11.3 Normal modes

To read this section you need to know the linear algebra concepts of eigenvalues and eigenvectors. Systems with many moving parts often move in complicated ways. Consider the two mass system shown in fig. 11.30. By drawing free-body diagrams and writing linear momentum balance for the two masses we can write the equations of motion in matrix form (see eqn. (10.44)) as

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}$$

where

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} 2k & -k \\ -k & 2k \end{bmatrix}.$$

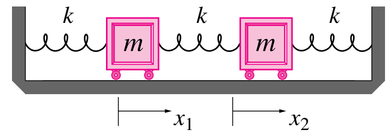


Figure 11.30: A two mass system. We define x_1 and x_2 so that the system is in equilibrium when $x_1 = x_2 = 0$.

Example: Complicated motion.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

we get the motion shown in fig. 11.31a. Both masses move in a complicated way and not synchronously with each other.

On the other hand, all such systems, if started in just the right way, will move in a simple way.

Example: Simple motion: a normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 11.31b. Both masses move in a simple sine wave, synchronously and *in phase* with each other.

That this system has this simple motion is intuitively apparent. If both of the equal masses are displaced equal amounts both have the same restoring force. So both move equal amounts in the ensuing motion. And nothing disturbs this symmetry as time progresses. In fact the frequency of vibration is exactly that of a single spring and mass (with the same k and m).

A given system can have more than one such simple motion.

Example: Another normal mode.
If we put the initial condition

$$\boldsymbol{x}_0 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \boldsymbol{v}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

we get the motion shown in fig. 11.31c. Both masses move in a simple sine wave, synchronously and exactly *out of phase* with each other. Being exactly out of phase is actually a form of being exactly in phase, but with a negative amplitude.

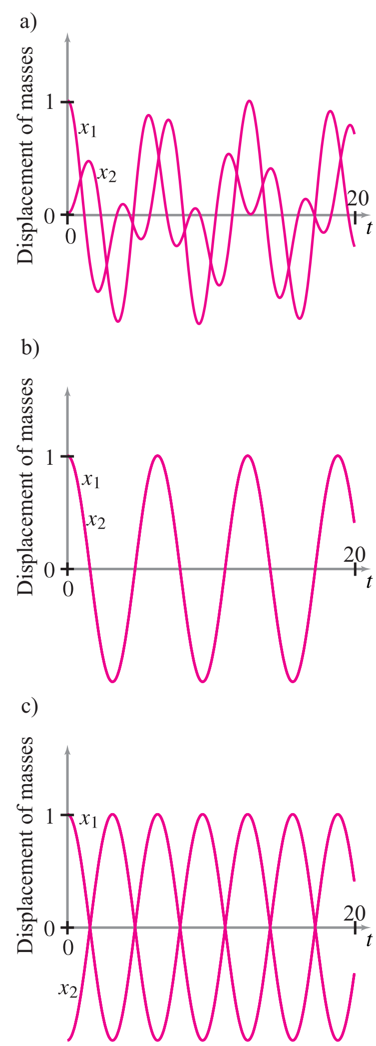


Figure 11.31: Motions of the masses from fig. 11.30 for three different initial conditions, all released from rest ($v_1 = v_2 = 0$)
a) $x_1 = 1,\ x_2 = 0$,
b) $x_1 = 1,\ x_2 = 1$, and
c) $x_1 = 1,\ x_2 = -1$.

This motion is also intuitive. Each mass has restoring force of $3k\Delta x$. One k from a spring at the end and $2k$ because each mass experiences a spring with half the length (and thus twice the stiffness) in the middle (because the middle of the middle spring doesn't move in this symmetric motion).

The system above is about the simplest for demonstration of *normal mode* vibrations. But more complicated elastic systems always have such simple normal mode vibrations.

All elastic systems with mass have *normal mode* vibrations in which all masses

- have simple harmonic motion
- with the same frequency as all the other masses, and
- exactly in (or out) of phase with all of the other masses

Thus the first and second normal modes from fig. 11.31b,c can be written as

$$\underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_1 t \\ \cos\lambda_1 t \end{bmatrix}}_{\text{First normal mode}} \quad \text{and} \quad \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \cos\lambda_2 t \\ -\cos\lambda_2 t \end{bmatrix}}_{\text{Second normal mode}}$$

where, by the physical reasoning in the examples we know that $\lambda_1 = \sqrt{k/m}$ and $\lambda_2 = \sqrt{3k/m}$. We could equally well have used the sine function instead of cosine.

