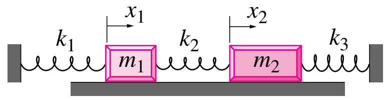
11.3.1 A two degree of freedom mass-spring system, made up of two unequal masses m_1 and m_2 and three springs with unequal stiffnesses k_1, k_2 and k_3, is shown in the figure. All three springs are relaxed in the configuration shown. Neglect friction.

a) Derive the equations of motion for the two masses.

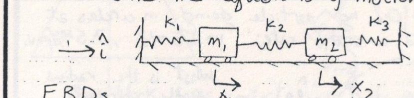
b) Does each mass undergo simple harmonic motion?



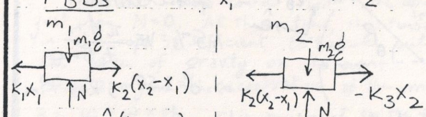
Problem 11.1

Answer:
(b) If we start off by assuming that each mass undergoes simple harmonic motion at the same frequency but different amplitudes, we will find that this two-degree-of-freedom system has two natural frequencies. Associated with each natural frequency is a fixed ratio between the amplitudes of each mass. Each mass will undergo simple harmonic motion at one of the two natural frequencies only if the initial displacements of the masses are in the fixed ratio associated with that frequency.

3.95 Derive the equations of motion.



FBDs



LMB: $\hat{\imath} \cdot (\sum \underset{\sim}{F} = \dot{\underset{\sim}{L}})$ | LMB: $\hat{\imath} \cdot (\sum \underset{\sim}{F} = \dot{\underset{\sim}{L}})$

$k_2(x_2 - x_1) - k_1 x_1 = m_1 a_1$ | $-k_3 x_2 - k_2(x_2 - x_1) = m_2 a_2$

therefore...

$$m_1 \ddot{x}_1 + (k_1 + k_2) x_1 - k_2 x_2 = 0$$
$$m_2 \ddot{x}_2 - k_2 x_1 + (k_2 + k_3) x_2 = 0$$

or, in matrix form, dividing thru by m:

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} \frac{k_1 + k_2}{m_1} & -\frac{k_2}{m_1} \\ -\frac{k_2}{m_2} & \frac{k_2 + k_3}{m_2} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

b) The solutions to the above are of the form:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = (A_1 \cos\omega_1 t + B_1 \sin\omega_1 t) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_1 + (A_2 \cos\omega_2 t + B_2 \sin\omega_2 t) \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}_2 .$$

(See ch. 10, sect. 4, for a discussion of how to get this.)

continued...

3.95 continued p.4

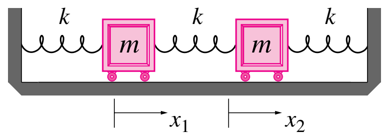
There are two natural frequencies (ω_1 & ω_2). The V's (normal modes) are constants that determine the relative amplitude of oscillations (they are the eigenvectors of the matrix of coefficients of x; the ω are the eigenvalues of that matrix.). This system will undergo SHM only at one of the two natural frequencies ω_1 or ω_2, and only if the initial ratios of the displacements are such that they correspond to one of the normal modes. Those of you who saw the "working model" demo may recall that the 3 degree of freedom spring-mass system underwent SHM; this was because the initial displacements of the masses corresponded to one of the normal modes of the system.

11.3.2 Normal Modes. Three equal springs (k) hold two equal masses (m) in place. There is no friction. x_1 and x_2 are the displacements of the masses from their equilibrium positions.

a) How many independent normal modes of vibration are there for this system?

b) Assume the system is in a normal mode of vibration and it is observed that $x_1 = A\sin(ct) + B\cos(ct)$ where A, B, and c are constants. What is $x_2(t)$? (The answer is not unique. You may express your answer in terms of any of A, B, c, m and k.)

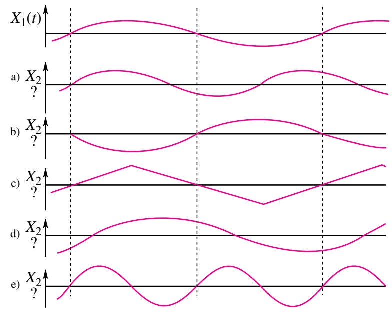
c) Find all of the frequencies of normal-mode-vibration for this system in terms of m and k.



Problem 11.2

Answer:
(a) Two normal modes.
(b) $x_2 = const * x_1 = const * (A\sin(ct) + B\cos(ct))$, where $const = \pm 1$.
(c) $\omega_1 = \sqrt{\frac{3k}{m}}, \omega_2 = \sqrt{\frac{k}{m}}$.

