SAMPLE 12.1 Velocity and acceleration from derivative of position:
The position vector of a particle is given as a function of time:

$$\vec{r}(t) = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}$$

where $C_1 = 1\,\text{m}, C_2 = 3\,\text{m/s}, C_3 = 1\,\text{m/s}^2$, and $C_4 = 2\,\text{m/s}$.

1. Find the position, velocity, and acceleration of the particle at $t = 2\,\text{s}$.
2. Find the change in the position of the particle between $t = 2\,\text{s}$ and $t = 3\,\text{s}$.

Solution We are given,

$$\vec{r} = (C_1 + C_2 t + C_3 t^2)\hat{\imath} + C_4 t\,\hat{\jmath}.$$

Therefore,

$$\begin{aligned} \vec{v} &\equiv \dot{\vec{r}} = \frac{d\vec{r}}{dt} = (C_2 + 2C_3 t)\hat{\imath} + C_4\hat{\jmath} \\ \vec{a} &\equiv \ddot{\vec{r}} = \frac{d^2\vec{r}}{dt^2} = 2C_3\hat{\imath}. \end{aligned}$$

1. Substituting the given values of the constants and $t = 2\,\text{s}$ in the equations above we get,

$$\begin{aligned} \vec{r}(t = 2\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}} \cdot 2\,\text{s} + 1\frac{\text{m}}{\text{s}^2} \cdot 4\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}} \cdot 2\,\text{s})\hat{\jmath} \\ &= 11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath} \\ \vec{v}(t = 2\,\text{s}) &= (3\frac{\text{m}}{\text{s}} + 2 \cdot 1\frac{\text{m}}{\text{s}^2} \cdot 2\,\text{s})\hat{\imath} + (2\frac{\text{m}}{\text{s}})\hat{\jmath} \\ &= 7\,\text{m/s}\hat{\imath} + 2\,\text{m/s}\hat{\jmath} \\ \vec{a}(t = 2\,\text{s}) &= (2 \cdot 1\frac{\text{m}}{\text{s}^2})\hat{\imath} = 2\,\text{m/s}^2\hat{\imath}. \end{aligned}$$

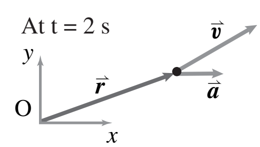


Figure 12.11

$$\boxed{\vec{r} = (11\hat{\imath} + 4\hat{\jmath})\,\text{m}, \quad \vec{v} = (7\hat{\imath} + 2\hat{\jmath})\,\text{m/s}, \quad \vec{a} = 2\,\text{m/s}^2\hat{\imath}}$$

2. The change in the position of the particle between the two time instants is,

$$\Delta\vec{r} = \vec{r}(t = 3\,\text{s}) - \vec{r}(t = 2\,\text{s}).$$

We already have $\vec{r}$ at $t = 2\,\text{s}$. We need to calculate $\vec{r}$ at $t = 3\,\text{s}$.

$$\begin{aligned} \vec{r}(t = 3\,\text{s}) &= (1\,\text{m} + 3\frac{\text{m}}{\text{s}} \cdot 3\,\text{s} + 1\frac{\text{m}}{\text{s}^2} \cdot 9\,\text{s}^2)\hat{\imath} + (2\frac{\text{m}}{\text{s}} \cdot 3\,\text{s})\hat{\jmath} \\ &= 19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}. \end{aligned}$$

Therefore,

$$\begin{aligned} \Delta\vec{r} &= (19\,\text{m}\hat{\imath} + 6\,\text{m}\hat{\jmath}) - (11\,\text{m}\hat{\imath} + 4\,\text{m}\hat{\jmath}) \\ &= 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}. \end{aligned}$$

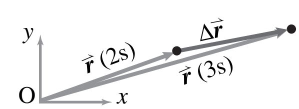


Figure 12.12

$$\boxed{\Delta\vec{r} = 8\,\text{m}\hat{\imath} + 2\,\text{m}\hat{\jmath}}$$

SAMPLE 12.2 Velocity and acceleration from position on a helix. Given that the position of a particle is

$$\vec{\boldsymbol{r}} = A\cos(\lambda t)\hat{\boldsymbol{\imath}} + B\sin(\lambda t)\hat{\boldsymbol{\jmath}} + Ct\hat{\boldsymbol{k}},$$

where A, B, C, and λ are constants, find

1. the velocity as a function of time,
2. the acceleration as a function of time,
3. a condition under which the acceleration vector is normal to the velocity vector.

