SAMPLE 11.12 **A two mass vibratory MEMS gyroscope:** A vibratory MEMS (microelectromechanical system) gyroscope employs two big plates as inertial masses, suspended by thin beams or 'springs' as shown in the figure. The two masses are made to vibrate (by electrical actuation) out of phase in the x-direction. Any rotation about the y-direction causes the masses to vibrate out-of-plane due to 'Coriolis acceleration' (you will learn about that in later chapters). We will restrict our attention to the planar motion of the gyroscope. A two degree of freedom spring-mass model is shown in the figure where $m = 34.5 \times 10^{-9}\,\text{kg}$, $k_1 = 25\,\text{N/m}$, and $k_2 = 3\,\text{N/m}$.

1. Write the equations of motion for the two masses.
2. For the out of phase motion of the two masses, assume that $x_1(t) = -x_2(t) = x_0 \sin\lambda_n t$. Determine the natural frequency λ_n corresponding to this mode of vibration.

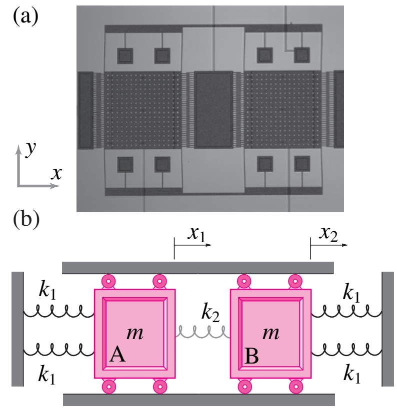

Figure 11.32: A MEMS vibratory gyroscope: (a) a micrograph of the two-mass structure. Each inertial mass (the plates with holes) is approximately $1\,\text{mm} \times 1\,\text{mm} \times 15\mu\text{m}$. The beams that hang the two masses act as springs. The comb drives on the left and right side of the two masses are used to drive the two masses to oscillate in the x-direction at their resonant frequencies. (b) a two degree of freedom spring-mass model of the driven gyroscope structure (without any damping).

Solution

1. The free-body diagram of each mass is shown in fig. 11.33. Assuming both x_1 and x_2 to be positive in the x-direction, and $x_2 > x_1$ at the instant shown in the figure, we can write the equations of motion using the balance of linear momentum as

$$\begin{aligned}
\text{Mass A:} \quad m\ddot{x}_1 &= k_2(x_2 - x_1) - 2k_1x_1 \\
&= -(2k_1 + k_2)x_1 + k_2x_2 \\
\text{Mass B:} \quad m\ddot{x}_2 &= -k_2(x_2 - x_1) - 2k_1x_2 \\
&= k_2x_1 - (2k_1 + k_2)x_2.
\end{aligned}$$

These two equations can be also written in a convenient matrix form as

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1 + k_2) & k_2 \\ k_2 & -(2k_1 + k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \qquad (11.46)$$

$m\ddot{x}_1 = -(2k_1 + k_2)x_1 + k_2x_2, \quad m\ddot{x}_2 = k_2x_1 - (2k_1 + k_2)x_2$

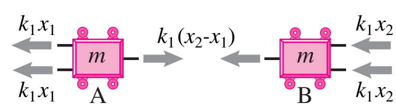

Figure 11.33: Partial free-body diagram of the two masses. Only relevant forces (in the x-direction) are shown in the diagram.

2. The out of phase normal mode of vibration of the two masses is such that $x_1(t) = x_0 \sin\lambda_n t$ and $x_2 = -x_0 \sin\lambda_n t$, *i.e.*, the two masses have out of phase displacements ($x_1 = -x_2$). If we substitute these values of the displacements, we see that both equations turn out to be the same and they give,

$$-\lambda_n^2 x_0 \sin\lambda_n t = \frac{1}{m}(-2k_1 - k_2 - k_2)x_0 \sin\lambda_n t$$

from which it follows that,

$$\lambda_n = \sqrt{\frac{2(k_1 + k_2)}{m}}.$$

Substituting the given values of m, k_1, and k_2, we get,

$$\lambda_n = \sqrt{\frac{2(25 + 3)\,\text{N/m}}{34.5 \times 10^{-9}\,\text{kg}}} = 40.29 \times 10^3\,\text{rad/s}.$$

Thus the natural frequency corresponding to the out of phase vibration mode is $40.29 \times 10^3\,\text{rad/s}$ which corresponds to $f_n = \lambda_n/2\pi = 6.4\,\text{kHz}$.