11.3.3 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice and briefly explain your answer (If a curve looks like it is meant to be a sine/cosine curve, it is.)



Problem 11.3

9.73

* Look at page 466 of text

a) could not because they have different freq.

b) could

c) could not because it is not simple harmonic motion

d) could not because not exactly in (or out) of phase

e) could not, reason same as (a)

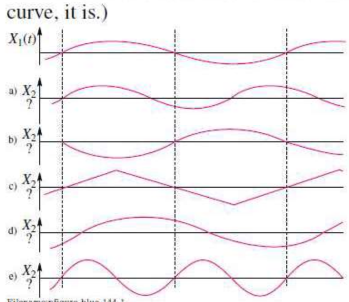
Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

HW4-P14-11.3.3

Tuesday, March 2, 2021 8:48 AM

Solution by Michael Zakoworotny

11.3.3 $x_1(t)$ and $x_2(t)$ are measured positions on two points of a vibrating structure. $x_1(t)$ is shown. Some candidates for $x_2(t)$ are shown. Which of the $x_2(t)$ could possibly be associated with a normal mode vibration of the structure? Answer "could" or "could not" next to each choice and briefly explain your answer (If a curve looks like it is meant to be a sine/cosine curve, it is.)



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Problem 11.3.3

Given the vibration of point 1, $x_1(t)$, determine which of these five vibrations of point 2 are associated with a normal mode.

Characteristics of normal modes:

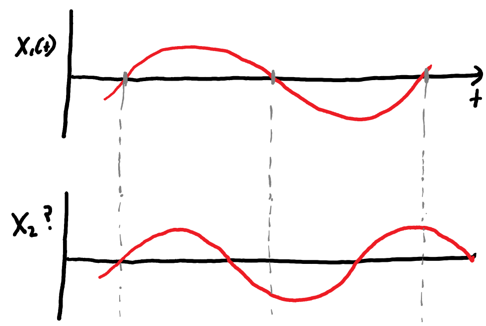
All elastic systems with mass have *normal mode* vibrations in which all masses

- have simple harmonic motion ~ SHM
- with the same frequency as all the other masses, and
- exactly in (or out) of phase with all of the other masses

From pg. 590 of textbook

Let's look at each case individually

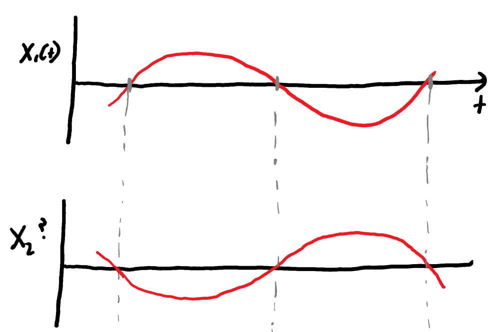
X a) x_2 has a different frequency than x_1, so is not a normal mode.



$\boxed{\text{Could not}}$

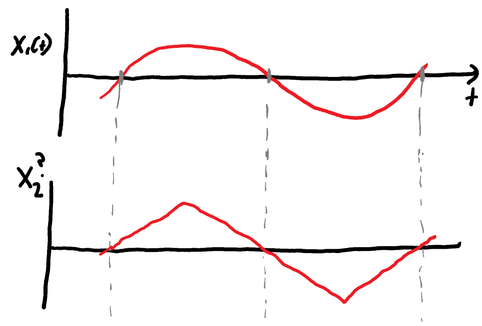
b) Is SHM, has same frequency as x_1, and is out of phase.

Therefore is a normal mode.



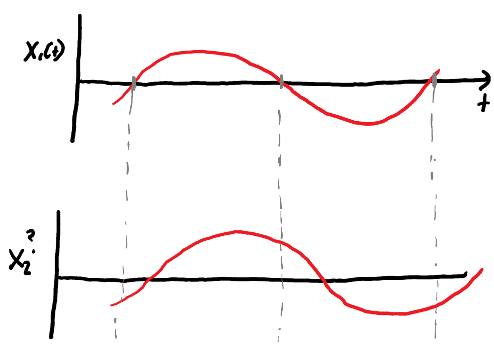
$\boxed{\text{Could}}$

X c) This is not SHM. Cannot be a normal mode



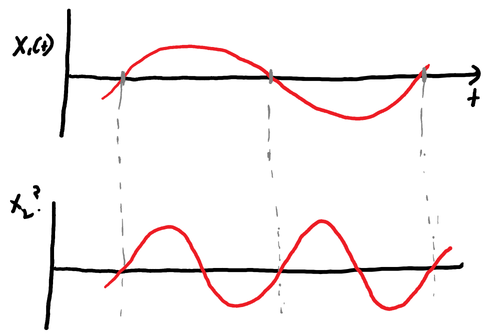
$\boxed{\text{Could not}}$

X d) Is SHM and x_2 has same frequency as x_1. However, is not exactly in/out of phase, so is not a normal mode.



