12.1 Dynamics of a particle in space

Preparatory Problems

12.1.1 Given $\vec{r}(t) = A\sin(\omega t)\hat{\imath} + Bt\hat{\jmath} + C\hat{k}$, find

1. $\vec{v}(t)$
2. $\vec{a}(t)$
3. $\vec{r}(t) \times \vec{a}(t)$.

12.1.2 A particle of mass =3 kg travels in space with its position known as a function of time, $\vec{r} = (\sin\frac{t}{\text{s}})\,\text{m}\hat{\imath} + (\cos\frac{t}{\text{s}})\,\text{m}\hat{\jmath} + te^{\frac{t}{\text{s}}}\,\text{m/s}\hat{k}$. At $t = 3\,\text{s}$, find the particle's

a) velocity and
b) acceleration.

12.1.3 A particle of mass $m = 2\,\text{kg}$ travels in the xy-plane with its position known as a function of time, $\vec{r} = 3t^2\,\text{m/s}^2\hat{\imath} + 4t^3\frac{\text{m}}{\text{s}^3}\hat{\jmath}$. At $t = 5\,\text{s}$, find the particle's

a) velocity and *
b) acceleration, and *
c) draw the three vectors.

12.1.4 The velocity of a particle of mass m on a frictionless surface is given as $\vec{v} = (0.5\,\text{m/s})\hat{\imath} - (1.5\,\text{m/s})\hat{\jmath}$. If the displacement is given by $\Delta\vec{r} = \vec{v}t$, find (a) the distance traveled by the mass in 2 seconds and (b) a unit vector along the displacement.

12.1.5 If $\dot{\vec{r}} = (u_0 \sin\Omega t)\hat{\imath} + v_0\hat{\jmath}$ and $\vec{r}(0) = x_0\hat{\imath} + y_0\hat{\jmath}$, with u_0, v_0, and Ω as constants, find $\vec{r}(t) = x(t)\hat{\imath} + y(t)\hat{\jmath}$. *

12.1.6 For $\vec{v} = v_x\hat{\imath} + v_y\hat{\jmath} + v_z\hat{k}$ and $\vec{a} = 2\,\text{m/s}^2\hat{\imath} - (3\,\text{m/s}^2 - 1\,\text{m/s}^3\,t)\hat{\jmath} - 5\,\text{m/s}^4\,t^2\hat{k}$, write the vector equation $\vec{v} = \int \vec{a}\,dt$ as three scalar equations (*i.e.*, find $v_x(t)$, $v_y(t)$, and $v_z(t)$).

12.1.7 Find $\vec{r}(5\,\text{s})$ given that $\dot{\vec{r}} = v_1\sin(ct)\hat{\imath} + v_2\hat{\jmath}$ and $\vec{r}(0) = 2\,\text{m}\hat{\imath} + 3\,\text{m}\hat{\jmath}$, and that v_1 is a constant $4\,\text{m/s}$, v_2 is a constant $5\,\text{m/s}$, and c is a constant $4\,\text{s}^{-1}$.

12.1.8 Let $\dot{\vec{r}} = v_0\cos\alpha\hat{\imath} + v_0\sin\alpha\hat{\jmath} + (v_0\tan\theta - gt)\hat{k}$, where v_0, α, θ, and g are constants. If $\vec{r}(0) = \vec{0}$, find $\vec{r}(t)$.

12.1.9 On a smooth circular helical path the velocity of a particle is $\dot{\vec{r}} = -R\sin t\hat{\imath} + R\cos t\hat{\jmath} + gt\hat{k}$. If $\vec{r}(0) = R\hat{\imath}$, find $\vec{r}((\pi/3)\,\text{s})$.

More-Involved Problems

12.1.10 A particle travels on a path in the xy-plane given by $y(x) = \sin^2(\frac{x}{\text{m}})\,\text{m}$. where $x(t) = t^3(\frac{\text{m}}{\text{s}^3})$. What are the velocity and acceleration of the particle in cartesian coordinates when $t = (\pi)^{\frac{1}{3}}\,\text{s}$?