Solution

1. The velocity:

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \frac{d\vec{\boldsymbol{r}}}{dt} \\
&= \frac{d}{dt}[A\cos(\lambda t)\hat{\boldsymbol{\imath}} + B\sin(\lambda t)\hat{\boldsymbol{\jmath}} + Ct\hat{\boldsymbol{k}}] \\
&= -A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}.
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{v}} = -A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}}}$$

2. The acceleration:

$$\begin{aligned}
\vec{\boldsymbol{a}} &= \frac{d\vec{\boldsymbol{v}}}{dt} \\
&= \frac{d}{dt}[-A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} \\
&= -A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}}
\end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = -A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}}}$$

3. The velocity vector and the acceleration vector will be orthogonal to each other if $\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = 0$. Taking the dot product of the two vectors, we find,

$$\begin{aligned}
\vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} &= (-A\lambda\sin(\lambda t)\hat{\boldsymbol{\imath}} + B\lambda\cos(\lambda t)\hat{\boldsymbol{\jmath}} + C\hat{\boldsymbol{k}})\cdot(-A\lambda^2\cos(\lambda t)\hat{\boldsymbol{\imath}} - B\lambda^2\sin(\lambda t)\hat{\boldsymbol{\jmath}}) \\
&= A^2\lambda^3\sin(\lambda t)\cos(\lambda t) - B^2\lambda^3\sin(\lambda t)\cos(\lambda t) \\
&= (A^2 - B^2)\lambda^3\sin(\lambda t)\cos(\lambda t).
\end{aligned}$$

Now, this dot product must be zero for all t if $\vec{\boldsymbol{a}}$ is normal to $\vec{\boldsymbol{v}}$. This is indeed the case if $A = B$. Thus, the condition for orthogonality of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$ is $A = B$.

$$\boxed{A = B \quad \Rightarrow \quad \vec{\boldsymbol{v}}\cdot\vec{\boldsymbol{a}} = 0}$$

Note: The path is an elliptical helix with axis in the z direction. The z-component of velocity is constant so the acceleration is entirely in the xy plane. In fact, the acceleration vector points from the particle towards the axis of the helix.

SAMPLE 12.3 Position from velocity. Assume the expression for velocity $\vec{v}$ of a particle is given: $\vec{v} = v_0\hat{\imath} - gt\,\hat{\jmath}$. Find the expressions for the x and y coordinates of the particle at a general time t, if the initial coordinates at $t = 0$ are (x_0, y_0). Plot the path of the particle taking $x_0 = 0$, $y_0 = 80\,\text{m}$, $v_0 = 2\,\text{m/s}$, $g = 10\,\text{m/s}^2$, and $t = 1 \ldots 4\,\text{s}$.

Solution The position vector of the particle at any time t is

$$\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}.$$

We are given that

$$\vec{r}(t=0) = x_0\hat{\imath} + y_0\hat{\jmath}.$$

Now

$$\begin{aligned} \vec{v} \equiv \frac{d\vec{r}}{dt} &= v_0\hat{\imath} - gt\,\hat{\jmath} \\ \text{or} \qquad dx\hat{\imath} + dy\,\hat{\jmath} &= (v_0\hat{\imath} - gt\,\hat{\jmath})\,dt. \end{aligned}$$