$\boxed{\text{Could not}}$

✗ e) x_2 has a different frequency than x_1, so is not a normal mode.



$\boxed{\text{Could not}}$

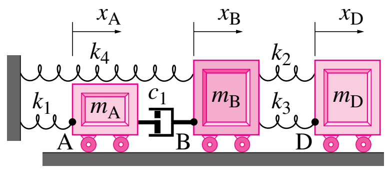
Summary

	Simple Harmonic Motion	Same frequency	Exactly in / out of phase
A)	✓	✗	✓
B)	✓	✓	✓
C)	✗	✓	✓
D)	✓	✓	✗
E)	✓	✗	✓

11.3.5 As in problem **??**, a system of three masses, four springs, and one damper are connected as shown. Assume that all the springs are relaxed when $x_A = x_B = x_D = 0$.

a) In the special case when $k_1 = k_2 = k_3 = k_4 = k$, $c_1 = 0$, and $m_A = m_B = m_D = m$, find a normal mode of vibration. Define it in any clear way and explain or show why it is a normal mode in any clear way.

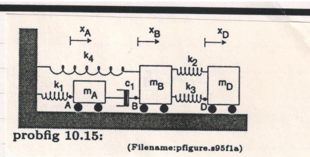
b) In the same special case as in (a) above, find another normal mode of vibration.



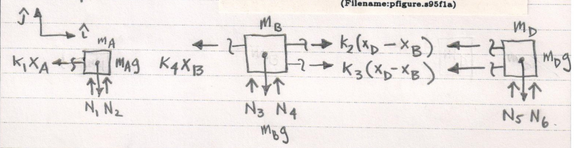
Problem 11.5

Answer:
(a) One normal mode: $[1, 0, 0]$.
(b) The other two normal modes: $[0, 1, \frac{1\pm\sqrt{17}}{4}]$.

10.15



FBD of each mass for $c_1 = 0$



LMB $\{ m\underline{a} = \Sigma \underline{F} \} \cdot \hat{\imath}$

note spring force simplification

$m_A \ddot{x}_A = -k_1 x_A$

$m_B \ddot{x}_B = -k_4 x_B + (k_2+k_3)(x_D - x_B)$

$m_D \ddot{x}_D = -(k_2+k_3)(x_D - x_B)$

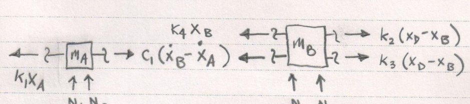
let $v_A = \dot{x}_A$, $v_B = \dot{x}_B$, $v_D = \dot{x}_D$

$$\dot{\begin{bmatrix} x_A \\ v_A \\ x_B \\ v_B \\ x_D \\ v_D \end{bmatrix}} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -k_1/m_A & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -(k_2+k_3+k_4)/m_B & 0 & (k_2+k_3)/m_B & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & (k_2+k_3)/m_D & 0 & -(k_2+k_3)/m_D & 0 \end{bmatrix} \begin{bmatrix} x_A \\ v_A \\ x_B \\ v_B \\ x_D \\ v_D \end{bmatrix}$$

How does matrix change when $c_1 \neq 0$?

$m_A \ddot{x}_A = -k_1 x_A + c_1(\dot{x}_B - \dot{x}_A)$

$m_B \ddot{x}_B = -k_4 x_B - c_1(\dot{x}_B - \dot{x}_A) + (k_2+k_3)(x_D - x_B)$



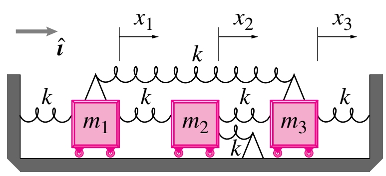
$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -\frac{k_1}{m_A} & -\frac{c_1}{m_A} & 0 & \frac{c_1}{m_A} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & \frac{c_1}{m_B} & \frac{-(k_2+k_3+k_4)}{m_B} & \frac{-c_1}{m_B} & \frac{k_2+k_3}{m_B} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{k_2+k_3}{m_D} & 0 & \frac{-(k_2+k_3)}{m_D} & 0 \end{bmatrix}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

11.3.7 Normal modes. All three masses have $m = 1\,\text{kg}$ and all 6 springs are $k = 1\,\text{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition
$$[x_0] = [\,0.1\,\text{m}\ \ 0\ \ 0\,],$$
$[\dot{x}_0] = [\,0\ \ 2\,\text{m/s}\ \ 0\,]$
what is the initial (immediately after the start) acceleration of mass 2?

