$\boxed{\begin{aligned} x(t) &= A\cos(\sqrt{2}s^{-1}t) + B\sin(\sqrt{2}s^{-1}t) \\ y(t) &= C\cos(\sqrt{2}s^{-1}t) + D\sin(\sqrt{2}s^{-1}t) - 5\,m \end{aligned}}$$

x and y are sine waves with same frequency but different phase shift.

The general result is elliptical, but solution becomes a straight line when the phase shift is the same or if one of them is constant. (When both are constant, system is at rest at one point in space.)

See next page for examples.

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d) cont.

Examples where solution is an ellipse :

i) IC : $z_0 = [1, -5, 3, -3]'$

Initial condition given in part b) with added initial velocity.

Since velocity components are not zero at equilibrium position , there are oscillations in both directions.

ii) $z_0 = [0, 0, 1, -1]'$

Starting at origin with initial velocity

Similar system with nonzero velocity components at equlibrium position.

Different initial conditions for this case change the shape and orientation of the ellipse due to the change in phase shift and amplitude of the sinusoidal waves.

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d) cont.

Examples of straight-line solutions:

iii) $z_0 = [1, -6, 0, 0]'$

Starting at a non-equilibrium position, with zero initial velocity

With zero initial velocity, the sine terms in both $x(t)$ and $y(t)$ disappear.

$x(t) = A\cos(\sqrt{3}\,s^{-1}t)$

$y(t) = C\cos(\sqrt{3}\,s^{-1}t) - 5\,m$

Therefore $y = Fx + G$ → linear relationship!

iv) $z_0 = [1, -5, 0, 0]'$

Conditions given in part b)

This condition given in part b) is special because the y component of position is at equilibrium with no initial velocity, while oscillation exists in x.

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v) $z0 = [0, -5, 0, 1]'$

Jumping straight up and down

This is the case where x component of position remains constant while y component oscillates, meaning if the person jumps up and down right under the pulley they wouldn't be swinging left or right (intuitive).

Solution at one point:

vi) $z0 = [0, -5, 0, 0]'$

Equilibrium position.

If starting at the equilibrium position with no initial velocity, the result is static.

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Contents


```
% MAE2030 HW5 P22 12.1.26
% Finding the position of bungy jumper using ODE45
```

Part b

Fining position at t = pi/sqrt(2)

```
% Parameters
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
r0 = [1;-5];
v0 = [0;0];
z0 = [r0;v0];

% Time
tend = pi/sqrt(2);
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Display Solution
fprintf('Position at t = pi/sqrt(2) is %.2f m i %.2f m j\n',zarray(end,1),zarray(end,2))
```

Part d

Finding general motion Parameters

```
p.m = 100; % [kg]
p.k = 200; % [N/m]
p.g = 10; % [kg/m^2]

% IC
% z0 = [1,-5,3,-3]';
% z0 = [1,-6,0,0]';
% z0 = [0,0,1,-1]';
% z0 = [1,-5,0,0]';
z0 = [0,-5,0,0]';
% z0 = [0,-5,0,1]';
```

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```
% Time
tend = 10;
tspan = [0 tend];

% Equation
eq = @(t,z) rhs(t,z,p);

% Options
smallnumber = 1e-6;
options = odeset('AbsTol',smallnumber,'RelTol',smallnumber);

% Solve
[tarray, zarray] = ode45(eq,tspan,z0,options);

% Plot
x = zarray(:,1);
y = zarray(:,2);
figure(1)
hold on
box on
grid minor
axis equal
plot(x,y,'bo')
xlabel('x(m)')
ylabel('y(m)')
title('Bungy Jumper Trajectory')

% rhs function
function zdot = rhs(t,z,p)
% Unit Vectors
i = [1;0]; j = [0;1];

% Unpack Variables
r = z(1:2);
v = z(3:4);
% Unpack Parameters
m = p.m; k = p.k; g = p.g;

% Equation
Ftot = -m*g*j-k*r;
vdot = Ftot/m;

% Return zdot
zdot = [v;vdot];

end
```

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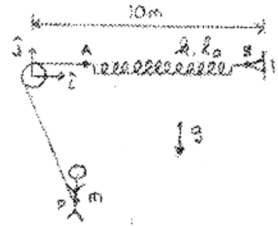
10.26



Given: $k = 200\ \mathrm{N/m}$
$\ell_0 = 2\ \mathrm{m}$
$\ell_{AP} = 8\ \mathrm{m} = \ell_r$
$m = 100\ \mathrm{kg}$
Assume no air drag.

a) $\vec{r} = x\hat{\imath} + y\hat{\jmath}$, $\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$, Find $\vec{a}$

PERSON: (1) PULLEY: (2)

$\hat{\lambda} = \frac{\vec{r}}{|\vec{r}|} = \frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}}$

δ = extension of spring = $\ell - \ell_0$, where $\ell = 10 - (\ell_r - \sqrt{x^2+y^2})$

$\therefore \ell = 10 - 8 + \sqrt{x^2+y^2} = 2 + \sqrt{x^2+y^2}$

$\therefore \delta = \ell - \ell_0 = \sqrt{x^2+y^2}$

From FBD 2: $T = k\delta$ (tension on both sides of pulley must be equal) $\therefore T = k\sqrt{x^2+y^2}$

From FBD 1: $\sum \vec{F} = m\vec{a} = -T\hat{\lambda} - mg\hat{\jmath}$

$\therefore -k\sqrt{x^2+y^2}\,\frac{x\hat{\imath} + y\hat{\jmath}}{\sqrt{x^2+y^2}} - mg\hat{\jmath} = m\vec{a}$

or $-kx\hat{\imath} - (ky + mg)\hat{\jmath} = m\vec{a}$ – plug in values

$\vec{a} = \frac{-200\ \mathrm{N/m}\ x}{100\ \mathrm{kg}}\hat{\imath} - \frac{200\ \mathrm{N/m}\ y + 100\ \mathrm{kg}(10\ \mathrm{m/s^2})}{100\ \mathrm{kg}}\hat{\jmath} = \boxed{2x\,[s^{-2}]\hat{\imath} - (2y + 10\ \mathrm{m})[s^{-2}]\hat{\jmath}}$

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Problem 10.26 (b).