Integrating both sides of the equation with appropriate limits, we get

$$\begin{aligned} \int_{x_0\hat{\imath}+y_0\hat{\jmath}}^{x\hat{\imath}+y\hat{\jmath}} (dx\hat{\imath} + dy\,\hat{\jmath}) &= \int_0^t (v_0\hat{\imath} - gt\,\hat{\jmath})dt \\ \int_{x_0}^{x} dx\hat{\imath} + \int_{y_0}^{y} dy\,\hat{\jmath} &= v_0\hat{\imath}\int_0^t dt - g\hat{\jmath}\int_0^t t\,dt \\ (x - x_0)\hat{\imath} + (y - y_0)\hat{\jmath} &= v_0t\hat{\imath} - \frac{1}{2}gt^2\,\hat{\jmath} \\ x\hat{\imath} + y\,\hat{\jmath} &= (x_0 + v_0t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}. \end{aligned}$$

Therefore,

$$\vec{r}(t) = (x_0 + v_0t)\hat{\imath} + (y_0 - \frac{1}{2}gt^2)\hat{\jmath}$$

and the (x, y) coordinates are

$$\begin{aligned} x(t) &= x_0 + v_0t \\ y(t) &= y_0 - \frac{1}{2}gt^2. \end{aligned}$$

$$\boxed{(x_0 + v_0t,\ y_0 - \tfrac{1}{2}gt^2)}$$

Plugging in $x0 = 0$, $y_0 = 80\,\text{m}$, $v_0 = 2\,\text{m/s}$, $g = 10\,\text{m/s}^2$, and taking 20 points between $t = 0$ to $t = 4$, we compute the values of x and y and plot them to get the path of the particle. The plot is shown in fig. 12.14 with a few intermediate positions marked on the path.

Comments: From the x and y coordinates, it is possible to get the equation of the path of the particle by eliminating the time from the two equations. From the expression for $x(t)$, we get $t = (x - x_0)/v_0$. Substituting this expression for t in the equation for $y(t)$, we get,

$$y - y_0 = \frac{g}{2v_0^2}(x - x_0)^2$$

which is the equation of the path. From this equation it should be clear that the path is parabolic. It is easier to see this if you shift the origin to (x_0, y_0) and use the new coordinates $\hat{x} = x - x_0$ and $\hat{y} = y - y_0$. Then, in terms of the new coordinates, the path becomes,

$$\hat{y} = \frac{g}{2v_0^2}\hat{x}^2.$$

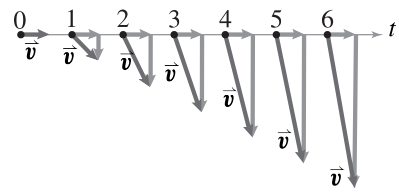


Figure 12.13: The given velocity vector at different time instants ($t = 0,\ 1\ 2$, etc.). Note that the x-component of the velocity remains constant (v_0) while the y-component grows linearly with time ($-gt$).

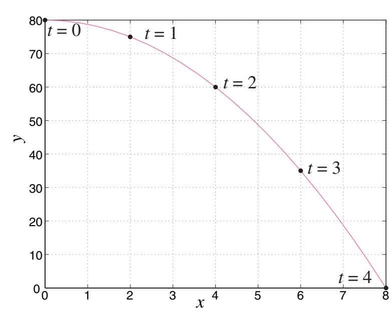


Figure 12.14: Path of the particle plotted by computing the coordinates $x(t)$ and $y(t)$ at 20 equal time intervals between $t = 0$ to 4 seconds. Five positions are marked on the path corresponding to $t = 0$, 1, 2, 3, and 4 seconds.

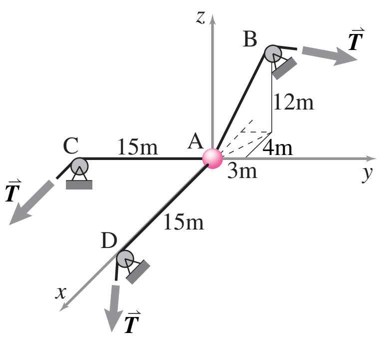


Figure 12.15: A ball in 3-D

SAMPLE 12.4 **Acceleration of a point mass in 3-D.** A ball of mass $m = 13\,\text{kg}$ is being pulled by three strings as shown in Fig. 12.15. The tension in each string is $T = 13\,\text{N}$. Find the acceleration of the ball.