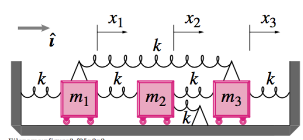


Problem 11.7

11.3.7 Normal modes. All three masses have $m = 1\,\text{kg}$ and all 6 springs are $k = 1\,\text{N/m}$. The system is at rest when $x_1 = x_2 = x_3 = 0$.

a) Find as many different initial conditions as you can for which normal mode vibrations result. In each case, find the associated natural frequency. (we will call two initial conditions $[v]$ and $[w]$ different if there is no constant c so that $[v_1\ v_2\ v_3] = c[w_1\ w_2\ w_3]$. Assume the initial velocities are zero.)

b) For the initial condition
$$[x_0] = [\,0.1\,\text{m}\ \ 0\ \ 0\,],$$
$[\dot{x}_0] = [\,0\ \ 2\,\text{m/s}\ \ 0\,]$
what is the initial (immediately after the start) acceleration of mass 2?



Problem 10.3.7

c) (Additional) For the same initial conditions in part b, plot x1, x2, & x3 vs t (all on the same plot) for 0<= t <= 20s.

a) Given:

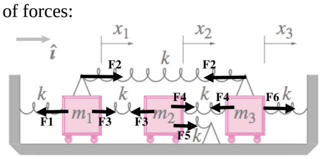
$k = 1\,\frac{\text{N}}{\text{m}}, m = 1\,\text{kg}$.

The system is at rest when $x_1 = x_2 = x_3 = 0 \rightarrow x_1 = x_2 = x_3 = 0$ is the equilibrium position.

Find:

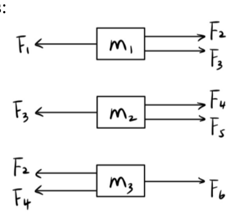
Initial conditions to result in normal mode

Sketch of forces:



All forces F_i are tensions. This is a sign convention. It does not mean that all springs are in tension. Rather it means that springs in compression have $F_i < 0$. Similarly, x_i are displacements to the right. A leftwards displacement corresponds to $x_i < 0$. For example the rightmost spring $F_6 = k \times$ (excess stretch of spring on right) $= -kx_3$ is only greater than zero (in tension) if $x_3 < 0$.

FBD's:



Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

Spring tensions are the excess tension above the tensions that may be present in the equilibrium position:

$$F_1 = kx_1$$
$$F_2 = k(x_3 - x_1)$$
$$F_3 = k(x_2 - x_1)$$
$$F_4 = k(x_3 - x_2)$$
$$F_5 = -kx_2$$
$$F_6 = -kx_3$$

LMB:

$$m\ddot{x}_1 = -F_1 + F_2 + F_3 = -3kx_1 + kx_2 + kx_3$$
$$m\ddot{x}_2 = -F_3 + F_4 + F_5 = kx_1 - 3kx_2 + kx_3$$
$$m\ddot{x}_3 = -F_2 - F_4 + F_6 = kx_1 + kx_2 - 3kx_3$$

LMB in matrix form:

$$\ddot{\vec{x}} = \begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{pmatrix} = -\frac{k}{m}\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} \quad ①$$

By definition, a normal mode vibration can be written as

$$\begin{pmatrix}x_1\\ x_2\\ x_3\end{pmatrix} = \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}\cos\left(\omega_n t + \phi\right) \quad ②$$

where ω_n is the natural frequency of oscillation. The magnitude and sign of u_1, u_2 and u_3 determine respectively the amplitude and relative direction of each mass's oscillation.

Plug ② into ①:

$$\begin{pmatrix}\ddot{x}_1\\ \ddot{x}_2\\ \ddot{x}_3\end{pmatrix} = -\omega_n^2\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}\cos(\omega_n t + \phi) = -\frac{k}{m}\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}\cos(\omega_n t + \phi)$$

Simplify:

$$\omega_n^2\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix} = \frac{k}{m}\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}\begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}$$

Further transform into the eigenvalue/eigenvector equation:

$$\lambda\vec{u} = A\vec{u}, \text{ where } \vec{u} = \begin{pmatrix}u_1\\ u_2\\ u_3\end{pmatrix}, \lambda = \omega_n^2, A = \frac{k}{m}\begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}$$