$f_n = 6.4\text{kHz}$

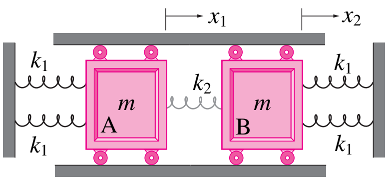

Figure 11.34: The two degree of freedom model of a two-mass MEMS vibratory gyroscope. Here, $m = 34.5 \times 10^{9}$ kg or 34.5μg, $k_1 = 25\,\mathrm{N/m}$, and $k_2 = 3\,\mathrm{N/m}$.

SAMPLE 11.13 Normal modes from eigen analysis: Consider the two-mass MEMS gyroscope of Sample 11.12 again. Using the equations of motion derived in Sample 11.12,

1. Find the natural frequencies and the corresponding normal modes of vibration of the system.
2. Using initial conditions based on the normal modes, solve the equations of motion numerically and plot $x_1(t)$ and $x_2(t)$ together for each normal mode. From the plots, show that the time period of oscillation conforms to the natural frequencies found above.

Solution

1. The equations of motion for the two degree of freedom model were obtained in eqn. (11.46) and are reproduced here:

$$\begin{pmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}. \tag{11.47}$$

Let us assume a normal mode of vibration in the form

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \sin\lambda_n t$$

where λ_n is the natural frequency of the system. Substituting this assumed motion in eqn. (11.47) and getting rid of $\sin\lambda_n t$ from both sides, we get,

$$-\lambda_n^2\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \frac{1}{m}\begin{bmatrix} -(2k_1+k_2) & k_2 \\ k_2 & -(2k_1+k_2) \end{bmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

Rearranging this equation a little bit, we can write it as $[\mathbf{A}]\,\mathbf{V} = \lambda\,\mathbf{V}$, the standard eigenvalue problem, where $\lambda (= \lambda_n^2)$ is the eigenvalue of the matrix $[\mathbf{A}]$ and $\mathbf{V}$ is the corresponding eigenvector. Here,

$$\mathbf{A} = \begin{bmatrix} (2k_1+k_2)/m & -k_2/m \\ -k_2/m & (2k_1+k_2)/m \end{bmatrix}.$$

Now, we can go to a computer and find the eigenvalues and eigenvectors of $[\mathbf{A}]$:

```
m = 34.5*10^(-9),  k1 = 25,  k2 = 3
A = [(2*k1+k2)/m     -k2/m;
     -k2/m           (2*k1+k2)/m]
lambda = eigenvalues(A)
v = eigenvectors(A)
```

By carrying out this computation, using appropriate commands in a computational package, we find the following two eigenvalues and the corresponding two eigenvectors:

$$\lambda^{(1)} = 1.449\times10^{9}, \qquad \mathbf{V}^{(1)} = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

$$\text{and} \quad \lambda^{(2)} = 1.623\times10^{9}, \qquad \mathbf{V}^{(2)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Now, since we know that $\lambda = \lambda_n^2$, we can find the natural frequencies of our system by taking the square root of the eigenvalues just found. Thus,

$$\lambda_n^{(1)} = 3.807\times10^{4}\ \mathrm{rad/s} \quad\Rightarrow\quad f_n^{(1)} = \lambda_n^{(1)}/2\pi = 6.06\ \mathrm{kHz},$$

$$\text{and} \quad \lambda_n^{(2)} = 4.029\times10^{4}\ \mathrm{rad/s} \quad\Rightarrow\quad f_n^{(2)} = \lambda_n^{(2)}/2\pi = 6.41\ \mathrm{kHz}.$$

The corresponding normal modes or mode shapes are given by $\mathbf{V}^{(1)}$ and $\mathbf{V}^{(2)}$. Please note that the components of an eigenvector are determined relative to each other, that is, the absolute numerical values are not unique, and any multiple of an eigenvector is also an eigenvector. For example, you could find $\mathbf{V}^{(1)} = [\sqrt{2} \quad \sqrt{2}]^T$, or $\mathbf{V}^{(1)} = [1/\sqrt{2} \quad 1/\sqrt{2}]^T$.