Solution The forces acting on the body are shown in the free-body diagram in Fig.12.16. From geometry:

$$\begin{aligned} \hat{\boldsymbol{\lambda}} &= \frac{\vec{\boldsymbol{r}}_{AB}}{|\vec{\boldsymbol{r}}_{AB}|} = \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{\sqrt{4^2 + 3^2 + 12^2}} \\ &= \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}. \end{aligned}$$

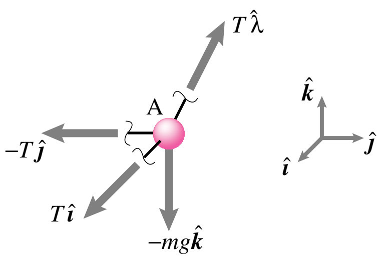


Figure 12.16: FBD of the ball

The balance of linear momentum for the ball gives

$$\sum \vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}} \tag{12.4}$$

where

$$\begin{aligned} \sum \vec{\boldsymbol{F}} &= T\hat{\boldsymbol{\imath}} - T\hat{\boldsymbol{\jmath}} + T\hat{\boldsymbol{\lambda}} - mg\hat{\boldsymbol{k}} \\ &= T\left(\hat{\boldsymbol{\imath}} - \hat{\boldsymbol{\jmath}} + \frac{-4\hat{\boldsymbol{\imath}} + 3\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}}{13}\right) - mg\hat{\boldsymbol{k}} \\ &= \frac{T}{13}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - mg\hat{\boldsymbol{k}}. \end{aligned}$$

Substituting $\sum \vec{\boldsymbol{F}}$ in eqn. (12.4):

$$\vec{\boldsymbol{a}} = \frac{T}{13m}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - g\hat{\boldsymbol{k}}.$$

Now plugging in the given values: $T = 13\,\text{N}$, $m = 13\,\text{kg}$, and $g = 10\,\text{m/s}^2$, we get

$$\begin{aligned} \vec{\boldsymbol{a}} &= \frac{\cancel{13}\,\text{N}}{\cancel{13}\cdot 13\,\text{kg}}(9\hat{\boldsymbol{\imath}} - 10\hat{\boldsymbol{\jmath}} + 12\hat{\boldsymbol{k}}) - 10\,\text{m/s}^2\hat{\boldsymbol{k}} \\ &= (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2. \end{aligned}$$

$$\boxed{\vec{\boldsymbol{a}} = (0.69\hat{\boldsymbol{\imath}} - 0.77\hat{\boldsymbol{\jmath}} - 9.08\hat{\boldsymbol{k}})\,\text{m/s}^2}$$

SAMPLE 12.5 Projectile motion with air drag. A projectile is fired into the air at an initial angle θ_0 and with initial speed v_0. The air resistance to the motion is proportional to the square of the speed of the projectile. Take the constant of proportionality to be k. Find the equations of motion of the projectile in the horizontal and vertical directions assuming the air resistance to be in the opposite direction of the velocity.

Solution The free-body diagram of the projectile is shown in the figure at some constant t during motion. At the instant shown, let the velocity of the projectile be $\vec{\boldsymbol{v}} = v\hat{\boldsymbol{e}}_t$ where

$$\hat{\boldsymbol{e}}_t = \vec{\boldsymbol{v}}/|\vec{\boldsymbol{v}}| = (v_x/|\vec{\boldsymbol{v}}|)\hat{\boldsymbol{\imath}} + (v_y/|\vec{\boldsymbol{v}}|)\hat{\boldsymbol{\jmath}}.$$

Then the force due to air resistance is

$$\vec{\boldsymbol{R}} = -kv^2\hat{\boldsymbol{e}}_t.$$

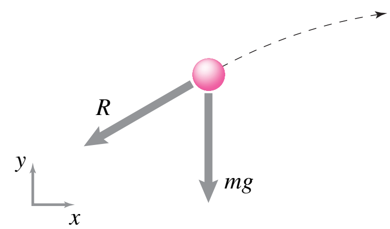


Figure 12.17: FBD of the projectile.