Plugging in values $k = 1\,\frac{\text{N}}{\text{m}}, m = 1$ kg and solve for the eigenvalues and eigenvectors of A.
Solve $|A - \lambda I| = 0$ and $(A - \lambda I)\vec{u} = \vec{0}$ to get

$$\lambda_1 = 1, \vec{u}_1 = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\ 1\\ 1\end{pmatrix}$$

$$\lambda_2 = 4, \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\ -1\\ 0\end{pmatrix}$$

$$\lambda_3 = 4, \vec{u}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

Can also use Matlab

```
A = [3, -1, -1; -1, 3, -1; -1, -1, 3];
[V, D] = eig(A)
V =
   -0.5774    0.1870    0.7948
   -0.5774   -0.7818   -0.2354
   -0.5774    0.5948   -0.5594
D =
    1.0000         0         0
         0    4.0000         0
         0         0    4.0000
```

where each column of V, which is the eigenvector, corresponds to each diagonal element of D, which is the eigenvalue.

Note: $\vec{u}_2$ and $\vec{u}_3$ can be any unit vector $\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$ such that $w_1 + w_2 + w_3 = 0$ since $A - \lambda_{2,3} I = \begin{bmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{bmatrix}$.

Initial condition to result in normal mode $\vec{u}_1$:

$$\left\{\vec{x}_0 = [1,1,1]'\,\text{m}, \dot{\vec{x}}_0 = [0,0,0]'\,\frac{\text{m}}{\text{s}}\right\}$$

$$\text{or}\left\{\vec{x}_0 = [0,0,0]'\,\text{m}, \dot{\vec{x}}_0 = [1,1,1]'\,\frac{\text{m}}{\text{s}}\right\}$$

Natural frequency $\omega_n = \sqrt{\lambda_1} = 1\,\frac{\text{rad}}{\text{s}}$

Initial condition to result in normal mode $\vec{u}_2$:

$$\left\{\vec{x}_0 = [1,-1,0]'\,\text{m}, \dot{\vec{x}}_0 = [0,0,0]'\,\frac{\text{m}}{\text{s}}\right\}$$

$$\text{or}\left\{\vec{x}_0 = [0,0,0]'\,\text{m}, \dot{\vec{x}}_0 = [1,-1,0]'\,\frac{\text{m}}{\text{s}}\right\}$$

Natural frequency $\omega_n = \sqrt{\lambda_2} = 2\,\frac{\text{rad}}{\text{s}}$

Initial condition to result in normal mode $\vec{u}_3$:

$$\left\{\vec{x}_0 = [1,0,-1]'\,\text{m}, \dot{\vec{x}}_0 = [0,0,0]'\,\frac{\text{m}}{\text{s}}\right\}$$

$$\text{or}\left\{\vec{x}_0 = [0,0,0]'\,\text{m}, \dot{\vec{x}}_0 = [1,0,-1]'\,\frac{\text{m}}{\text{s}}\right\}$$

Natural frequency $\omega_n = \sqrt{\lambda_3} = 2\,\frac{\text{rad}}{\text{s}}$

b) Given: $\vec{x}_0 = [0.1, 0, 0]'$ m, $\dot{\vec{x}}_0 = [0, 2, 0]'\,\frac{\text{m}}{\text{s}}$
Find: $\ddot{x}_2$

Use $\ddot{x}_2 = \frac{k}{m}(x_1 - 3x_2 + x_3)$ from part (a)

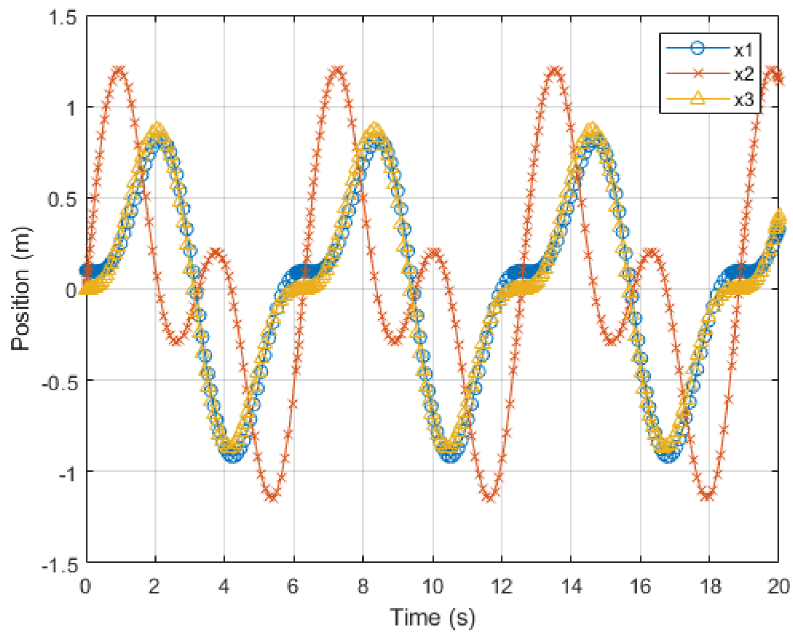
Plug in $\vec{x}_0 = [x_1, x_2, x_3]' = [0.1, 0, 0]'$ m