12.1.11 The position of a particle is given by $\vec{r}(t) = (t^2\,\text{m/s}^2\hat{\imath} + e^{\frac{t}{\text{s}}}\,\text{m}\hat{\jmath})$. What are the velocity and acceleration of the particle as functions of time? Draw the path of the particle and show the vectors $\vec{v}$ and $\vec{a}$ at $t = 1\,\text{s}$. *

12.1.12 A particle travels on an elliptical path given by $y^2 = b^2(1 - \frac{x^2}{a^2})$ with constant speed v. Find the velocity of the particle when $x = a/2$ and $y > 0$ in terms of a, b, and v.

12.1.13 A particle travels on a path in the xy-plane given by $y(x) = (1 - e^{-\frac{x}{\text{m}}})\,\text{m}$. Make a plot of the path. It is known that the x coordinate of the particle is given by $x(t) = t^2\,\text{m/s}^2$. What is the rate of change of speed of the particle? What angle does the velocity vector make with the positive x axis when $t = 3\,\text{s}$?

12.1.14 A particle starts at the origin in the xy-plane, $(x_0 = 0, y_0 = 0)$ and travels only in the positive xy quadrant. Its speed and x coordinate are known to be $v(t) = \sqrt{1 + (\frac{4}{\text{s}^2})t^2}\,\text{m/s}$ and $x(t) = t\,\text{m/s}$, respectively. What is $\vec{r}(t)$ in cartesian coordinates? What are the velocity, acceleration, and rate of change of speed of the particle as functions of time? What kind of path is the particle on? What is the distance of the particle from the origin and its velocity and acceleration when $x = 3\,\text{m}$?

12.1.15 For a particle, $\sum\vec{F} = m\vec{a}$. Two forces $\vec{F}_1$ and $\vec{F}_2$ act on a mass P as shown in the figure. P has mass 2 lbm. The acceleration of the mass is somehow measured to be $\vec{a} = -2\,\text{ft/s}^2\hat{\imath} + 5\,\text{ft/s}^2\hat{\jmath}$.

a) Write the equation

$$\sum\vec{F} = m\vec{a}$$

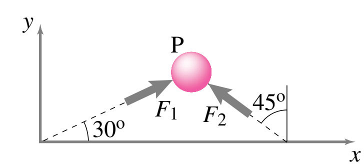
in vector form (evaluating each side as much as possible).

b) Write the equation in scalar form (use any method you like to get two scalar equations in the two unknowns F_1 and F_2).

c) Write the equation in matrix form.

d) Find $F_1 = |\vec{F}_1|$ and $F_2 = |\vec{F}_2|$ by the following methods:

1. from the scalar equations using hand algebra,
2. from the matrix equation using a computer, and
3. from the vector equation using a cross product.



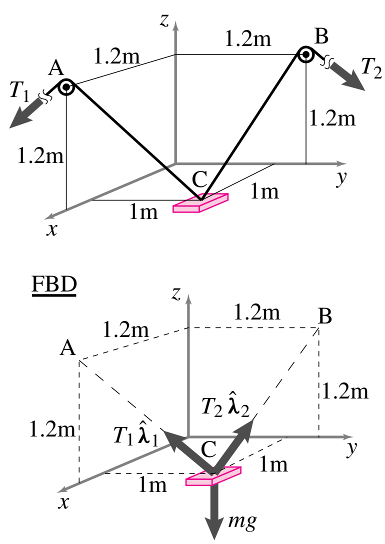
Problem 12.1.15

12.1.16 Three forces, $\vec{F}_1 = 20\,\text{N}\hat{\imath} - 5\,\text{N}\hat{\jmath}$, $\vec{F}_2 = F_{2x}\hat{\imath} + F_{2z}\hat{k}$, and $\vec{F}_3 = F_3\hat{\lambda}$, where $\hat{\lambda} = \frac{1}{2}\hat{\imath} + \frac{\sqrt{3}}{2}\hat{\jmath}$, act on a body with mass 2 kg. The acceleration of the body is $\vec{a} = -0.2\,\text{m/s}^2\hat{\imath} + 2.2\,\text{m/s}^2\hat{\jmath} + 1.7\,\text{m/s}^2\hat{k}$. Write the equation $\sum\vec{F} = m\vec{a}$ as scalar equations and solve them (most conveniently on a computer) for F_{2x}, F_{2z}, and F_3.