```
function Prob1026()
% Problem 10.26 Solution
% March 11, 2002

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%  VARIABLES (Assume consistent units)
%   r  = displacement (vector with x and y components)
%   v  = dr/dt

% INITIAL CONDITIONS
r0= [1 -5]'; % initial position
v0= [0 0]'; % initial velocity
z0= [r0;v0]; % pack variables

tspan =[0 pi/sqrt(2)]; % time interval of integration

[t zarray] = ode45(@rhs,tspan, z0);

% Unpack Variables
r= zarray(:,1:2);

disp(r(end,:));

% ANSWER:
% ans =
%
%     -1.0000     -5.0000    (meters)
%

end

% THE DIFFERENTIAL EQUATION 'The Right Hand Side'
function zdot = rhs(t,z)

%Unpack variables
r= z(1:2);
v= z(3:4);

%The equations
rdot= v;
vdot= [-2*r(1) -2*r(2)-10]';

% Pack the rate of change of r and v
zdot= [rdot; vdot];

end
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

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b) See Matlab code on previous page

c) From part (a),

$$\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath} = -2x\hat{\imath} - (2y+10)\hat{\jmath}$$

$$\therefore \quad (\ddot{x} = -2x)\hat{\imath}\cdot\hat{\imath} \Rightarrow \ddot{x} = -2x \quad (1)$$

$$(\ddot{y} = -2y-10)\hat{\jmath}\cdot\hat{\jmath} \Rightarrow \ddot{y} = -2y-10 \quad (2)$$

Solve (1) and (2) with $\vec{r}(0) = \hat{\imath} - 5\hat{\jmath}$, $\vec{v}(0) = \vec{0}$

(1) $\ddot{x} = -2x$, so $x(t) = A\sin(\sqrt{2}t) + B\cos(\sqrt{2}t)$

$$\dot{x}(t) = \sqrt{2}A\cos(\sqrt{2}t) - \sqrt{2}B\sin(\sqrt{2}t)$$

$$\dot{x}(0) = 0 = \sqrt{2}A \quad \therefore A = 0$$

$$x(0) = 1 = B\cos(0) \quad \therefore B = 1$$

$$\therefore x(t) = \cos(\sqrt{2}t)$$

(2) $\ddot{y} = -2y - 10$, so $y(t) = C\sin(\sqrt{2}t) + D\cos(\sqrt{2}t) - 5$ y_p

$$\dot{y}(0) = 0 = \sqrt{2}C \quad \therefore C = 0$$

$$y(0) = -5 = D\cos(0) - 5 \quad \therefore D = 0$$

$$\therefore y(t) = -5$$

$$\vec{r}(t) = \cos\sqrt{2}t\,\hat{\imath} - 5\hat{\jmath} = x(t)\hat{\imath} + y(t)\hat{\jmath}$$

$$\therefore \vec{r}\left(\tfrac{\pi}{\sqrt{2}}\right) = \cos\left(\sqrt{2}\tfrac{\pi}{\sqrt{2}}\right)\hat{\imath} - 5\hat{\jmath} = \boxed{-\hat{\imath} - 5\hat{\jmath}\ [\mathrm{m}]}$$

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12.1.26 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

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11.1.27

a) Equation of motion

$$m\ddot{a} = -mg\hat{j}$$

$$\ddot{y} = -g\hat{j}$$

$$\ddot{x} = 0$$

$$\vec{V}_y = \dot{y} = (V_0 \sin\theta_0 - gt)\hat{j}$$

$$\vec{V}_x = V_0\cos\theta_0\,\hat{i}$$

$$\vec{V} = V_0\cos\theta_0\,\hat{i} + (V_0\sin\theta_0 - gt)\hat{j}$$

b) acceleration

$$\vec{a} = -g\hat{j}$$

Velocity

~~$\vec{V} = V_0 t \cos$~~

$$\vec{V} = V_0\cos\theta_0\,\hat{i} + (V_0\sin\theta_0 - gt)\hat{j}$$

Position

$$\vec{r} = V_0 t\cos\theta_0\,\hat{i} + (V_0 t\sin\theta_0 - \tfrac{1}{2}gt^2)\hat{j}$$

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2c

c) maximum height-

$$\frac{d}{dt}(v_y) = 0 \Rightarrow v_0 \sin\theta_0 - gt = 0 \Rightarrow t = \frac{v_0 \sin\theta_0}{g}$$

$$h_{max} = \left(\frac{v_0 \sin\theta_0}{g}\right) v_0 \sin\theta_0 - \frac{1}{2} g \left(\frac{v_0 \sin\theta_0}{g}\right)^2$$

$$h_{max} = \frac{v_0^2 \sin^2\theta_0}{2g}$$

d) $y = 0 \Rightarrow v_0 t \sin\theta_0 - \frac{1}{2} g t^2 = 0$

$$t = \frac{2v_0 \sin\theta_0}{g} = t_0$$

$$r_x\Big|_{t=t_0} = v_0 \cos\theta_0 \left(\frac{2v_0 \sin\theta_0}{g}\right) = \frac{v_0^2 \cos 2\theta_0}{g}$$

e) $y = v_0 t \sin\theta_0 - \frac{1}{2} g t^2$

$x = v_0 t \cos\theta_0$

eleminating 't' we get-

$$y = v_0 \sin\theta_0 \left(\frac{x}{v_0 \cos\theta_0}\right) - \frac{1}{2} g \left(\frac{x}{v_0 \cos\theta_0}\right)^2$$

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$$y = \frac{x}{V_0 \cos\theta_0}\left[V_0 \sin\theta_0 - \frac{gx}{2V_0\cos\theta_0}\right]$$

$$-y = \frac{gx^2}{2V_0^2\cos^2\theta_0} - x\tan\theta_0$$

$$-\frac{2V_0^2\cos^2\theta_0}{g}\, y = x^2 - 2x\left(\frac{V_0^2\cos^2\theta_0 \tan\theta_0}{g}\right)$$

$$\left(x - \frac{V_0^2 \sin 2\theta}{2g}\right)^2 = -\frac{2V_0^2\cos^2\theta}{g}\, y + \left(\frac{V_0^2 \sin^2\theta}{2g}\right)^2$$

$$\left(x - \frac{V_0^2 \sin 2\theta}{2g}\right)^2 = -\frac{2V_0^2\cos^2\theta}{g}\left[y - \frac{V_0^2 \sin^2\theta}{2g}\right]$$

This equation represents a parabola

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.27 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\mathrm{m/s}$, the drag proportionality constant is $C = 5\,\mathrm{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free-body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