Superposition of normal modes

Note that the governing equation (eqn. (11.3)) is 'linear' in that the sum of any two solutions is a solution. If we add the two solutions from fig. 11.31b,c we have a solution. And if we divide that sum by two we get a solution. And not just any solution, but the solution in fig. 11.31a. The top curve is the sum of the bottom two divided by two (The curves for $x_1(t)$ and $x_2(t)$ need to be added separately).

For more complicated systems it is not so easy to guess the normal modes. Most any initial condition will result in a complicated motion. Nonetheless the concept of normal modes applies to any system governed by the system of equations (eqn. (11.3)):

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0}.$$

Any collection of springs and masses connected any which way has normal mode vibrations. And because elastic solids are the continuum equivalent of a collection of springs and masses, the concept applies to all elastic structures. Here are the basic facts

- An elastic system with n degrees of freedom has n independent normal modes.

- In each normal mode i all the points move with the same angular frequency λ_i and exactly in phase.
- Any motion of the system is a superposition of normal modes (a sum of motions each of which is a normal mode).

Example: Musical instruments
The pitch of a bell is determined by that normal mode of the bell that has the lowest natural frequency. Similarly for violin and piano strings, marimba keys, kettle drums and the air-column in a tuba.

A recipe for finding the normal modes of more complex systems is given in box 11.3 on page 3.

Normal modes and single-degree-of-freedom systems

Any complex elastic system has simple normal mode motions. And all motions of the system can be represented as a superposition of normal modes. Hence sometimes we can think of every system as if it *is* a single degree of freedom system. For example, if a complex elastic system is forced, it will resonate if the frequency of forcing matches any of its normal mode (or natural) frequencies.

The math of, and how to find, normal modes

Consider a collection of n masses connected by springs whose motions are governed by eqn. (11.3)

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \boldsymbol{0},$$

where the positions of the masses are $\boldsymbol{x} = \boldsymbol{x}(t) = [x_1(t), x_2(t), ..., x_n(t)]'$. The matrices $[M]$ and $[K]$ have to do with the masses and the network of springs, respectively. At this point in the book the examples are masses in a line, but the concepts are more general.

How to find a solution. The approach used by the professionals is to *guess* that there are "normal mode" solutions and then see if they are. A normal mode solution, with all masses moving sinusoidally and synchronously, is

$$\boldsymbol{x} = \begin{bmatrix} V_1 \cos\lambda t \\ V_2 \cos\lambda t \\ \vdots \end{bmatrix} = \boldsymbol{V} \cos\lambda t.$$

Upper case bold $\boldsymbol{V}$ (to distinguish it from lower case velocity) is a list of constants $[V_1, V_2, ...]'$. We could have used sin just as well as cos for our guess. Now we plug our guess into the governing equations to see if it is a good guess:

$$
\begin{aligned}
[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} &= \boldsymbol{0} \\
[M]\frac{d^2}{dt^2}\{\boldsymbol{V}\cos\lambda t\} + [K]\{\boldsymbol{V}\cos\lambda t\} &= \boldsymbol{0} \\
-\lambda^2[M]\boldsymbol{V}\cos\lambda t + [K]\boldsymbol{V}\cos\lambda t &= \boldsymbol{0} \\
\left\{-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V}\right\}\cos\lambda t &= \boldsymbol{0}.
\end{aligned}
$$

This equation has to hold true for all t therefore the constant column vector inside the brackets $\{\}$ must be zero:

$$
\begin{aligned}
-\lambda^2[M]\boldsymbol{V} + [K]\boldsymbol{V} &= \boldsymbol{0} \\
\left[-\lambda^2[M] + [K]\right]\boldsymbol{V} &= \boldsymbol{0}
\end{aligned}
$$

The matrix $[M]$ is usually invertible. If $[M]$ is diagonal its inverse is $[M]$ with each element replaced by its reciprocal. Assuming $[M]^{-1}$ exists we can multiply through by $[M]^{-1}$ to get:

$$[M]^{-1}[K]\boldsymbol{V} = \lambda^2\boldsymbol{V},$$

where we used that $[M]^{-1}[M] = [1] =$ the identity matrix, and that $[1]\boldsymbol{V} = \boldsymbol{V}$. Defining the product $[B] = [M]^{-1}[K]$ and substituting we get the classic eigenvalue problem:

$$[B]\boldsymbol{V} = \lambda^2\boldsymbol{V}. \tag{11.45}$$

There is a lot to know about eqn. (11.45). It's a famous equation. Equation (11.45) says that $\boldsymbol{V}$ is a vector that, when multiplied by $[B]$ gives itself back again, multiplied by a constant. For the special vector $\boldsymbol{V}$, being multiplied by the matrix $[B]$ is equivalent to being multiplied by the scalar λ^2.