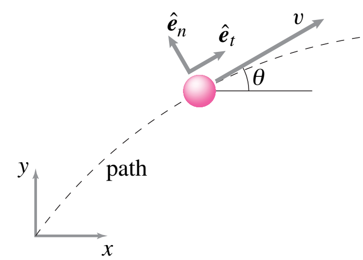


Figure 12.18

Now applying the linear momentum balance on the projectile, we get

$$\begin{aligned} \vec{\boldsymbol{R}} + m\vec{\boldsymbol{g}} &= m\vec{\boldsymbol{a}} \\ \text{or} \qquad -kv^2\hat{\boldsymbol{e}}_t - mg\hat{\boldsymbol{\jmath}} &= m\overbrace{(\ddot{x}\hat{\boldsymbol{\imath}} + \ddot{y}\hat{\boldsymbol{\jmath}})}^{\vec{\boldsymbol{a}}}. \end{aligned} \tag{12.5}$$

Noting that $v = |\vec{\boldsymbol{v}}| = |\dot{x}\hat{\boldsymbol{\imath}} + \dot{y}\hat{\boldsymbol{\jmath}}| = \sqrt{\dot{x}^2 + \dot{y}^2}$, and dotting both sides of eqn. (12.5) with $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ we get

$$\begin{aligned} -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\imath}}) &= m\ddot{x} \\ -k(\dot{x}^2 + \dot{y}^2)\cdot(\hat{\boldsymbol{e}}_t\cdot\hat{\boldsymbol{\jmath}}) - mg &= m\ddot{y}. \end{aligned}$$

Rearranging terms and carrying out the dot products, we get

$$\begin{aligned} \ddot{x} &= -\frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_x/|\vec{\boldsymbol{v}}|) \\ \ddot{y} &= -g - \frac{k}{m}(\dot{x}^2 + \dot{y}^2)(v_y/|\vec{\boldsymbol{v}}|). \end{aligned}$$

Substituting these expressions in to the equations for $\ddot{x}$ and $\ddot{y}$ we get

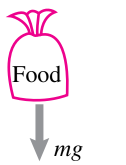
$$\boxed{\ddot{x} = -\frac{k}{m}\dot{x}\sqrt{\dot{x}^2 + \dot{y}^2}, \quad \ddot{y} = -\frac{k}{m}\dot{y}\sqrt{\dot{x}^2 + \dot{y}^2} - g}$$

For initial conditions we could use

$$x_0 = 0, y_0 = 0 \qquad \text{and} \qquad \dot{x}_0 = v_{x0} = v_0\cos\theta_0, \qquad \text{and} \qquad \dot{y}_0 = v_{y0} = v_0\sin\theta_0.$$

Note that θ changes with time. We can express θ in terms of $\dot{x}$ and $\dot{y}$. We can calculate the slope of the trajectory from

$$\tan\theta = \frac{\dot{y}}{\dot{x}}.$$



FBD of the bag

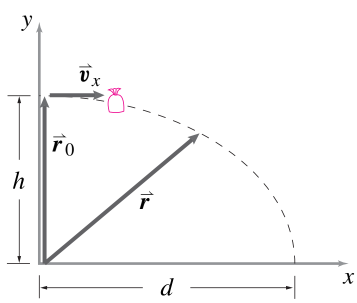


Figure 12.19: Free-body diagram of the bag and the geometry of its motion.

SAMPLE 12.6 Trajectory of a food-bag. In a flood hit area relief supplies are dropped in a 20 kg bag from a helicopter. The helicopter is flying parallel to the ground at 200 km/h and is 80 m above the ground when the package is dropped. How much horizontal distance does the bag travel before it hits the ground? Take the value of g, the gravitational acceleration, to be $10\,\mathrm{m/s^2}$. Ignore air drag.

Solution You must have solved such problems in elementary physics courses. Usually, in all projectile motion problems the equations of motion are written separately in the x and y directions, realizing that there is no force in the x direction, and then the equations are solved. Here we show you how to write and keep the equations in vector form all the way through.