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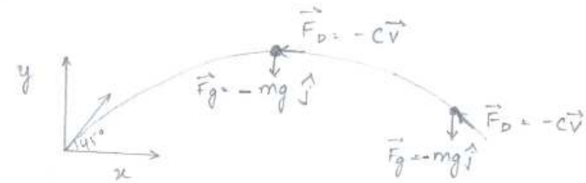
11.1.28

a)



b) Linear momentum balance equation

$$m a_x = -c v_x \qquad m\ddot{x} = -c\dot{x}$$

$$m a_y = -mg - c v_y \qquad m\ddot{y} = -mg - c\dot{y}$$

Where $\vec{V} = v_x\hat{i} + v_y\hat{j}$

c) We can put in the state space form as

$$\begin{bmatrix} \dot{x} \\ \dot{y} \\ \ddot{x} \\ \ddot{y} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{c}{m} & 0 \\ 0 & 0 & 0 & -\frac{c}{m} \end{bmatrix} \begin{bmatrix} x \\ y \\ \dot{x} \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ -g \end{bmatrix}$$

d) Matlab

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

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e) $m\ddot{x} = -c\dot{x} \Rightarrow \dfrac{m\,dv_x}{dt} = -cv_x$

$$\int \frac{dv_x}{v_x} = -\frac{c}{m}\int dt$$

$$\Rightarrow \log v_x = -\frac{ct}{m} + c_1$$

$$v_x = Ae^{-ct/m}$$

at $t=0$, $v_x\big|_{t=0} = u\cos\theta \Rightarrow v_x = u\cos\theta\, e^{-\frac{ct}{m}} \longrightarrow (1)$

Similarly

$$\int \frac{dv_y}{v_y + \frac{mg}{c}} = \frac{-c}{m}\int dt$$

$$\log\left(v_y + \frac{mg}{c}\right) = -\frac{ct}{m} + c_1$$

$$v_y = Ae^{-\frac{ct}{m}} - \frac{mg}{c}$$

$$A - \frac{mg}{c} = u\sin\theta$$

$$A = u\sin\theta + \frac{mg}{c}$$

$$v_y = u\sin\theta\, e^{-ct/m} + \frac{mg}{c}\left(e^{-ct/m} - 1\right)$$

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Further Integration

$$\frac{dx}{dt} = u\cos\theta\, e^{-ct/m}$$

$$x = u\cos\theta\left(-\frac{m}{c}\right)e^{-ct/m} + B$$

$$x\big|_{t=0} = 0 \Rightarrow B = u\cos\theta\left(\frac{m}{c}\right)$$

$$x = u\cos\theta\left(\frac{m}{c}\right)\left[1 - e^{-ct/m}\right]$$

Similarly

$$\dot{y} = (u\sin\theta + mg/c)\, e^{-ct/m} - \frac{mg}{c}$$

$$y = (u\sin\theta + mg/c)\left(-\frac{m}{c}\right)e^{-ct/m} - \frac{mgt}{c} + B$$

$$y\big|_{t=0} = 0 \Rightarrow B = \left(u\sin\theta + \frac{mg}{c}\right)\left(\frac{m}{c}\right)$$

$$y = \frac{m}{c}\left(u\sin\theta + \frac{mg}{c}\right)\left(1 - e^{-ct/m}\right) - \frac{mgt}{c}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.28 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{e}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates.

Answer:
Equation of motion: $-mg\hat{\jmath} - b(\dot{x}^2 + \dot{y}^2)\left(\frac{\dot{x}\hat{\imath}+\dot{y}\hat{\jmath}}{\sqrt{\dot{x}^2+\dot{y}^2}}\right) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})$.

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11.1.29

Equation of motion

$$m\ddot{x} = \frac{-b(\dot{x}^2+\dot{y}^2)\dot{x}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -b\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{y} = -mg - \frac{b(\dot{x}^2+\dot{y}^2)\dot{y}}{\sqrt{\dot{x}^2+\dot{y}^2}} = -mg - b\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\ddot{x} = -\frac{b\dot{x}}{m}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$\ddot{y} = -g - \frac{b\dot{y}}{m}\sqrt{\dot{x}^2+\dot{y}^2}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.29 The equations of motion from problem 12.1.28 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations.

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plot of its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for the ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

Answer:
(a) System of equations:

$$\begin{aligned}\dot{x} &= v_x\\ \dot{y} &= v_y\\ \dot{v}_x &= -\frac{b}{m}v_x\sqrt{v_x^2+v_y^2}\\ \dot{v}_y &= -g-\frac{b}{m}v_y\sqrt{v_x^2+v_y^2}\end{aligned}$$

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11·1·30

a) System of first order differential Equation

$$\dot{x} = u$$
$$\dot{y} = v$$
$$\dot{u} = -\frac{b}{m} u \sqrt{u^2+v^2}$$
$$\dot{v} = -g - \frac{b}{m} v \sqrt{u^2+v^2}$$

b) using $g = 10\,m/sec^2$, $m = 10\,kg$

$b = 5N/(m/sec)^2$, $V_0 = 100\,m/sec$

c) Matlab

d) From Matlab we get initial angle that maximizes range is

Theta = 15·2432°

without Air resistance (b=0)

Theta = 45°

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

12.1.30 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{\boldsymbol{v}}|$, where $\vec{\boldsymbol{v}}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{\boldsymbol{r}} = \vec{\boldsymbol{0}}$ with $\vec{\boldsymbol{v}} = v_{x0}\hat{\boldsymbol{\imath}} + v_{y0}\hat{\boldsymbol{\jmath}}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

HW6-P24-12.1.31

Tuesday, March 16, 2021 2:22 PM

Solution by Michael Zakaworotny

12.1.31 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of v_{x0}, v_{y0}, m, g, c, and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

24| 12.1.31 This problem should expand your understanding of parabolic-flight ballistics to the more realistic ballistics of things where air drag is important. Add part (e): Assume, in some consistent units, that $m = 1$, $g = 10$ and $c = 1$ in some consistent unit system. Assume the initial launch speed is 1000. Forget the wall. Find the best angle to achieve the maximum range. What is that angle and range?

Given: m, c, g, linear drag

Find: a) Terminal speed in freefall

b) Two 2^{nd} order scalar ODE's for 2D motion

c) Given $\vec{r}_0 = \vec{0}$, $\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$, and a wall distance d away, find height, h, of collision with wall

d) Computer simulation, including time and height of wall collision

e) Given $m=1$, $g=10$, $c=1$, $|\vec{v}_0| = 1000$, find angle to optimize range

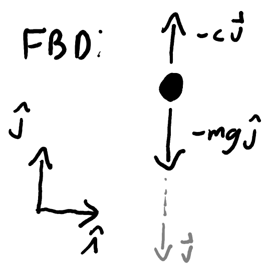
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Solution:

a) Find terminal velocity in freefall

Freefall - $\vec{v} \cdot \hat{\imath} = 0$

FBD:



In steady state, mass does not accelerate

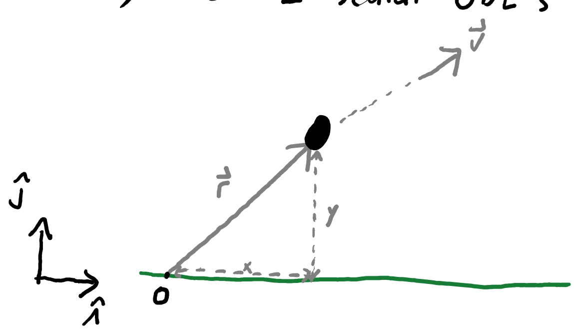
$$\Sigma \vec{F} = m\vec{a}$$

$$\{-mg\hat{\jmath} - c\vec{v} = m\cancel{\vec{a}}^{\,0}\}$$

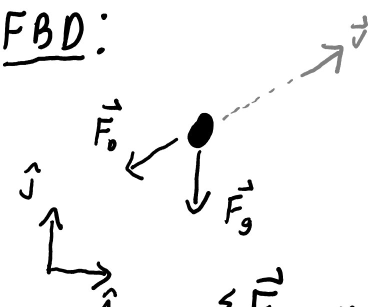
$$\{\}\cdot\hat{\jmath}: \; -mg - cv_t = 0$$

$$\boxed{v_t = \frac{-mg}{c}}$$ ← Terminal speed

b) Get 2 scalar ODE's for particle motion



FBD:



$$\vec{F}_D = \underbrace{c|\vec{v}|}_{\text{scalar magnitude}} \cdot \underbrace{-\frac{\vec{v}}{|\vec{v}|}}_{\text{direction - unit vector}} = -c\vec{v}$$

$$\vec{F}_g = -mg\hat{\jmath}$$

$$\Sigma \vec{F} = m\vec{a}$$

$$\vec{F}_g + \vec{F}_D = m\vec{a}$$

$$\vec{r} = x\hat{\imath} + y\hat{\jmath}$$

$$\vec{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

$$\vec{a} = \ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

$$\{-mg\hat{\jmath} - c(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\}$$

$$\{\}\cdot\hat{\imath}: \quad -c\dot{x} = m\ddot{x}$$

$$\{\}\cdot\hat{\jmath}: \quad -mg - c\dot{y} = m\ddot{y}$$

$$\boxed{\begin{aligned} m\ddot{x} + c\dot{x} &= 0 \\ m\ddot{y} + c\dot{y} &= -mg \end{aligned}}$$

c) Get height, h, when $x = d$

Solve for $x(t)$ and $y(t)$

$$\vec{r}_0 = \vec{0} \Rightarrow x(0) = 0, \; y(0) = 0$$

$$\vec{v}_0 = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath} \Rightarrow v_x(0) = v_{x0}, \; v_y(0) = v_{y0}$$

$\underline{x}$: $m\ddot{x} + c\dot{x} = 0 \Rightarrow m\dot{v}_x + cv_x = 0$

$$\frac{dv_x}{dt} + \frac{c}{m}v_x = 0$$ ← solve as a 1st order separable ODE

$$v_x(t) = Ae^{-c/m \cdot t}$$

Solve for A using $v_x(0) = v_{x0}$

$$v_{x0} = A\cancel{e^{0}}^{\,1} \Rightarrow A = v_{x0}$$

$$\boxed{v_x(t) = v_{x0}e^{-c/m \cdot t}}$$

$$v_x(t) = \frac{dx}{dt} \Rightarrow x = \int v_x(t)\,dt$$

$$x(t) = v_{x0} \cdot -\frac{m}{c}e^{-c/m \cdot t} + B$$

Solve for B using $x(0) = 0$

$$0 = -v_{x0}\frac{m}{c} \cdot \cancel{e^{0}}^{\,1} + B \Rightarrow B = v_{x0} \cdot \frac{m}{c}$$

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$$\boxed{x(t) = \frac{v_{x0} m}{c}\left[1 - e^{-c/m \cdot t}\right]}$$

$\underline{y}$: $m\ddot{y} + c\dot{y} = -mg \Rightarrow m\dot{v}_y + cv_y = -mg$

Homogeneous: $\frac{dv_y}{dt} + \frac{c}{m} v_y = 0$ Particular: $v_y = B$

$v_y(t) = Ae^{-c/m\, t}$

$0 + cB = -mg$

$B = -mg/c$

$$v_y(t) = Ae^{-c/m\, t} - mg/c$$

Solve for A using $v_y(0) = v_{y0}$

$v_{y0} = A \cdot \cancel{e^0}^{1} - mg/c$

$A = v_{y0} + mg/c$

$$\boxed{v_y(t) = (v_{y0} + mg/c)\, e^{-c/m \cdot t} - mg/c}$$

$v_y(t) = \frac{dy}{dt} \Rightarrow y = \int v_y(t)\, dt$

$y(t) = (v_{y0} + mg/c)(-m/c)\, e^{-c/m \cdot t} - mg/c\, t + B$

Solve for B using $y(0) = 0$

$0 = -\frac{m}{c}(v_{y0} + mg/c)\cancel{e^0}^{1} - 0 + B$

$B = \frac{m}{c}(v_{y0} + mg/c)$

$$\boxed{y(t) = \frac{m}{c}(v_{y0} + mg/c)\left[1 - e^{-c/m \cdot t}\right] - mg/c \cdot t}$$

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* To solve for y position as function of x, combine $x(t)$ and $y(t)$

$$t = \frac{-m}{c}\ln\left[1 - \frac{x(t)\,c}{m\,V_{x0}}\right]$$

$$y(x(t)) = \frac{m}{c}\left(v_{y0} + \frac{mg}{c}\right)\left[1 - \left(1 - \frac{x(t)c}{mv_{x0}}\right)\right] - \frac{mg}{c}\cdot\left(\frac{-m}{c}\right)\ln\left(1 - \frac{x(t)c}{mv_{x0}}\right)$$

$$\boxed{y(x(t)) = \left(v_{y0} + \frac{mg}{c}\right)\left(\frac{x(t)}{v_{x0}}\right) + \frac{m^2 g}{c^2}\ln\left(1 - \frac{x(t)\,c}{mv_{x0}}\right)}$$

Let $x(t) = d$

$$\boxed{y(x=d) = \left(v_{y0} + \frac{mg}{c}\right)\left(\frac{d}{v_{x0}}\right) + \frac{m^2 g}{c^2}\ln\left(1 - \frac{dc}{mv_{x0}}\right)}$$

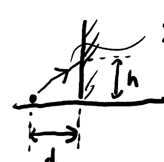
Check: $$\lim_{d \to 0} y(x=d) = \frac{v_{y0}d}{v_{x0}} + \frac{mgd}{cV_{x0}} + \frac{m^2 g}{c^2}\cdot\frac{-dc}{mV_{x0}}$$

$$= \frac{v_{y0}d}{v_{x0}} + \cancel{\frac{mgd}{cV_{x0}}} - \cancel{\frac{mgd}{cV_{x0}}}$$

$$= \boxed{\frac{v_{y0}d}{v_{x0}}}$$

From: $\ln(1-x) \approx -x$ for small x

This represents case where particle is launched very close to wall. Trajectory is approximately a straight line in this limit