Given $[B]$ there are generally n linear independent *eigen vectors* $\boldsymbol{V}^1, \boldsymbol{V}^2, ...\boldsymbol{V}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. Note, $[B]$ is generally not symmetric.

In the case of our vibration problem the eigen vectors are called *modes* or *eigen modes* or *mode shapes* or *normal modes*.

Recipe for finding normal modes

Given the matrices $[M]$ and $[K]$ proceed as follows.

- Calculate $[B] = [M]^{-1}K$
- Use a math computer program to find the eigenvalues and eigenvectors of $[B]$, call these $\boldsymbol{V}^i$ and λ_i^2. Usually this is a single command, like:

```
eig(B)
```

- For each i between 1 and n write each normal mode as $\boldsymbol{x}(t) = \boldsymbol{V}\cos(\lambda_i t)$ or as $\boldsymbol{x}(t) = \boldsymbol{V}\sin(\lambda_i t)$

For example, if

$$[M] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix} \quad \text{and} \quad [K] = \begin{bmatrix} -2k & k \\ k & -2k \end{bmatrix}.$$

then, for any values of k and m, the computer will return for the eigen values and eigenvectors of $[B] = [M]^{-1}[K]$:

$$\boldsymbol{V}^1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \text{with } \lambda_1^2 = k/m \text{ and } \boldsymbol{V}^2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{with } \lambda_2^2 = 3k/m$$

Why are they called 'normal' modes?: Another recipe

The math here is relatively advanced, so trust or skip it if you don't have the needed linear algebra background. In math speak 'normal' sometimes means orthogonal. Here is the sense that 'normal' modes are orthogonal to each other. First, the matrix $[M]$ is generally both symmetric, non-singular and even positive definite, so $[M]$

- Has a symmetric inverse $[M]^{-1}$ with

$$[M]^{-1}[M] = [M][M]^{-1} = [\mathbf{1}] = [\text{identity matrix}],$$

- Has a unique positive definite square root $\sqrt{[M]}$ with

$$\sqrt{[M]}\sqrt{[M]} = [M],$$

- Has a unique inverse square root $[M]^{-1/2}$ with

$$[M]^{-1/2}[M][M]^{-1/2} = [\mathbf{1}].$$

First we use $[M]^{-1/2}$ to change coordinates from $\boldsymbol{x}$ to $\boldsymbol{y}$ as

$$\boldsymbol{x} = [M]^{-1/2}\boldsymbol{y} \quad \text{or} \quad \boldsymbol{y} = [M]^{1/2}\boldsymbol{x}.$$

Now we substitute this into the basic vibration equation ($[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = \mathbf{0}$) and pre-multiply the whole equation by $[M]^{-1/2}$ to get

$$\underbrace{[M]^{-1/2}[M][M]^{-1/2}}_{[1]}\ddot{\boldsymbol{y}} + \underbrace{[M]^{-1/2}[K][M]^{-1/2}}_{[A]}\boldsymbol{y} = \mathbf{0}$$

$$\Rightarrow \quad \ddot{\boldsymbol{y}} + [A]\boldsymbol{y} = \mathbf{0}.$$

Now we look for solutions for $\boldsymbol{y}$ exactly as we did for $\boldsymbol{x}$ before. But, as opposed to $[B]$, $[A]$ is symmetric. So $[A]$ has n linear independent and mutually orthogonal *eigen vectors* $\boldsymbol{W}^1, \boldsymbol{W}^2, ...\boldsymbol{W}^n$ with associated *eigen values* $\lambda_1^2, \lambda_2^2, ..., \lambda_n^2$. These $\boldsymbol{W}_i$ give the $\boldsymbol{V}_i$ by $\boldsymbol{V}_i = [M]^{-1/2}\boldsymbol{W}_i$. The λ_i are the same.