The free-body diagram of the bag during its free flight is shown in Fig. 12.19. The only force acting on the bag is its weight. Therefore, from the linear momentum balance for the bag we get

$$m\vec{\boldsymbol{a}} = -mg\hat{\boldsymbol{\jmath}}.$$

Let us choose the origin of our coordinate system on the ground exactly below the point at which the bag is dropped from the helicopter. Then, the initial position of the bag $\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} = 80\,\mathrm{m}\,\hat{\boldsymbol{\jmath}}$. The fact that the bag is dropped from a helicopter flying horizontally gives us the initial velocity of the bag:

$$\vec{\boldsymbol{v}}(0) \equiv \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}} = 200\,\mathrm{km/h}\hat{\boldsymbol{\imath}}.$$

So now we have a 2nd order differential equation (from linear momentum balance):

$$\ddot{\vec{\boldsymbol{r}}} = -g\hat{\boldsymbol{\jmath}}$$

with two initial conditions:

$$\vec{\boldsymbol{r}}(0) = h\hat{\boldsymbol{\jmath}} \quad \text{and} \quad \dot{\vec{\boldsymbol{r}}}(0) = v_x\hat{\boldsymbol{\imath}}$$

which we can solve to get the position vector of the bag at any time. Since the basis vectors $\hat{\boldsymbol{\imath}}$ and $\hat{\boldsymbol{\jmath}}$ do not change with time, solving the differential equation is a matter of simple integration:

$$\begin{aligned}
\ddot{\vec{\boldsymbol{r}}} &\equiv \frac{d\dot{\vec{\boldsymbol{r}}}}{dt} = -g\hat{\boldsymbol{\jmath}} \\
\int d\dot{\vec{\boldsymbol{r}}} &= -\hat{\boldsymbol{\jmath}}\int g\,dt \\
\text{or} \qquad \dot{\vec{\boldsymbol{r}}} &= -gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1
\end{aligned} \tag{12.6}$$

and integrating once again, we get

$$\begin{aligned}
\vec{\boldsymbol{r}} &= \int(-gt\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1)\,dt \\
&= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + \vec{\boldsymbol{c}}_1 t + \vec{\boldsymbol{c}}_2
\end{aligned} \tag{12.7}$$

where $\vec{\boldsymbol{c}}_1$ and $\vec{\boldsymbol{c}}_2$ are constants of integration and are vector quantities. Now substituting the initial conditions in eqn. (12.6) and eqn. (12.7) we get

$$\begin{aligned}
\dot{\vec{\boldsymbol{r}}}(0) &= v_x\hat{\boldsymbol{\imath}} = \vec{\boldsymbol{c}}_1, \quad \text{and} \\
\vec{\boldsymbol{r}}(0) &= h\hat{\boldsymbol{\jmath}} = \vec{\boldsymbol{c}}_2.
\end{aligned}$$

Therefore, the solution is

$$\begin{aligned}
\vec{\boldsymbol{r}}(t) &= -\frac{1}{2}gt^2\hat{\boldsymbol{\jmath}} + v_x t\hat{\boldsymbol{\imath}} + h\hat{\boldsymbol{\jmath}} \\
&= v_x t\hat{\boldsymbol{\imath}} + (h - \frac{1}{2}gt^2)\hat{\boldsymbol{\jmath}}.
\end{aligned}$$

So how do we find the horizontal distance traveled by the bag from our solution? The distance we are interested in is the x-component of $\vec{r}$, *i.e.,* $v_x t$. But we do not know t. However, when the bag hits the ground, its position vector has no y-component, *i.e.,* we can write $\vec{r} = d\hat{\imath} + 0\hat{\jmath}$ where d is the distance we are interested in. Now equating the components of $\vec{r}$ with the obtained solution, we get

$$d = v_x t \quad \text{and} \quad 0 = h - \frac{1}{2}gt^2.$$

Solving for t from the second equation and substituting in the first equation we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

$$\boxed{d = 222\,\text{m}}$$

Comments: Here we have tried to show you that solving for position from the given acceleration in vector form is not really any different than solving in scalar form provided the unit vectors involved are fixed in time. As long as the right hand side of the differential equation is integrable, the solution can be obtained. If the method shown above seems too "*mathy*" or intimidating to you then follow the usual scalar way of doing this problem.