Slope: v_{y0}/v_{x0} - tangent to path

$$\frac{v_{y0}}{v_{x0}} = \frac{h}{d} \Rightarrow h = \frac{v_{y0}d}{v_{x0}}$$

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$$\lim_{c\to 0} y(x=d) = \frac{v_{y0}d}{v_{x0}} + \lim_{c\to 0}\left[\frac{mgd}{v_{x0}}\cdot\frac{1}{c} + \frac{m^2 g}{c^2}\ln\left(1-\frac{dc}{mv_{x0}}\right)\right]$$

$$\hookrightarrow \frac{1}{v_{x0}}\cdot\frac{mgdc + m^2 g v_{x0}\ln\left(1-\frac{dc}{mv_{x0}}\right)}{c^2} \to \frac{0}{0}$$

L'Hopital's: $$\frac{1}{v_{x0}}\cdot\frac{mgd + m^{\cancel{2}} g \cancel{v_{x0}}\cdot\frac{1}{1-\frac{dc}{mv_{x0}}}\cdot -\frac{d}{\cancel{m}\cancel{v_{x0}}}}{2c}$$

$$= \frac{mgd}{v_{x0}}\cdot\frac{1-\frac{mv_{x0}}{mv_{x0}-dc}}{2c}$$

$$= \frac{mgd}{v_{x0}}\cdot\frac{\cancel{mv_{x0}} - d\cancel{c} - \cancel{mv_{x0}}}{2\cancel{c}(mv_{x0}-dc)}$$

$$\Rightarrow \lim_{c\to 0}\frac{mgd}{v_{x0}}\cdot\frac{-d}{2(mv_{x0}-dc)}$$

$$= \frac{-\cancel{m}gd^2}{2\cancel{m}v_{x0}^2} = \frac{-gd^2}{2v_{x0}^2}$$

$$= \boxed{\frac{v_{y0}d}{v_{x0}} - \frac{gd^2}{2v_{x0}^2}}$$

This represents the case with no drag

(i.e. a standard ballistic problem, from Kinematics)

d) MATLAB code plotting trajectory

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
% % MAE 2030
% HW 6, Q24 Solution
% Solution by Michael Zakoworotny

clear all; clc; close all;
set(0,'DefaultLineLineWidth',2); % set all plots lines to size of 2
set(0,'DefaultAxesFontSize',16); % set plot font size to 16

% Define parameters - mass, linear drag, gravity and position of wall
m = 5; c = 1; g = 10; d = 5;
p.m = m; p.c = c; p.g = g; p.d = d;

% Time span
t0 = 0; tend = 10;
tspan = [t0 tend];

% Initial conditions
r0 = [0 0];
% v0 = [10 10]; % parameterize by x and y velocity components
v0 = 10*[cosd(45) sind(45)]; % paramaterize by speed and angle
z0 = [r0 v0]';

% Solution step
fun = @(t,z) myRHS(z,t,p);
tol = 1e-8;
wall_event = @(t,z) wallEvent(t,z,p);
opts = odeset('RelTol',tol,'AbsTol',tol,'Events',wall_event);
[t_array, z_array] = ode45(fun, tspan, z0, opts);

% Unpack solution
x_array = z_array(:,1); y_array = z_array(:,2);
y_wall = y_array(end);
fprintf('The y position when the particle collides with the wall is %.4f\n',y_wall);

% Plot solution - x vs y
fig = figure; axis equal; box on; hold on;
plot(x_array, y_array, '-k');
plot([d d],[min(xlim),max(xlim)],'-r');
xlabel('x'); ylabel('y');
title('Trajectory of particle');


function zdot = myRHS(z,t,p)
    % Unpack parameters
    m = p.m; c = p.c; g = p.g;
    % Get position and velocity (column vectors) from current state
    r = z(1:2);
    v = z(3:4);

    % Solve for rdot and vdot
    rdot = v;
    vdot = 1/m*(-c*v - m*g*[0;1]);
    % Store zdot vector from rdot and vdot
    zdot = [rdot; vdot];
```

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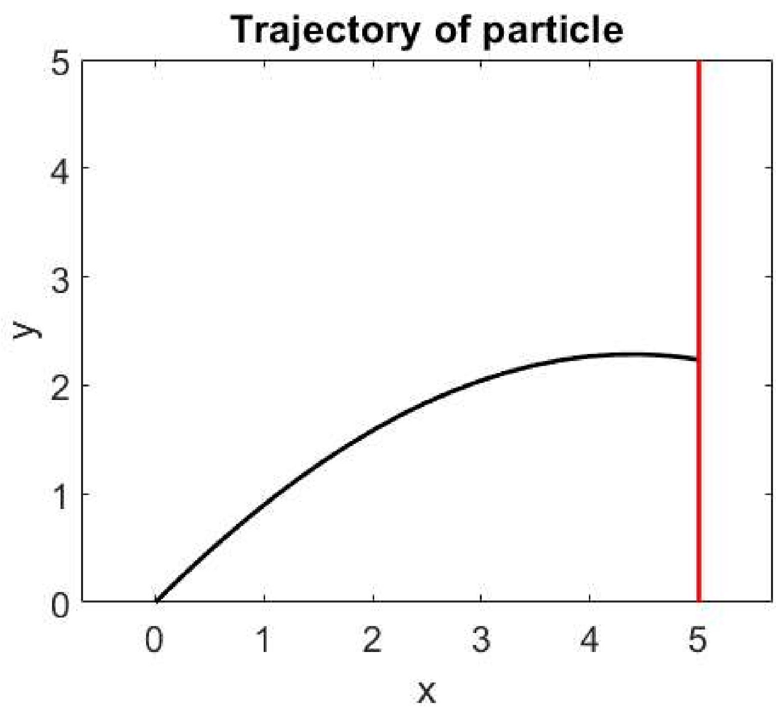
```
end

function [value,isterminal,direction] = wallEvent(t,z,p)
    % Returns parameters to be used in computing an event in ode45.
    x = z(1);
    d = p.d;
    value = x - d; % event occurs when this is 0
    isterminal = 1; % stop integration at event
    direction = 1; % only meaningful solution is when z increases to d