12.1.17 In three-dimensional space with no gravity a particle with $m = 3\,\text{kg}$ at A is pulled by three strings which pass through points B, C, and D respectively. The acceleration is known to be $\vec{a} = (1\hat{\imath} + 2\hat{\jmath} + 3\hat{k})\,\text{m/s}^2$. The position vectors of B, C, and D relative to A are given in the first few lines of code below. Complete the pseudo-code to find the three tensions. The last line should read `T = ...` with `T` being assigned to be a 3-element column vector with the three tensions in Newtons. [Hint: If `x`, `y`, and `z` are three column vectors then `A=[x y z]` is a matrix with `x`, `y`, and `z` as columns.]

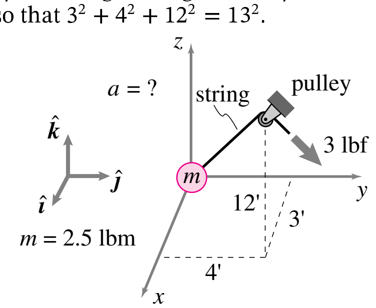
```
% Incomplete PSEUDO-CODE file
m   = 3;
a   = [ 1 2 3]';
~
rAB = [ 2 3 5]';
rAC = [-3 4 2]';
rAD = [ 1 1 1]';
uAB = rAB/(magnitude of rAB);
     .
     .
     .
T   =
```

12.1.18 A block of mass 100 kg is pulled with two strings AC and BC. Given that the tensions $T_1 = 1200\,\mathrm{N}$ and $T_2 = 1500\,\mathrm{N}$, find the magnitude and direction of the acceleration of the block. [$\sum \vec{F} = m\vec{a}$]



Problem 12.1.18

12.1.19 Neglecting gravity, the only force acting on the mass shown in the figure is from the string. Find the acceleration of the mass. Use the dimensions and quantities given. Recall that lbf is a pound force, lbm is a pound mass, and $\mathrm{lbf}/\mathrm{lbm} = g$. Use $g = 32\,\mathrm{ft/s^2}$. Note also that $3^2 + 4^2 + 12^2 = 13^2$.

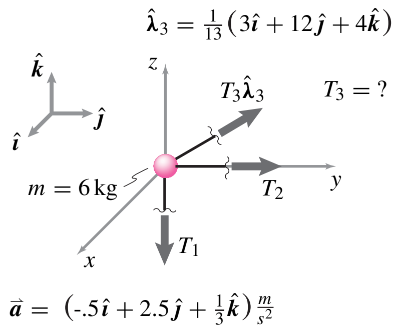


Problem 12.1.19

12.1.20 Three strings are tied to the mass shown with the directions indicated in the figure. They have unknown tensions T_1, T_2, and T_3. There is no gravity. The acceleration of the mass is given as $\vec{a} = (-0.5\hat{\imath} + 2.5\hat{\jmath} + \frac{1}{3}\hat{k})\,\mathrm{m/s^2}$.

a) Given the free-body diagram in the figure, write the equations of linear momentum balance for the mass.

b) Find the tension T_3. *



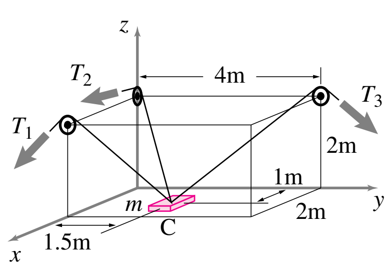
Problem 12.1.20

12.1.21 An object C of mass 2 kg is pulled by three strings as shown. The acceleration of the object at the position shown is $\boldsymbol{a} = \left(-0.6\hat{\imath} - 0.2\hat{\jmath} + 2.0\hat{k}\right)\mathrm{m/s^2}$.

a) Draw a free-body diagram of the mass.

b) Write the equation of linear momentum balance for the mass. Use $\boldsymbol{\lambda}$'s as unit vectors along the strings.

c) Find the three tensions T_1, T_2, and T_3 at the instant shown. You may find these tensions by using hand algebra with the scalar equations, using a computer with the matrix equation, or by using a cross product on the vector equation.



Problem 12.1.21

12.1.22 Particle moves on a strange path. Given that a particle moves in the xy plane for 1.77 s obeying

$$\vec{r} = (5\,\mathrm{m})\cos^2(t^2/\mathrm{s}^2)\hat{\imath} + (5\,\mathrm{m})\sin(t^2/\mathrm{s}^2)\cos(t^2/\mathrm{s}^2)\hat{\jmath}$$

where x and y are the horizontal distance in meters and t is measured in seconds.

a) Accurately plot the trajectory of the particle.

b) Mark on your plot where the particle is going fast and where it is going slow. Explain how you know these points are the fast and slow places.