$\ddot{x}_2 = \frac{1\text{N/m}}{1\text{kg}}(0.1\text{m} - 3 \times 0 + 0) = 0.1\,\frac{\text{m}}{\text{s}^2}$

c) Given: $\vec{x}_0 = [0.1, 0, 0]'$ m, $\dot{\vec{x}}_0 = [0, 2, 0]'\,\frac{\text{m}}{\text{s}}$
Find: $[x_1, x_2, x_3]'$ for 0<= t <= 20s

See attached Matlab.

The motion $\vec{x}(t) = [x_1, x_2, x_3]'$ is a superposition of the three normal modes. If you squint just right, you can see in the complex motion in the plot some remnants of oscillation with 2 frequencies: 1 rad/s for the waves and 2 rad/s for the envelope of the waves.



Published with MATLAB® R2020b

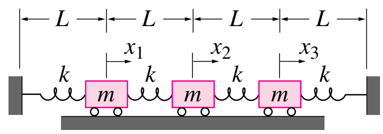
```matlab
% MAE 2030 SP21, HW P16(c)
% Duan Li

% parameters
tend = 20; % end time, s
p.m = 1; % mass, kg
p.k = 1; % spring constant, N/m

% initial conditions
x0 = [0.1, 0, 0]'; % position, m
v0 = [0, 2, 0]'; % velocity, m/s
z0 = [x0; v0]; % augmented state

% solve ODEs
rhs = @(t, z) myRHS(t, z, p); % set up anonymous function
[t, zarray] = ode45(rhs, [0 tend], z0); % solve ODEs

% plot
close all;
x1 = zarray(:, 1);
x2 = zarray(:, 2);
x3 = zarray(:, 3);
plot(t, x1, '-o', t, x2, '-x',t, x3, '-^')
legend('x1', 'x2', 'x3')
xlabel('Time (s)')
ylabel('Position (m)')
grid on;

% RHS function
function zdot = myRHS(t, z, p)
    % unpack parameters
    names = fieldnames(p);
    for i = 1:length(names)
        eval([names{i} '= p.' names{i} ';']);
    end

    % 1st order ODEs
    x = z(1:3);
    v = z(4:6);
    M = -k/m * [3, -1, -1; -1, 3, -1; -1, -1, 3];
    xdot = v;
    vdot = M * x;
    zdot = [xdot; vdot];
end
```

11.3.8 For the three-mass system shown, assume $x_1 = x_2 = x_3 = 0$ when all the springs are fully relaxed. One of the normal modes is described with the initial condition $(x_{1_0}, x_2, x_3) = (1, 0, -1)$.

a) What is the angular frequency ω for this mode? Answer in terms of L, m, k, and g. (Hint: Note that in this mode of vibration the middle mass does not move.)

b) Make a neat plot of x_3 versus x_1 for one cycle of vibration with this mode.

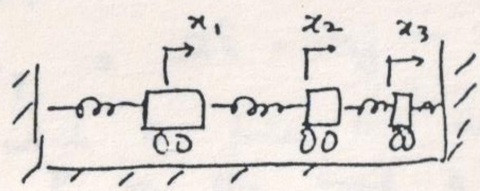
c) Make a single plot with both x_1 and x_3 vs t for one cycle of this mode.



Problem 11.8

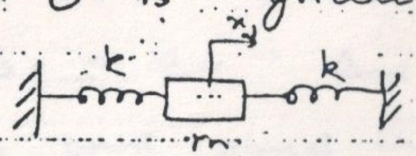
Answer:
(a) $\omega = \sqrt{\frac{2k}{m}}$.

$\boxed{10.26}$ Mode $(1, 0, -1)$ x_1, x_2, x_3 measured from equilibrium, find ω.



All masses and springs are equal.