end
```

```
The y position when the particle collides with the wall is 2.2361
```



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```
% % MAE 2030
% HW 6, Q24 Solution
% Solution by Michael Zakoworotny
%
% Solve for height of wall collision symbolically

clear all; close all; clc;

% Define symbolic variables for position (as function of time), and
% parameters
syms x(t) y(t) m c g d
% Symbolic variables for initial conditions
syms x0 y0 vx0 vy0 real

% Unit vectors
i_ = [1;0]; j_ = [0;1];

% Position, and its derivatives
r = [x; y];
v = diff(r);
a = diff(v);

% Define governing ODE and initial conditions for position and velocity
odes = m*a == -c*v - m*g*j_;
ic_pos = r(0) == [0; 0];
ic_vel = v(0) == [vx0; vy0];

% Solve system for x and y as functions of time
sol = dsolve(odes,ic_pos,ic_vel); % solve a symbolic ODE. sol is a struct
                                  % which contains the x and y scalar
                                  % solutions
x = simplify(sol.x); % extract the x solution
y = simplify(sol.y); % extract the y solution

% Solve for the time, and y position, at which x = d
t_wall = solve(x == d, t); % solve for the t for which x = d
y_wall = simplify(subs(y,t,t_wall)); % substitute t_wall into the equation for y

% Printout of solutions
fprintf('x(t) = \n');
pretty(x);
fprintf('\ny(t) = \n');
pretty(y);
% Printout of time and y position of wall collision
fprintf('\nt at wall collision = \n');
pretty(t_wall);
fprintf('\ny at wall collision = \n');
pretty(y_wall);
% Limit as d->0. MATLAB shows computes this as zero because it does not
% take into account small angle approximation
fprintf('\nThe limit of y as d approaches zero is \n');
pretty(simplify(limit(y_wall,d,0)));
% Limit as c->0. Consistent with our expected kinematic solution
fprintf('\nThe limit of y as c approaches zero is \n');
pretty(simplify(limit(y_wall,c,0)));
```

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

```
x(t) =
                 /   c t \
m vx0 - m vx0 exp| - --- |
                 \    m  /
--------------------------
             c


y(t) =
                    /          /   c t \               \
                    |       exp| - --- | (g m + c vy0) |
                    |          \    m  /               |
                  m | g t + -------------------------- |
m (g m + c vy0)     \                    c             /
--------------- - --------------------------------------
        2                            c
       c


t at wall collision =
       /   c d - m vx0 \
  m log| - ----------- |
       \      m vx0    /
- ----------------------
             c


y at wall collision =
 2            2         /   c d - m vx0 \
c  d vy0 + g m  vx0 log| - ----------- | + c d g m
                       \      m vx0    /
--------------------------------------------------
                        2
                       c  vx0


The limit of y as d approaches zero is
0


The limit of y as c approaches zero is
  d (d g - 2 vx0 vy0)
- -------------------
              2
         2 vx0
```

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Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

e) Get angle of maximum range

$m=1$, $g=10$, $c=1$, $|\vec{v}_0|=1000$

<u>Approximation method:</u>

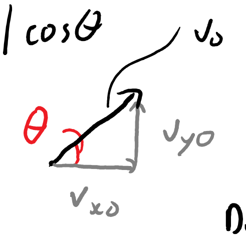
Since initial velocity is so large, drag force is much larger than gravity. So, in limit as $|\vec{v}|\to\infty$, we can ignore gravity.

Consider time to reach ground:

$$y(t) \simeq \frac{v_{y0}m}{c}\left[1-e^{-c/m\,t}\right]=0$$

Set $t\to\infty$ b/c $t>0$ → Under this assumption, particle rapidly flies up, stops due to large drag force, and then drops in freefall when gravity is dominant again

$v_{x0}=|\vec{v}_0|\cos\theta$



$$x(t)=\frac{v_{x0}m}{c}[1-0]$$

$$x(t\to\infty)=|\vec{v}_0|\cos\theta\cdot\frac{m}{c}$$ ← maximized when $\boxed{\theta=0}$

<u>Exact method</u>

* For more precise answer solve for t for given θ

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

* For more precise answer solve for t for given θ

$$y(t) = \frac{m}{c}\left(v_{y0} + \frac{mg}{c}\right)\left[1 - e^{-c/m \cdot t}\right] - \frac{mg}{c} \cdot t = 0$$

$\hookrightarrow v_{y0} = |\vec{v}_0| \sin\theta$

Then solve for θ that maximizes x

$v_{x0} = |\vec{v}_0| \cos\theta$ $\quad$ $t(\theta)$, from eqn. above

$$x(t) = \frac{v_{x0} m}{c}\left[1 - e^{-c/m \cdot t}\right]$$

This can be done in MATLAB using the fzero and fminbnd functions.

$$\boxed{\begin{gathered}\theta_{max} = \text{argmax}(x(\theta)) = 3.649^\circ \\ x_{max} = 997.34\end{gathered}}$$

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```
% % MAE 2030
% HW 6, Q24 part e Solution
% Solution by Michael Zakoworotny

clear all; close all; clc;
set(0,'DefaultLineLineWidth',2); % set all plots lines to size of 2
set(0,'DefaultAxesFontSize',16); % set plot font size to 16

% Given parameters
m = 1; g = 10; c = 1;
v0 = 1000;

% Anonymous function handle for y(t, th), since y is dependent on launch
% angle as well
y = @(t, th) m/c*(v0*sin(th) + m*g/c) .* (1-exp(-c/m*t)) - m*g/c*t;
% Anonymous function handle for x(t, th)
x = @(t, th) v0*cos(th)*m/c .* (1-exp(-c/m*t));

% Anonymous function which returns the time for the particle to reach y=0,
% given a launch angle, th. fzero finds the roots of a single-variable
% function, which in this case is the function y(t,th), for a given value
% of th. Start search near t = 10 s
t_drag = m/c*log(v0*c/m/g); % appr. time such that drag force = gravity
t_fall = @(th)sqrt(2*v0*t_drag*sin(th)/g); % approximate freefall time from
                           % top of trajectory, as function of launch angle
% Begin search around t = 1.5*(t_drag+t_fall), where 1.5 is a safety factor
% on the apprxomate time of trajectory
t_ = @(th) fzero(@(t) y(t,th), 1.5*(t_drag+t_fall(th)), ...
            optimset('Display','off','TolX',1e-6)); % tolerance on solution
% Anonymous function which returns the range, given a launch angle th. Uses
% the previous function t_ to obtain the launch time
range = @(th) x(t_(th), th);
% Solve for the angle for which the range is maximized. Since fminbnd
% searches for local minima, we are finding the minimum of -range. Search
% between 0 and 10 degrees, since we showed the optimal angle is close to 0
angle_opt = fminbnd(@(th) -range(th), 0, 10*pi/180);
% Get the range using this angle
range_opt = range(angle_opt);
% Flight time
t_opt = t_(angle_opt);

% Printout
fprintf('The optimal angle is %.4f degrees\n',angle_opt*180/pi);
fprintf('The range at this angle is %.4f \n',range_opt);

% Plot flight path
t_plot = linspace(0, t_opt, 1000);
figure; hold on; box on;
plot(x(t_plot,angle_opt),y(t_plot,angle_opt),'-k');
plot(xlim,[0,0],'--g'); % ground
axis equal;
title('Optimal range trjactory');
xlabel('x'); ylabel('y');