These coordinate-changed $\boldsymbol{W}_i = [M]^{1/2}\boldsymbol{V}_i$ are 'normal' (mutually orthogonal) but the more physical $\boldsymbol{V}_i$ are not, even though the $\boldsymbol{V}_i$ are called 'normal' modes. Or you can say that the $\boldsymbol{V}_i$ are mutually orthogonal with respect to the weighting $[M]$, *e.g.*, $\boldsymbol{V}_2'[M]\boldsymbol{V}_5 = 0$.

SAMPLE 11.12 **A two mass vibratory MEMS gyroscope:** A vibratory MEMS (microelectromechanical system) gyroscope employs two big plates as inertial masses, suspended by thin beams or 'springs' as shown in the figure. The two masses are made to vibrate (by electrical actuation) out of phase in the x-direction. Any rotation about the y-direction causes the masses to vibrate out-of-plane due to 'Coriolis acceleration' (you will learn about that in later chapters). We will restrict our attention to the planar motion of the gyroscope. A two degree of freedom spring-mass model is shown in the figure where $m = 34.5 \times 10^{-9}$ kg, $k_1 = 25\,\mathrm{N/m}$, and $k_2 = 3\,\mathrm{N/m}$.

1. Write the equations of motion for the two masses.
2. For the out of phase motion of the two masses, assume that $x_1(t) = -x_2(t) = x_0 \sin \lambda_n t$. Determine the natural frequency λ_n corresponding to this mode of vibration.

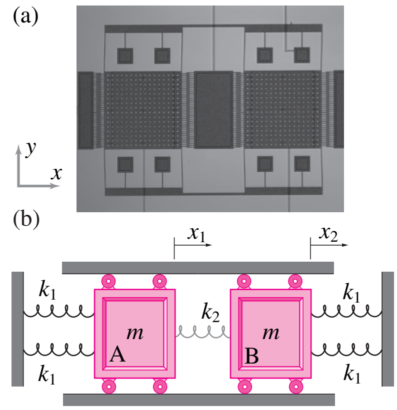


Figure 11.32: A MEMS vibratory gyroscope: (a) a micrograph of the two-mass structure. Each inertial mass (the plates with holes) is approximately 1 mm × 1 mm × 15μm. The beams that hang the two masses act as springs. The comb drives on the left and right side of the two masses are used to drive the two masses to oscillate in the x-direction at their resonant frequencies. (b) a two degree of freedom spring-mass model of the driven gyroscope structure (without any damping).

Solution

1. The free-body diagram of each mass is shown in fig. 11.33. Assuming both x_1 and x_2 to be positive in the x-direction, and $x_2 > x_1$ at the instant shown in the figure, we can write the equations of motion using the balance of linear momentum as

$$\begin{aligned}
\text{Mass A:} \quad m\ddot{x}_1 &= k_2(x_2 - x_1) - 2k_1x_1 \\
&= -(2k_1 + k_2)x_1 + k_2x_2 \\
\text{Mass B:} \quad m\ddot{x}_2 &= -k_2(x_2 - x_1) - 2k_1x_2 \\
&= k_2x_1 - (2k_1 + k_2)x_2.
\end{aligned}$$

These two equations can be also written in a convenient matrix form as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1 + k_2) & k_2 \\ k_2 & -(2k_1 + k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \quad (11.46)$$

$$m\ddot{x}_1 = -(2k_1 + k_2)x_1 + k_2x_2, \quad m\ddot{x}_2 = k_2x_1 - (2k_1 + k_2)x_2$$

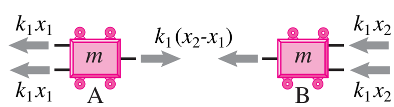


Figure 11.33: Partial free-body diagram of the two masses. Only relevant forces (in the x-direction) are shown in the diagram.