The scalar method:

From the linear momentum balance, $-mg\hat{\jmath} = m\vec{a}$, writing the acceleration as $\vec{a} = a_x\hat{\imath} + a_y\hat{\jmath}$ and equating the x and y components from both sides, we get

$$a_x = 0 \quad \text{and} \quad a_y = -g.$$

Now using the formula for distance under uniform acceleration from Chapter 3, $x = x_0 + v_0 t + \frac{1}{2}at^2$, in both x and y directions, we get

$$\begin{aligned}
d &= \overbrace{x_0}^{0} + v_x t + \frac{1}{2}\overbrace{a_x}^{0} t^2 \\
&= v_x t \\
0 &= \overbrace{y_0}^{h} + \overbrace{v_y}^{0} t + \frac{1}{2}\overbrace{a_y}^{-g} t^2 \\
&= h - \frac{1}{2}gt^2 \\
\Rightarrow \quad t &= \sqrt{\frac{2h}{g}}.
\end{aligned}$$

Substituting for t in the equation for d we get

$$d = v_x\sqrt{\frac{2h}{g}} = \frac{200\,\text{km}}{3600\,\text{s}} \cdot \sqrt{\frac{2 \cdot 80\,\text{m}}{10\,\text{m/s}^2}} = \frac{2}{9}\,\text{km} \approx 222\,\text{m}.$$

as above.

SAMPLE 12.7 Cartoon mechanics: The cannon. It is sometimes claimed that students have trouble with dynamics because they built their intuition by watching cartoons. This claim could be rebutted on many grounds.

1) Students don't have trouble with dynamics! They love the subject.
2) Nowadays many cartoons are made using 'correct' mechanics, and
3) the cartoons are sometimes more accurate than the pedagogues anyway.

Problem: What is the path of a cannon ball? In the cartoon world the cannon ball goes in a straight line out the cannon then comes to a stop and then starts falling. Of course a good physicist knows the path is a parabola. Or is it?

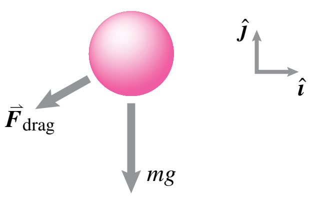


Figure 12.20

Solution The drag force on a cannon ball moving through air is approximately proportional to the speed squared and resists motion. Gravity is approximately constant. Then

$$
\begin{aligned}
\vec{F}_{\text{drag}} &= cv^2 \cdot (\text{unit vector opposing motion}) \\
&= cv^2 \cdot \left(\frac{-\vec{v}}{|\vec{v}|}\right) \\
&= -c|\vec{v}|\vec{v} \\
&= -c\sqrt{\dot{x}^2 + \dot{y}^2}\,(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})
\end{aligned}
$$

So the linear momentum balance gives

$$\sum \vec{F} = \dot{\vec{L}}$$
$$\left\{-mg\hat{\jmath} - c\sqrt{\dot{x}^2 + \dot{y}^2}(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath}) = m(\ddot{x}\hat{\imath} + \ddot{y}\hat{\jmath})\right\}$$

$$
\begin{aligned}
\{\}\cdot\hat{\imath} \quad &\Rightarrow \quad \ddot{x} \quad = \quad \left[-c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{x}/m\right] \\
\{\}\cdot\hat{\jmath} \quad &\Rightarrow \quad \ddot{y} \quad = \quad -c\sqrt{\dot{x}^2 + \dot{y}^2}\,\dot{y}/m - g
\end{aligned}
$$

Solving these equations numerically with reasonable values ① of $\dot{x}_0$, $\dot{y}_0$, m and c gives which is closer to a cartoon's triangle than to a naive physicist's parabola.

① To be precise, if the launch speed is much faster than the 'terminal velocity' of the falling ball.

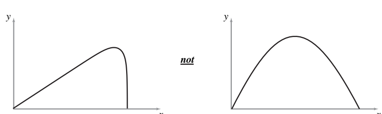


Figure 12.21