% Plot range as a function of launch angle. Ideally, we could use the fplot
```

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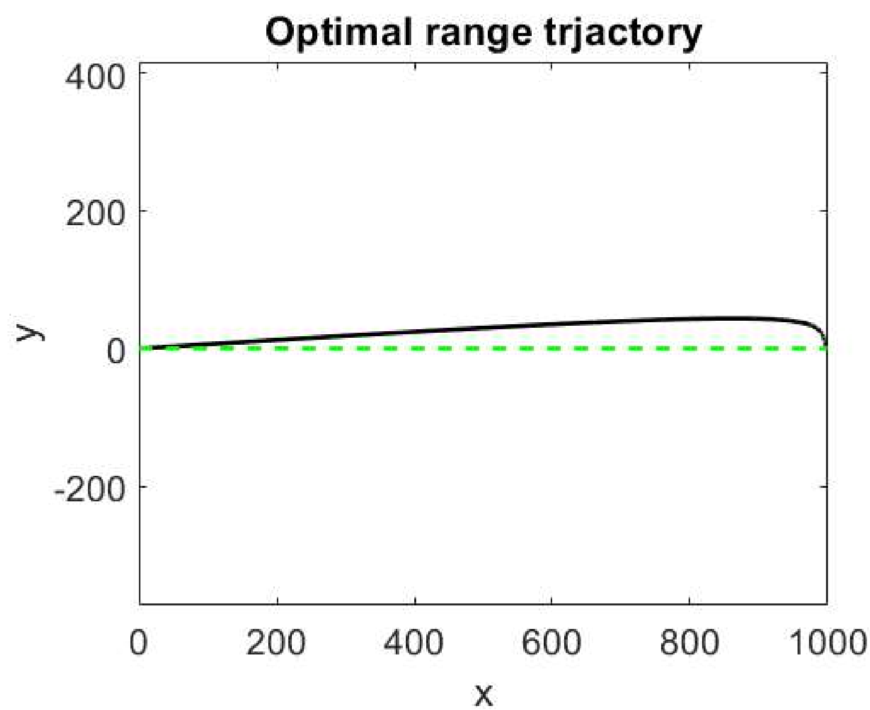
```
% function to plot an anonymous function handle. However, since x is
% defined as a function of t and th in our code, the fplot function is not
% equipped to deal with this. Instead, we precompute the ranges over a set
% of launch angles
th = linspace(0,pi/3,1000); % list of angles to plot - from 0 to 60 deg.
ranges = zeros(1,length(th)); % empty list of ranges corresponding to th's
for i = 1:length(th) % compute each range and add to list
    ranges(i) = range(th(i));
end
% Plot
figure; hold on; box on;
plot(th*180/pi,ranges,'-k');
plot(angle_opt*180/pi,range_opt,'bd') % show optimal point
xlabel('$\theta ({}^{\circ})$','interpreter','latex');
ylabel('Range');
title('Range over launch angles');
```