12.1.23 *Computer question:* **What's the plot? What's the mechanics question?** Shown are shown some pseudo computer commands that are not commented adequately, unfortunately, and no computer is available at the moment.

a) Draw as accurately as you can, assigning numbers etc, the plot that results from running these commands.

b) See if you can guess a mechanical situation that is described by this program. Sketch the system and define the variables to make the script file agree with the problem stated.

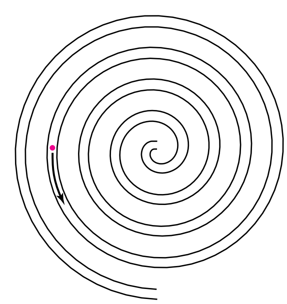
```
ODEs = {z1dot = z2
        z2dot = 0}
ICs = {z1 =1, z2=1}

Solve ODEs with ICs from t=0 to t=5

plot z2 and z1 vs t on the same plot
```

Problem 12.1.23

12.1.24 A particle is blown out through the uniform spiral tube shown, which lies flat on a horizontal frictionless table. Draw the particle's path after it is expelled from the tube. Defend your answer.



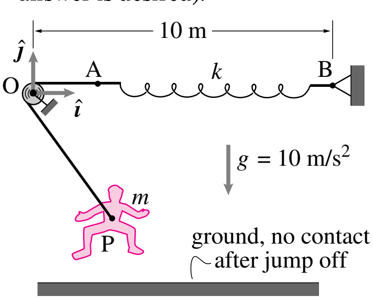
Problem 12.1.24

12.1.25 Bungy Jumping. In a relatively safe bungy jumping system, people jump up from the ground while being pulled up by a rope that runs over a pulley at O and is connected to a stretched spring anchored at B. The ideal pulley has negligible size, mass, and friction. For the situation shown the spring AB has rest length $\ell_0 = 2\,\text{m}$ and a stiffness of $k = 200\,\text{N/m}$. The inextensible massless rope from A to P has length $\ell_r = 8\,\text{m}$, the person has a mass of 100 kg. Take O to be the origin of an xy coordinate system aligned with the unit vectors $\hat{\imath}$ and $\hat{\jmath}$

a) Assume you are given the position of the person $\vec{r} = x\hat{\imath} + y\hat{\jmath}$ and the velocity of the person $\boldsymbol{v} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$. Find her acceleration in terms of some or all of her position, her velocity, and the other parameters given. Then use the numbers given, where supplied, in your final answer.

b) Given that bungy jumper's initial position and velocity are $\vec{r}_0 = 1\,\text{m}\hat{\imath} - 5\,\text{m}\hat{\jmath}$ and $\boldsymbol{v}_0 = \boldsymbol{0}$ write computer commands to find her position at $t = \pi/\sqrt{2}\,\text{s}$.

c) Find the answer to part (b) with pencil and paper (that is, find an analytic solution to the differential equations, a final numerical answer is desired).



Problem 12.1.25: Conceptual setup for a bungy jumping system.

12.1.26 A softball pitcher releases a ball of mass m upwards from her hand with speed v_0 and angle θ_0 from the horizontal. The only external force acting on the ball after its release is gravity.

a) What is the equation of motion for the ball after its release?

b) What are the position, velocity, and acceleration of the ball?

c) What is its maximum height?

d) At what distance does the ball return to the elevation of release?

e) What kind of path does the ball follow and what is its equation y as a function of x?

12.1.27 Find the trajectory of a not-vertically-fired cannon ball assuming the air drag is proportional to the speed. Assume the mass is 10 kg, $g = 10\,\text{m/s}$, the drag proportionality constant is $C = 5\,\text{N/(m/s)}$. The cannon ball is launched at 100 m/s at a 45 degree angle.

- Draw a free-body diagram of the mass.
- Write linear momentum balance in vector form.
- Solve the equations on the computer and plot the trajectory.
- Solve the equations by hand and then use the computer to plot your solution.
- Compare the two plots and comment on the differences, if any.