2. The out of phase normal mode of vibration of the two masses is such that $x_1(t) = x_0 \sin \lambda_n t$ and $x_2 = -x_0 \sin \lambda_n t$, *i.e.*, the two masses have out of phase displacements ($x_1 = -x_2$). If we substitute these values of the displacements, we see that both equations turn out to be the same and they give,

$$-\lambda_n^2 x_0 \sin \lambda_n t = \frac{1}{m}(-2k_1 - k_2 - k_2)x_0 \sin \lambda_n t$$

from which it follows that,

$$\lambda_n = \sqrt{\frac{2(k_1 + k_2)}{m}}.$$

Substituting the given values of m, k_1, and k_2, we get,

$$\lambda_n = \sqrt{\frac{2(25 + 3)\,\mathrm{N/m}}{34.5 \times 10^{-9}\,\mathrm{kg}}} = 40.29 \times 10^3\,\mathrm{rad/s}.$$

Thus the natural frequency corresponding to the out of phase vibration mode is 40.29×10^3 rad/s which corresponds to $f_n = \lambda_n/2\pi = 6.4$ kHz.

$$f_n = 6.4\mathrm{kHz}$$

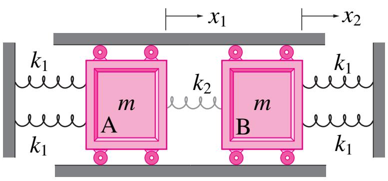


Figure 11.34: The two degree of freedom model of a two-mass MEMS vibratory gyroscope. Here, $m = 34.5 \times 10^{9}$ kg or 34.5μg, $k_1 = 25\,\text{N/m}$, and $k_2 = 3\,\text{N/m}$.

SAMPLE 11.13 Normal modes from eigen analysis: Consider the two-mass MEMS gyroscope of Sample 11.12 again. Using the equations of motion derived in Sample 11.12,

1. Find the natural frequencies and the corresponding normal modes of vibration of the system.
2. Using initial conditions based on the normal modes, solve the equations of motion numerically and plot $x_1(t)$ and $x_2(t)$ together for each normal mode. From the plots, show that the time period of oscillation conforms to the natural frequencies found above.

Solution

1. The equations of motion for the two degree of freedom model were obtained in eqn. (11.46) and are reproduced here:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.47}$$

Let us assume a normal mode of vibration in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sin\lambda_n t$$

where λ_n is the natural frequency of the system. Substituting this assumed motion in eqn. (11.47) and getting rid of $\sin\lambda_n t$ from both sides, we get,

$$-\lambda_n^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Rearranging this equation a little bit, we can write it as $[\mathbf{A}]\,\mathbf{V} = \lambda\,\mathbf{V}$, the standard eigenvalue problem, where $\lambda (= \lambda_n^2)$ is the eigenvalue of the matrix $[\mathbf{A}]$ and $\mathbf{V}$ is the corresponding eigenvector. Here,

$$\mathbf{A} = \begin{bmatrix} (2k_1+k_2)/m & -k_2/m \\ -k_2/m & (2k_1+k_2)/m \end{bmatrix}.$$

Now, we can go to a computer and find the eigenvalues and eigenvectors of $[\mathbf{A}]$:

```
m = 34.5*10^(-9),  k1 = 25,  k2 = 3
A = [(2*k1+k2)/m     -k2/m;
     -k2/m           (2*k1+k2)/m]
lambda = eigenvalues(A)
v = eigenvectors(A)
```

By carrying out this computation, using appropriate commands in a computational package, we find the following two eigenvalues and the corresponding two eigenvectors:

$$\lambda^{(1)} = 1.449\times 10^9, \qquad \mathbf{V}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$\text{and} \quad \lambda^{(2)} = 1.623\times 10^9, \qquad \mathbf{V}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, since we know that $\lambda = \lambda_n^2$, we can find the natural frequencies of our system by taking the square root of the eigenvalues just found. Thus,

$$\lambda_n^{(1)} = 3.807\times 10^4 \text{ rad/s} \quad\Rightarrow\quad f_n^{(1)} = \lambda_n^{(1)}/2\pi = 6.06 \text{ kHz},$$

$$\text{and} \quad \lambda_n^{(2)} = 4.029\times 10^4 \text{ rad/s} \quad\Rightarrow\quad f_n^{(2)} = \lambda_n^{(2)}/2\pi = 6.41 \text{ kHz}.$$

The corresponding normal modes or mode shapes are given by $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$. Please note that the components of an eigenvector are determined relative to each other, that is, the absolute numerical values are not unique, and any multiple of an eigenvector is also an eigenvector. For example, you could find $\mathbf{V}^{(1)} = [\sqrt{2} \quad \sqrt{2}]^T$, or $\mathbf{V}^{(1)} = [1/\sqrt{2} \quad 1/\sqrt{2}]^T$.