```
The optimal angle is 3.6490 degrees
The range at this angle is 997.3376
```



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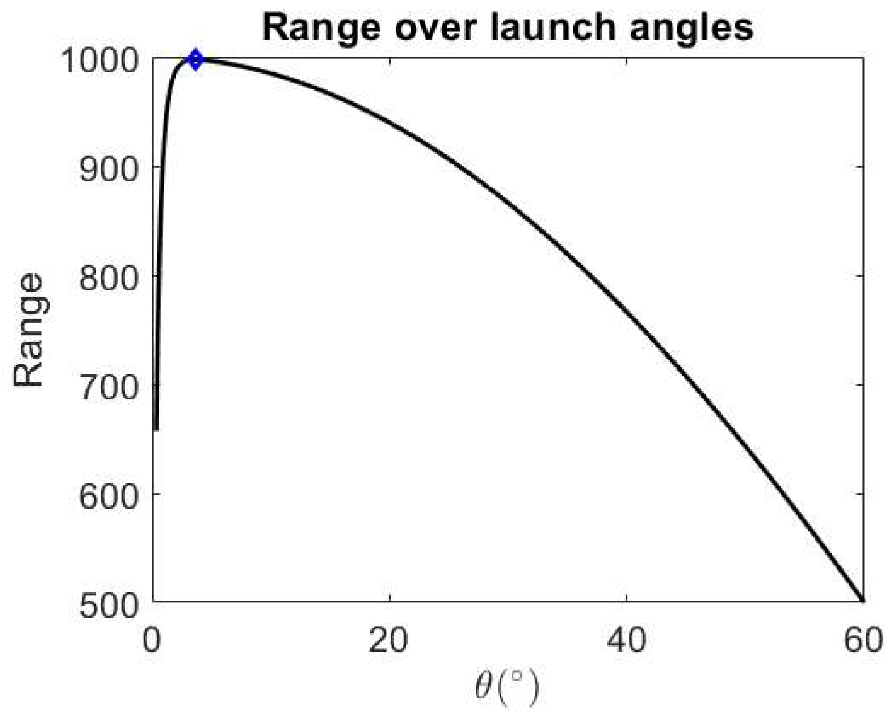

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12.1.31 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01$ kg / m.
Gravity g = 10 m/s.

a) Write and execute computer code to find the height at $t = 1$ s. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

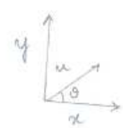
b) Estimate the height at $t = 1$ s using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

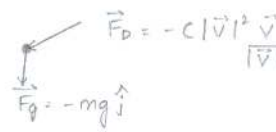
11.1.32 (27)

$m = 1$ kg , $\theta = 30°$, $u = 172$ m/sec

$F_D \propto cV^2$, $c = 0.01$ kg/m , $g = 10$ m/sec^2

FBD





Linear momentum balance

$$m\vec{a} = \vec{F}_D + \vec{F}_G$$

$$= -mg\hat{j} - cV_x\sqrt{V_x^2+V_y^2}\,\hat{i} - cV_y\sqrt{V_x^2+V_y^2}\,\hat{j}$$

$$m\ddot{x} = -c\dot{x}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$m\ddot{y} = -mg - c\dot{y}\sqrt{\dot{x}^2+\dot{y}^2}$$

$$u = \dot{x} \;,\; v = \dot{y} \longrightarrow \text{(i),(ii)}$$

$$\dot{u} = -\frac{c}{m}u\sqrt{u^2+v^2} \longrightarrow \text{(iii)}$$

$$\dot{v} = -g - \frac{c}{m}v\sqrt{u^2+v^2} \longrightarrow \text{(iv)}$$

Matlab :- height at $t = 1$ sec

from matlab solution height at $t = 1$.

is $h = 46.5047$ m

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(28)

b) Assume 'g' is negligible

If 'g' is assumed to be negligible drag force is always in opposite direction of velocity. Hence no force component $\perp$ to the current trajectory acts on it.

ie it is a straight line motion, with $\theta = 30^\circ$ along the line

$$ma = -cv^2$$

$$m\frac{dv}{dt} = -cv^2$$

$$\int \frac{dv}{v^2} = -\int \frac{c}{m}\,dt$$

$$-\frac{1}{v} = -\frac{ct}{m} + A$$

$$v\Big|_{t=0} = u_0 \Rightarrow A = -\frac{1}{u_0}$$

$$\frac{1}{v} = \frac{ct}{m} + \frac{1}{u_0}$$

$$v = \frac{mu_0}{cu_0t+1} = \frac{ds}{dt}$$

$$S = \int \frac{mu_0}{cu_0t+1}\,dt = \frac{mu_0}{cu_0}\log(cu_0t+1) = \frac{m}{c}\log(cu_0t+1)$$

for $t = 1$ $\quad S = 100\log(2-72)$

$$h = 100\sin 30^\circ = 50\,m$$

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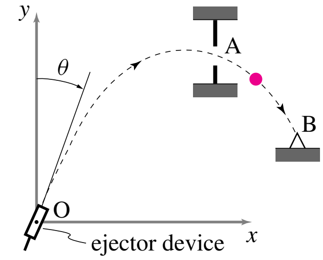
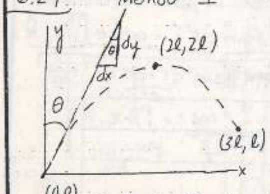
12.1.32 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that it passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 12.32

6.27 Method I



x-y plane not necessarily vertical

find the angle θ which causes the projectile to pass through $(2\ell, 2\ell)$ and end up at $(3\ell, \ell)$

as long as the x-y plane is not horizontal, the particle's trajectory is parabolic

gen'l eqn of parabola $\rightarrow$ $y = Ax^2 + Bx + C$

we know 3 pts. on the curve, so we can eval. the consts. A, B, C: $(0,0)$: $0 = A(0) + B(0) + C \Rightarrow C = 0$

$(2\ell, 2\ell)$: $2\ell = A(4\ell^2) + 2B\ell$

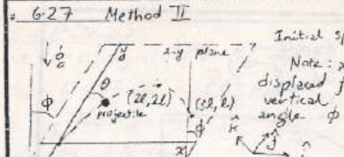
$(3\ell, \ell)$: $\ell = A(9\ell^2) + 3B\ell$

$\Rightarrow$ (after some algebra), $A = -\frac{2}{3\ell}$, $B = 7/3$

Thus, $y = \frac{-2}{3\ell}x^2 + 7/3\, x$; (path of particle)

$\theta = \cot^{-1}\left(\frac{dy}{dx} \text{ at } x=0\right) = \cot^{-1}(7/3) = \boxed{23.2^\circ}$

6.27 Method II p. 11



Initial speed v_0

Note: x-y plane displaced from the vertical by an angle ϕ

$\hat{k}$ = unit normal to x-y plane

FBD of projectile

$\sum \underset{\sim}{F} = N\hat{k} + mg(-\cos\phi\,\hat{j} - \sin\phi\,\hat{k})$

LMB of particle

$N\hat{k} + mg(-\cos\phi\,\hat{j} - \sin\phi\,\hat{k}) = m(\ddot{x}\,\hat{i} + \ddot{y}\,\hat{j} + \underset{0}{\ddot{z}}\,\hat{k})$ 0 particle moves in x-y plane

$\hat{i}\cdot(\text{LMB}) = m\ddot{x} = 0$ (1)

$\hat{j}\cdot(\text{LMB}) = m\ddot{y} = -mg\cos\phi$ (2)

Integrate eqns of motion (1), (2) twice to get $x(t)$, $y(t)$ (ϕ does not change w/ time!)

$\ddot{x} = 0$ $\qquad$ $\ddot{y} = -g\cos\phi = \text{const.}$

$\Rightarrow \dot{x} = A$ $\qquad$ $\Rightarrow \dot{y} = -g\cos\phi\, t + C$

$\Rightarrow x = At + B$ $\qquad$ $\Rightarrow y = -g\cos\phi \frac{t^2}{2} + Ct + D$

Initial conditions $\qquad$ Initial conditions

$x(t=0) = 0$ $\qquad$ $y(t=0) = 0$

$\dot{x}(t=0) = V_0 \sin\theta$ $\qquad$ $\dot{y}(t=0) = v_0\cos\theta$

$\Rightarrow B = 0,\ A = V_0\sin\theta$ $\qquad$ $\Rightarrow D = 0\ \ C = V_0\cos\theta$

so $x = (v_0\sin\theta)t$ (3), $y = (-g\cos\phi)\frac{t^2}{2} + (v_0\cos\theta)t$ (4)

put (3) $\rightarrow$ (4) to eliminate t:

$y = \frac{-g\cos\phi\, x^2}{2v_0^2\sin^2\theta} + \frac{x\cos\theta}{\sin\theta}$ $\qquad$ note that $\frac{g\cos\phi}{2v_0^2}$ is a constant.

let $K = -\frac{g\cos\phi}{2v_0^2}$ (to clean things up. also, we want to avoid having θ be a fn. of these quantities in our answer.)

$\therefore\ y = \frac{Kx^2}{\sin^2\theta} + \frac{x\cos\theta}{\sin\theta}$ (5)

use points $(2\ell, 2\ell)$ & $(3\ell, \ell)$ in (5):

$2\ell = \frac{K(4\ell^2)}{\sin^2\theta} + 2\ell\frac{\cos\theta}{\sin\theta} \rightarrow \sin\theta = \frac{K(2\ell)}{\sin\theta} + \cos\theta$ (6)

cont'd

6.27 cont'd p. 12

now for $(3\ell, \ell)$

$\Rightarrow \ell = \frac{K(9\ell^2)}{\sin^2\theta} + \frac{3\ell\cos\theta}{\sin\theta}$

$\Rightarrow \sin\theta = \frac{9K\ell}{\sin\theta} + 3\cos\theta$ (7)

using eqns (6) + (7), eliminate $K\ell$:

from (6) $\quad K\ell = \frac{(\sin\theta - \cos\theta)\sin\theta}{2}$

put into (7)

$\sin\theta = \frac{9}{2}(\sin\theta - \cos\theta) + 3\cos\theta$

$\Rightarrow \frac{7}{2}\sin\theta = \frac{3}{2}\cos\theta$

$\Rightarrow \theta = \tan^{-1}(3/7) = \boxed{23.2^\circ = \theta}$

The same result as method I but more work!

(Note that there is one unique initial velocity v_0 (that depends on the angle ϕ that the x-y plane makes with the vertical) for which the particle will follow this path. Using eqns (6) & (7), we could solve for this velocity in terms of g & ϕ).

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