12.1.28 A baseball pitching machine releases a baseball of mass m from its barrel with speed v_0 and angle θ_0 from the horizontal. The only external forces acting on the ball after its release are gravity and air resistance. The speed of the ball is given by $v^2 = \dot{x}^2 + \dot{y}^2$. Taking into account air resistance on the ball proportional to its speed squared, $\vec{F}_d = -bv^2\hat{\boldsymbol{e}}_t$, find the equation of motion for the ball, after its release, in cartesian coordinates. *

12.1.29 The equations of motion from problem 12.1.28 are nonlinear and cannot be solved in closed form for the position of the baseball. Instead, solve the equations numerically. Make a computer simulation of the flight of the baseball, as follows.

a) Convert the equation of motion into a system of first order differential equations. *

b) Pick values for the gravitational constant g, the coefficient of resistance b, and initial speed v_0, solve for the x and y coordinates of the ball and make a plot of its trajectory for various initial angles θ_0.

c) Use Euler's, Runge-Kutta, or other suitable method to numerically integrate the system of equations.

d) Use your simulation to find the initial angle that maximizes the distance of travel for the ball, with and without air resistance.

e) If the air resistance is very high, what is a qualitative description for the curve described by the path of the ball? Show this with an accurate plot of the trajectory. (Make sure to integrate long enough for the ball to get back to the ground.)

12.1.30 A particle of mass m moves in a viscous fluid which resists motion with a force of magnitude $F = c|\vec{v}|$, where $\vec{v}$ is the velocity. Do not neglect gravity.

a) (easy) In terms of some or all of g, m, and c, what is the particle's terminal (steady-state) falling speed?

b) Starting with a free-body diagram and linear momentum balance, find two second order scalar differential equations that describe the two-dimensional motion of the particle.

c) (Challenge, long calculation) Assume the particle is thrown from $\vec{r} = \vec{0}$ with $\vec{v} = v_{x0}\hat{\imath} + v_{y0}\hat{\jmath}$ at a vertical wall a distance d away. Find the height h along the wall where the particle hits. (Answer in terms of some or all of $v_{x0}, v_{y0}, m, g, c,$ and d.) [Hint: i) find $x(t)$ and $y(t)$, ii) eliminate t, iii) substitute $x = d$. The answer is not tidy. In the limit $d \to 0$ the answer reduces to a sensible dependence on d (The limit $c \to 0$ is also sensible.).]

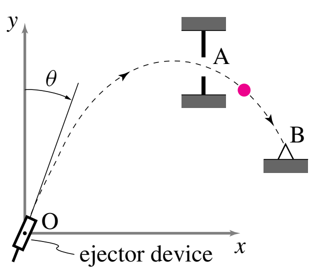
d) (Challenge, computer simulation). Do a computer simulation of the problem and find the solution in your simulation. Choose non-trivial numbers for all constants. To get an accurate solution you need an accurate interpolation to find at what time the particle hits the wall.

12.1.31 Someone in a violent part of the world shot a projectile at someone else. The basic facts:
Launched from the origin.
Projectile mass = 1 kg.
Launch angle 30° above horizontal.
Launch speed 172 m/s.
Drag force $\propto cv^2$ with $c = .01\,\text{kg/m}$.
Gravity $g = 10\,\text{m/s}$.

a) Write and execute computer code to find the height at $t = 1\,\text{s}$. [Hints: sketch of problem, FBD, write drag force in vector form, LMB, 1st order equations, numerical setup, find height at 1 s].

b) Estimate the height at $t = 1\,\text{s}$ using pencil and paper. An answer in meters is desired. [Hints: Assume g is negligible. Good calculus skills are needed but no involved arithmetic is needed. $1 + 1.72 = 2.72 \approx e$. After you have found a solution check that the force of gravity is a small fraction of the drag force throughout the duration of one second of your solution.]

12.1.32 In the arcade game shown, the object of the game is to propel the small ball from the ejector device at O in such a way that it passes through the small aperture at A and strikes the contact point at B. The player controls the angle θ at which the ball is ejected and the initial velocity v_o. The trajectory is confined to the frictionless xy-plane, which may or may not be vertical. Find the value of θ that gives success. The coordinates of A and B are $(2\ell, 2\ell)$ and $(3\ell, \ell)$, respectively, where ℓ is your favorite length unit.



Problem 12.1.32