CHAPTER 12

Particle dynamics in space (unconstrained)

This chapter is about the vector equation $\vec{F} = m\vec{a}$ *for one particle. Concepts and applications include ballistics and planetary motion. The differential equations of motion are set up in cartesian coordinates and integrated either numerically, or for special simple cases, by hand. Constraints, forces from ropes, rods, chains, floors, rails and guides that can only be found once one knows the acceleration, are not considered.*

The previous chapter was about particles that move in a straight line. Now we will consider particles that move in more complicated ways. More specifically, in this chapter we will consider the curving motion of a single particle using cartesian coordinates. We will be able to calculate the path of a hit baseball (perhaps taking account of air friction), a satellite, or a bungee jumper.

The key tool is, in Newton's words,

Figure 12.1: Some small blobs of water fly in nearly (neglecting air friction) parabolic arcs. This fountain is in the Detroit airport.

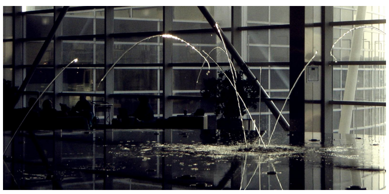

Figure 12.2: Small streams start to show the arcs in a still photo.

Figure 12.3: A full parabola shows.

Figure 12.4: The water runs in continuous streams. Some fancy valve work (under the visible part of the fountain) is required to get such a laminar stream that holds together for the whole flight. This water draws its own graph of the trajectory of its particles. You can do the same kind of thing with a squirt gun next to a blackboard.

> *"Any change of motion is proportional to the force that acts, and it is made in the direction of the straight line in which that force is acting."*

Realizing that the quantification of motion is the product of mass and velocity, and that the rate of change of velocity is acceleration, in modern language we could rephrase Newton's law as:

> *'the net force on a particle is its mass times its acceleration.'*

Informally we think 'force causes motion in the direction of the force'. Then, thinking more carefully we fill in the details that in this context 'motion' means acceleration and that the amount of force needed for a given acceleration is also proportional to the mass.

You can also think of $\vec{F} = m\vec{a}$ as a special case of the more general principle of linear momentum balance (LMB) for a system, where the system of interest is just a single particle. If we start with the general form of LMB given in the front cover, and discussed in general terms in chapter 1, we get:

$$\begin{aligned}
\sum \vec{F}_{\mathrm{i}} &= \dot{\vec{L}} && \text{Linear momentum balance for any system} \\
&= \sum m_i \vec{a}_{\mathrm{i}} && \text{for a system of particles} \\
&= m\vec{a} && \text{for one particle}
\end{aligned}$$

If we define $\vec{F}$ to be the net force on the particle ($\vec{F} = \sum \vec{F}_{\mathrm{i}}$) then linear momentum balance becomes 'Newton's second law',

$$\vec{F} = m\vec{a}. \tag{12.1}$$

Does force cause acceleration or is it the other way around? Whether force causes acceleration or acceleration necessitates force is a question of *causality*. This is a philosophical question that has no consequence when doing calculations. All that shows up in the math, and in any problem solution, is that when there is a net force there is acceleration of mass, and when there is acceleration of mass there is a net force. When a car crashes into a pole there is a big force and a big deceleration of the car. You could think of the force on the bumper as causing the car to slow down rapidly. Or you could think of the rapid car deceleration as necessitating a force. It is only a matter of personal taste because in both cases the same eqn. (12.1) applies. Equations don't have a 'cause' side and a 'result' side (If $A = B$ does A cause B or does B cause A?).

Acceleration is the second derivative of position

What is acceleration? If $\vec{r}(t)$ is the position of a particle relative to some origin, the particle's acceleration is

$$\vec{a} \equiv \ddot{\vec{r}}.$$

As for scalars, one or two dots over a vector is a short hand notation for the first or second time derivative. In the next section we'll explain how to take the derivative of a vector. As explained in box 12.1 the vector differentiation has to be done using an appropriate coordinate system.

12.1 Newton's laws are accurate in a Newtonian reference frame

Acceleration is calculated from position using a particular coordinate system. For our purposes here, a coordinate system is also a *reference frame*. The calculation of acceleration of a particle depends on how the coordinate system itself is moving. So the simple equation

$$\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$$

has as many different interpretations as there are differently moving coordinate systems (and there are an infinite number of those). In each different coordinate system, the coordinates of a given particle are different from the coordinates in another system. And the calculated accelerations are also different. Sir Isaac Newton was sitting on earth contemplating position relative to the ground at his feet when he noticed that his second law accurately described things like falling apples. So the equation $\vec{\boldsymbol{F}} = m\vec{\boldsymbol{a}}$ is valid using coordinate systems that are fixed to the earth. Well, not quite. Isaac noticed that the motion of the planets around the sun only followed his law if the acceleration was calculated using a coordinate system that was still relative to 'the fixed stars.' With a fixed-star coordinate system you calculate slightly (about 0.25%) different accelerations for things like falling apples than you do using a coordinate system that is stuck to the earth. And nowadays when astrophysicists try to figure out how the laws of mechanics explain the shapes of spiral galaxies, they realize that none of the so-called 'fixed stars' are so totally fixed. They need even more care to pick a coordinate system where eqn. (12.1) is accurate.

Despite all this confusion, it is generally agreed that no matter where you are there exists some coordinate system for which Newton's laws are incredibly accurate.

Further, once you know one 'good' coordinate system you know many others. Any system which translates (has no relative rotation) with constant velocity relative to a 'good' system is also a 'good' system. Why? Because the difference between the accelerations measured in the two frames is the relative acceleration of the frames, which is zero. Mechanics is the same on a constant velocity train or plane as on a stationary plane or train. Any reference frame in which Newton's laws are accurate is called a $\mathcal{N}$*ewtonian reference frame*. Sometimes people also call such a frame a $\mathcal{F}$*ixed frame*, as in 'fixed to the earth' or 'fixed to the stars'. But a Newtonian frame could also be 'fixed' to a constant velocity train or plane.

For most engineering purposes a coordinate system attached to the ground under your feet is a good approximation to a $\mathcal{N}$ewtonian frame. Fortunately. Or else apples would fall differently. Imagine Newton's apple having fallen on some crazy curved path leaving Newton confounded and the subject of mechanics still a mystery. The fall of apples, both in Newton's day and now, is well predicted using Newton's laws and treating the ground as a $\mathcal{N}$ewtonian frame. However, if you are interested in trajectory control of satellites, you need to use something more like the 'fixed stars' as your (even more accurate) Newtonian reference frame in order to make accurate predictions using Newton's laws.