$$\lambda_n^{(1)} = 3.807 \times 10^4 \text{ rad/s}, \quad \mathbf{V}^{(1)} = [1 \quad 1]^T$$
$$\lambda_n^{(2)} = 4.029 \times 10^4 \text{ rad/s}, \quad \mathbf{V}^{(2)} = [1 \quad -1]^T$$

2. The normal modes thus found indicate that as long as we set the initial conditions for the two masses in the same proportion as one of the mode shapes (eigenvectors), the two masses will vibrate synchronously with the same frequency (corresponding to the chosen mode shape). So, we now simulate the motion of the two masses by solving the equations of motion numerically, using appropriate initial conditions.

We first write the equations of motion as a set of first order equations:

$$\begin{aligned}
\dot{x}_1 &= u_1 \\
\dot{u}_1 &= -\frac{(2k_1 + k_2)}{m} x_1 + \frac{k_2}{m} x_2 \\
\dot{x}_2 &= u_2 \\
\dot{u}_2 &= \frac{k_2}{m} x_2 - \frac{(2k_1 + k_2)}{m} x_2.
\end{aligned}$$

For the first mode, we set the initial conditions $x_1(0) = x_2(0) = 1\mu$m corresponding to the first eigenvector $\mathbf{V}^{(1)} = [1 \quad 1]^T$. Now, we are ready to solve the equations numerically.

```
ODEs = {x1dot = u1,
        u1dot = -(2*k1+k2)/m * x1 + k2/m * x2,
        x2dot = u2,
        u2dot = k2/m * x1 - (2*k1+k2)/m * x2}
ICs   = {x1(0)=1E-6, u1(0)=0, x2(0)=1E-6, u2(0)=0}
Set m = 34.5E-9, k1 = 25, k2 = 3,
Solve ODEs with ICs for t=0 to t=0.6E-3
Plot x1(t) and x2(t)
```

Note that we are solving the equations for only 0.6 milliseconds, that is, less than a millisecond. This is because we already know that the frequency is very high, roughly about 6 kHz, which means we can get six oscillations in one millisecond. The plot of $x_1(t)$ and $x_2(t)$ are shown in fig. 11.35. From this plot, we find that the time period of one oscillation is approximately 1.64×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 3.8 \times 10^4$ rad/s.

Similarly, using the initial conditions $x_1(0) = 1\mu$m, and $x_2(0) = -1\mu$m (corresponding to the second eigenvector $\mathbf{V}^{(2)}$, we get the plot shown in fig. 11.36. From this plot, we find that the time period of one oscillation is approximately 1.57×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 4 \times 10^4$ rad/s. Thus, the results of the numerical solution match the results obtained from the eigenvalue analysis.

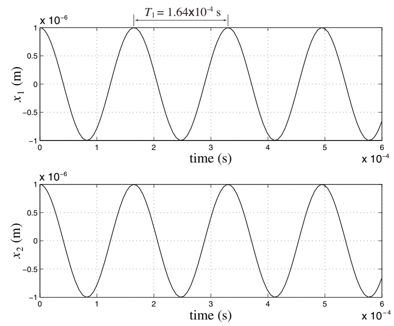


Figure 11.35: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the first normal mode. Note that both masses move together (in phase) and oscillate with the same frequency, $\lambda_n^{(1)}$. In this mode, the middle spring, k_2 plays no role as it remains unstretched at all times $x_2(t) - x_1(t) = 0$ for all t.

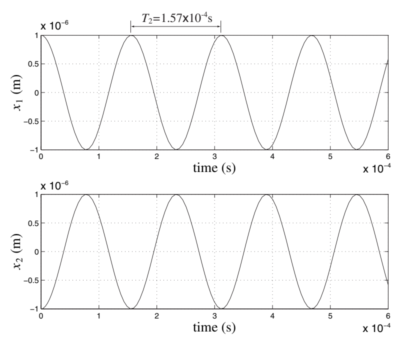


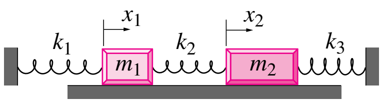
Figure 11.36: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the second normal mode. In this mode, each mass moves opposite to the other (out of phase) but both oscillate with the same frequency, $\lambda_n^{(2)}$.