$$\begin{aligned} \lambda_n^{(1)} &= 3.807 \times 10^4 \,\text{rad/s}, \quad \mathbf{V}^{(1)} = [1 \quad 1]^T \\ \lambda_n^{(2)} &= 4.029 \times 10^4 \,\text{rad/s}, \quad \mathbf{V}^{(2)} = [1 \quad -1]^T \end{aligned}$$

2. The normal modes thus found indicate that as long as we set the initial conditions for the two masses in the same proportion as one of the mode shapes (eigenvectors), the two masses will vibrate synchronously with the same frequency (corresponding to the chosen mode shape). So, we now simulate the motion of the two masses by solving the equations of motion numerically, using appropriate initial conditions.

We first write the equations of motion as a set of first order equations:

$$\begin{aligned} \dot{x}_1 &= u_1 \\ \dot{u}_1 &= -\frac{(2k_1 + k_2)}{m} x_1 + \frac{k_2}{m} x_2 \\ \dot{x}_2 &= u_2 \\ \dot{u}_2 &= \frac{k_2}{m} x_2 - \frac{(2k_1 + k_2)}{m} x_2. \end{aligned}$$

For the first mode, we set the initial conditions $x_1(0) = x_2(0) = 1\mu$m corresponding to the first eigenvector $\mathbf{V}^{(1)} = [1 \quad 1]^T$. Now, we are ready to solve the equations numerically.

```
ODEs = {x1dot = u1,
        u1dot = -(2*k1+k2)/m * x1 + k2/m * x2,
        x2dot = u2,
        u2dot = k2/m * x1 - (2*k1+k2)/m * x2}
ICs   = {x1(0)=1E-6, u1(0)=0, x2(0)=1E-6, u2(0)=0}
Set m = 34.5E-9, k1 = 25, k2 = 3,
Solve ODEs with ICs for t=0 to t=0.6E-3
Plot x1(t) and x2(t)
```

Note that we are solving the equations for only 0.6 milliseconds, that is, less than a millisecond. This is because we already know that the frequency is very high, roughly about 6 kHz, which means we can get six oscillations in one millisecond. The plot of $x_1(t)$ and $x_2(t)$ are shown in fig. 11.35. From this plot, we find that the time period of one oscillation is approximately 1.64×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 3.8 \times 10^4$ rad/s.

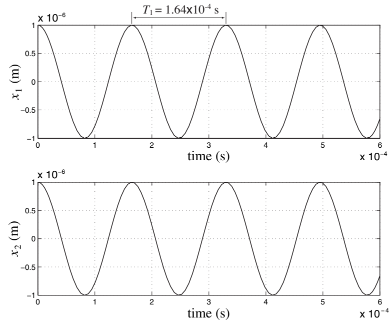

Figure 11.35: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the first normal mode. Note that both masses move together (in phase) and oscillate with the same frequency, $\lambda_n^{(1)}$. In this mode, the middle spring, k_2 plays no role as it remains unstretched at all times $x_2(t) - x_1(t) = 0$ for all t.

Similarly, using the initial conditions $x_1(0) = 1\mu$m, and $x_2(0) = -1\mu$m (corresponding to the second eigenvector $\mathbf{V}^{(2)}$, we get the plot shown in fig. 11.36. From this plot, we find that the time period of one oscillation is approximately 1.57×10^{-4} seconds, which gives a frequency of $\lambda_n = 2\pi/T = 4 \times 10^4$ rad/s. Thus, the results of the numerical solution match the results obtained from the eigenvalue analysis.

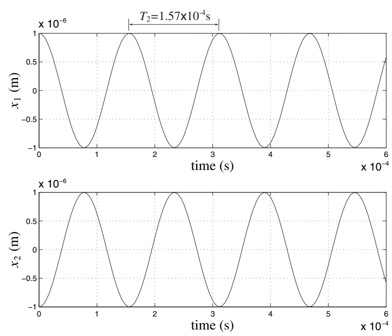

Figure 11.36: Motions of mass A and mass B, $x_1(t)$ and $x_2(t)$ in the second normal mode. In this mode, each mass moves opposite to the other (out of phase) but both oscillate with the same frequency, $\lambda_n^{(2)}$.