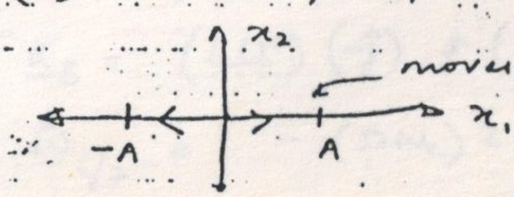
Trick: Since the middle mass does not move in this mode, the motion of mass (1) is governed by:



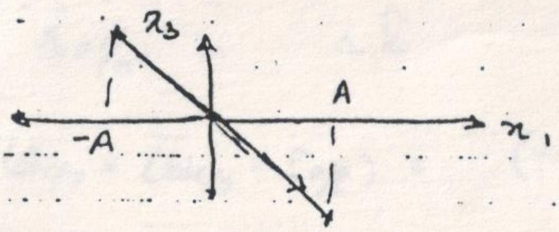
and $x_3 = -x_1$
$x_2 = 0$

$\therefore$ by inspection $\boxed{\omega = \sqrt{\frac{2k}{m}}}$

(b) $x_2 = 0$, $x_1 = A \sin \omega t$.

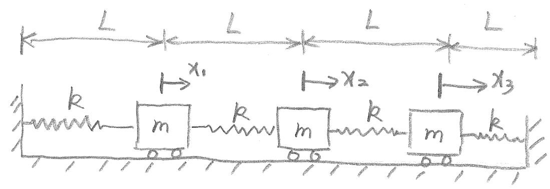


moves up and down x-axis.



①

9.82 Normal modes



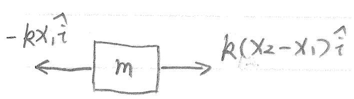
3-Mass system, connected by springs with stiffness k. All the mass is m.
Assume $x_1 = x_2 = x_3$ where all springs are at rest.
A normal mode of this system can be described as eigenvector $(1, 0, -1)$.

Q: 1). What's the frequency corresponding to this normal mode?
2). Plot x_2 versus x_1 for one cycle of vibration in this mode.

Solution: 1). First, we want to derive the equations of motion for this system, starting from FBD.

FBD: $\longrightarrow \hat{\imath}$ Eq. of motions

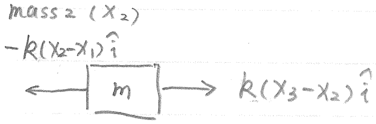
mass 1 (x_1)



$$\{m\ddot{x}_1\hat{\imath} = k(x_2-x_1)\hat{\imath} - kx_1\hat{\imath}\}$$

$$\{\}\cdot\hat{\imath} \Rightarrow \boxed{m\ddot{x}_1 = -2kx_1 + kx_2}$$

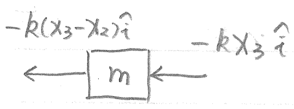
mass 2 (x_2)



$$\{m\ddot{x}_2\hat{\imath} = k(x_3-x_2)\hat{\imath} - k(x_2-x_1)\hat{\imath}\}$$

$$\{\}\cdot\hat{\imath} \Rightarrow \boxed{m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3}$$

mass 3 (x_3)



$$\{m\ddot{x}_3\hat{\imath} = -kx_3\hat{\imath} - k(x_3-x_2)\hat{\imath}\}$$

$$\{\}\cdot\hat{\imath} \Rightarrow \boxed{m\ddot{x}_3 = kx_2 - 2kx_3}$$

(9.82. cont'd) ②

Second, by definition of normal mode, for this normal mode with (1, 0, −1), we can write

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \underbrace{A}_{\text{amplitude}} \underbrace{\cos}_{\text{simple harmonic function}} (\underbrace{\omega}_{\text{same frequency, we want to solve it}} t + \underbrace{\phi}_{\text{exactly in or out of phase.}})$$

$$\Rightarrow \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} A \frac{d^2 \cos(\omega t + \phi)}{dt^2} = -\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} A \omega^2 \cos(\omega t + \phi)$$

Third, substitute $[x]$, $[\ddot{x}]$ for this normal mode into equations of motions:

$$\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} A\omega^2 \cos(\omega t + \phi) + \begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} A\cos(\omega t + \phi) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -\begin{bmatrix} m \\ 0 \\ -m \end{bmatrix} \omega^2 + \begin{bmatrix} 2k \\ 0 \\ -2k \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow -m\omega^2 + 2k = 0 \quad \Rightarrow \quad \boxed{\omega = \sqrt{\frac{2k}{m}}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

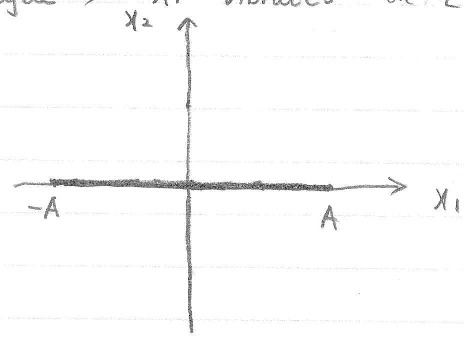
(9.82, cont'd) ③

(2.) From (1.), we know in this mode

$x_1 = A\cos(\sqrt{\frac{2k}{m}}t + \phi)$

$x_2 = 0$

During one cycle, x_1 vibrates in $[-A, A]$ and x_2 remains 0.