11.3 Normal Modes

11.3.1 A two degree of freedom mass-spring system, made up of two unequal masses m_1 and m_2 and three springs with unequal stiffnesses k_1, k_2 and k_3, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses.

b) Does each mass undergo simple harmonic motion? *



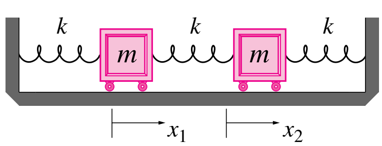
Problem 11.3.1

11.3.2 Normal Modes. Three equal springs (k) hold two equal masses (m) in place. There is no friction. x_1 and x_2 are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system? *

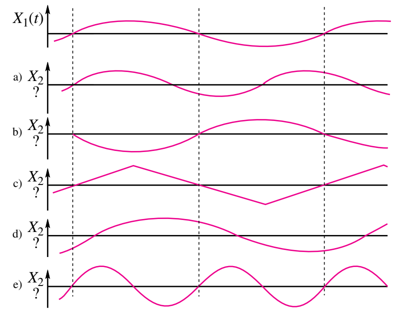
b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where A, B, and c are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of A, B, c, m and k.) *

c) Find all of the frequencies of normal-mode-vibration for this system in terms of m and k. *



Problem 11.3.2

11.3.3 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice and briefly explain your answer (If a curve looks like it is meant to be a sine/cosine curve, it is.)

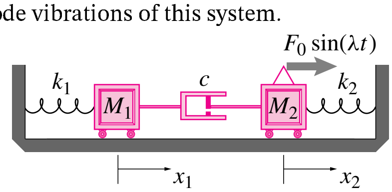


Problem 11.3.3

More-Involved Problems

11.3.4 Two masses are connected to fixed supports and each other with the two springs and dashpot shown. The displacements x_1 and x_2 are defined so that $x_1 = x_2 = 0$ when both springs are unstretched.

For the special case that $C = 0$ and $F_0 = 0$ clearly define two different sets of initial conditions that lead to normal mode vibrations of this system.

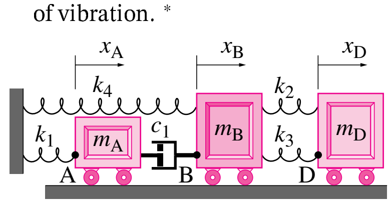


Problem 11.3.4

11.3.5 As in problem 10.4.11, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

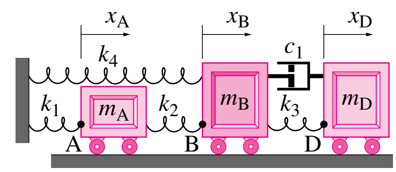
a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way. *

b) In the same special case as in (a) above, find another normal mode of vibration. *



Problem 11.3.5

11.3.6 As in problem 10.4.10, a system of three masses, four springs, and one damper are connected as shown. In the special case when $c_1 = 0$, find the normal modes of vibration.

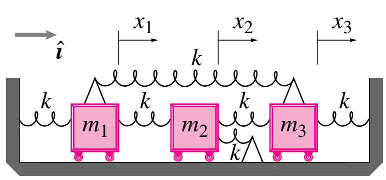


Problem 11.3.6

11.3.7 Normal modes. All three masses have $m = 1\,\mathrm{kg}$ and all 6 springs are $k = 1\,\mathrm{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition
$$[x_0] = [\,0.1\,\mathrm{m}\ \ 0\ \ 0\,],$$
$[\dot{x}_0] = [\,0\ \ 2\,\mathrm{m/s}\ \ 0\,]$
what is the initial (immediately after the start) acceleration of mass 2?



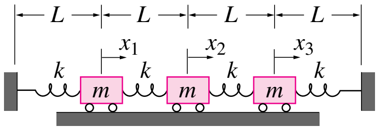
Problem 11.3.7

11.3.8 For the three-mass system shown, assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed. One of the normal modes is described with the initial condition $(x_{1_0}, x_2, x_3) = (1, 0, -1)$.

a) What is the angular frequency ω for this mode? Answer in terms of L, m, k, and g. (Hint: Note that in this mode of vibration the middle mass does not move.) *

b) Make a neat plot of x_3 versus x_1 for one cycle of vibration with this mode.

c) Make a single plot with both x_1 and x_3 vs t for one cycle of this mode.



Problem 11.3.8