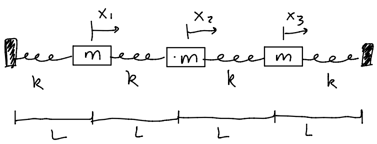


Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

#15 Solution - Michelle deSouza

Friday, February 26, 2021 10:56 AM

$\boxed{11.3.8}$



$x_1 = x_2 = x_3 = 0$ when all springs at rest length

a) Normal mode IC $V = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$. Find ω. $\boxed{V \neq \dot{x}}$

FBDs:

$-kx_1\hat{\imath}$ $k(x_2-x_1)\hat{\imath}$

$-k(x_2-x_1)\hat{\imath}$ $k(x_3-x_2)\hat{\imath}$

$-k(x_3-x_2)\hat{\imath}$ $-kx_3\hat{\imath}$

put into matrix form

$m\ddot{x}_1 = -kx_1 + k(x_2-x_1) \longrightarrow m\ddot{x}_1 = -2kx_1 + kx_2$

$m\ddot{x}_2 = -k(x_2-x_1) + k(x_3-x_2) \longrightarrow m\ddot{x}_2 = kx_1 - 2kx_2 + kx_3$

$m\ddot{x}_3 = -k(x_3-x_2) - kx_3 \longrightarrow m\ddot{x}_3 = kx_2 - 2kx_3$

$$① \underbrace{\begin{bmatrix} m & 0 & 0 \\ 0 & m & 0 \\ 0 & 0 & m \end{bmatrix}}_{M} \underbrace{\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix}}_{\ddot{x}} \underbrace{=}_{=} \underbrace{-\begin{bmatrix} 2k & -k & 0 \\ -k & 2k & -k \\ 0 & -k & 2k \end{bmatrix}}_{-K} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x}$$ matrix form

Normal Mode Equations:

plug in to ①

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} A\cos(\omega t + \phi) \quad \frac{d^2}{dt^2}$$

$$\begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \\ \ddot{x}_3 \end{bmatrix} = \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} -\omega^2 A\cos(\omega t + \phi)$$

Definition of normal mode:

- simple harmonic motion ✓
- same frequency ✓
- masses exactly in/out of phase ✓

Notes:

- V is an amplitude for a given x, not velocity!
- Because all x are in phase, amplitudes are I.C.s where $\dot{x} = 0$, aka. V

$$① \; M\ddot{X} = -KX$$

$$M \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \cancel{-}\omega^2 \cancel{A\cos(\omega t + \phi)} = \cancel{-}K \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \cancel{A\cos(\omega t + \phi)}$$

$$M \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \omega^2 = K \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

eigenvalue problem! $v\lambda = Av$, $v\omega^2 = M^{-1}Kv$ $\rightarrow$

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} \omega^2 = M^{-1}K \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}$$

plug in: $M = m\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ $K = k\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

$V = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \leftarrow$ given normal mode/amplitudes

Note

- M is diagonal - take reciprocal of each element to find M^{-1}

$M^{-1} = \frac{1}{m}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \omega^2 = \frac{k}{m} \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \omega^2 = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix} \frac{k}{m}$$

$$\omega^2 = 2\frac{k}{m}$$

$$\boxed{\omega = \sqrt{\frac{2k}{m}}}$$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

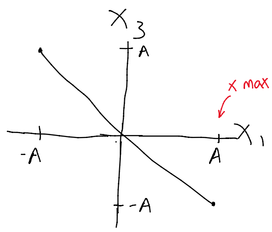
Sketches

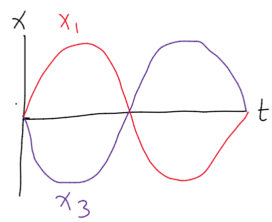
b) x_3 vs. x_1

c) x_1 vs. t and x_3 vs. t

$$x_1 = A\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi\right)$$
$$x_3 = -A\cos\left(\sqrt{\frac{2k}{m}}\,t + \phi\right)$$

$x_1 = -x_3$ for all t





$$T = \frac{2\pi}{\omega} = \pi\sqrt{\frac{2m}{k}}$$

What's actually happening? masses 1 and 3 are oscillating in exact mirrors of each other!