11.1 Damped vibrations

In the real world, macroscopic oscillators that are not pumped have motions that decay in time due to dissipation. Our goal is to understand what happens to a harmonic oscillator when we add friction. In particular we are interested in the simplest kind of friction caused by an ideal linear dashpot. What we will find is exponentially decaying solutions that may, or may not have, an oscillatory nature depending on the amount of friction.

Damping

Dashpots are used to absorb energy. One is shown schematically in fig. 11.1. Often springs and dashpots are light in comparison to the machinery to which they are attached so their mass and weight are neglected. They are usually attached with pin joints, ball and socket joints, or other kinds of flexible connections so only forces are transmitted. Because they only have forces at their ends they are 'two-force' bodies so (see section 5.2) the forces at their ends are equal, opposite, and along the line of connection. The most familiar examples are the shock absorbers of a car or the damper for screen doors.

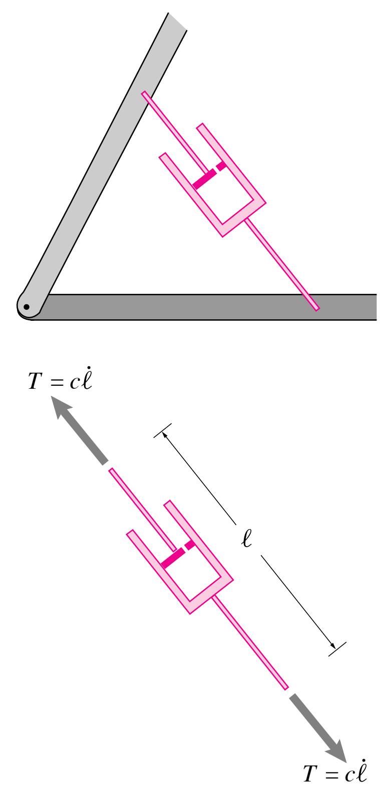


Figure 11.1: **A dashpot.** A dashpot (or damper) is shown here connecting two parts of a mechanism. The tension in the dashpot is proportional to the rate at which it lengthens. The symbol shown represents any device which resists the relative motion of its endpoints. The schematic is supposed to suggest a plunger in a cylinder. For the plunger to move, fluid must leak around the cylinder. This leakage happens for either direction of motion. Thus the damper resists relative motion in either direction; i. e., for $\dot{\ell} > 0$ and $\dot{\ell} < 0$.

The dashpot provides resistance to motion by sucking or pushing air or oil in and out of the cylinder through a small opening. Due to the viscosity of the air or oil, a pressure drop is created across the small opening. Ideally, this viscous resistance causes a force that is exactly proportional to the velocity, giving so-called *linear* damping. The same proportionality, of force and speed, is assumed to hold for both lengthening and shortening. So, the compression force (negative tension) is also proportional to the rate at which the dashpot shortens (negative lengthens).

The tension in the dashpot is usually assumed to be proportional to the rate at which it lengthens, although this approximation is not especially accurate for most dampers one can buy. In a physical dashpot nonlinearities, from the fluid flow and from friction between the piston and the cylinder, are often significant. Also, dashpots that use air as a working fluid may have compressibility that introduces extra springiness to the system.

The defining equation for an ideal linear dashpot is:

$$T = C\,\dot{\ell}$$

where C is the dashpot constant.

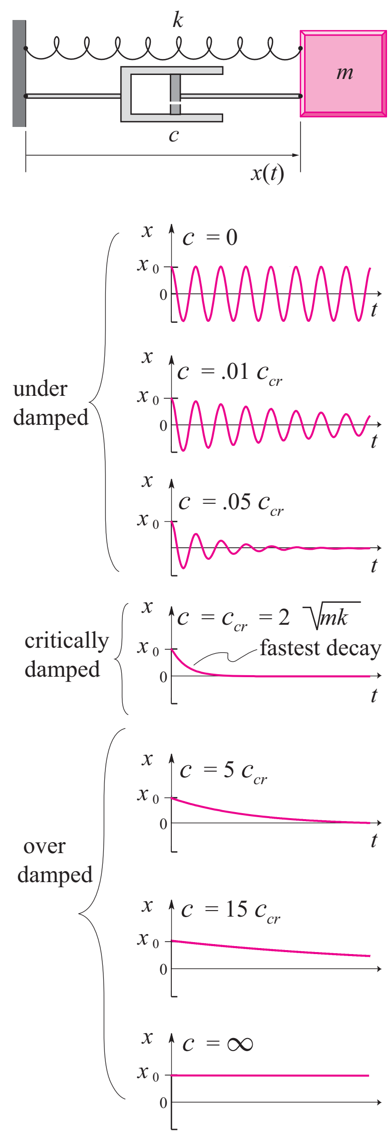


Figure 11.2: The effect of varying the damping with a fixed mass and spring. In all the plots the mass is released from rest at $x = x_0$. In the case of under-damping, oscillations persist for a long time, forever if there is no damping. In the case of over-damping, the dashpot doesn't relax for a long time; it stays locked up forever in the limit of $c \to \infty$. The fastest relaxation occurs for critical damping.

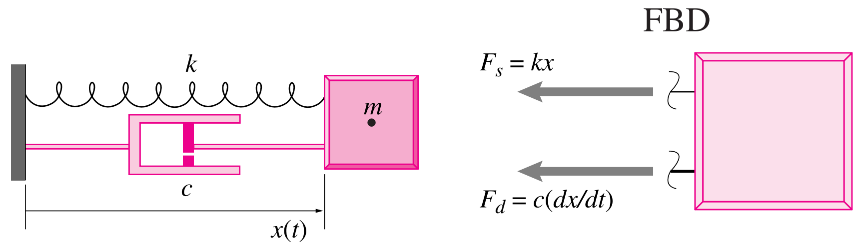


Figure 11.3: A mass spring dashpot system, or damped harmonic oscillator. The free-body diagram of the mass shows the tension in the dashpot.

Damped oscillations

Adding a dashpot in parallel with the spring of a mass-spring system creates a *mass-spring-dashpot* system, or *damped harmonic oscillator*. The system is shown in fig. 11.3. Also in fig. 11.3 is a free-body diagram of the mass. It has two forces acting on it, neglecting gravity:

$F_s \;=\; kx$ is the spring force, assuming a linear spring, and

$F_d \;=\; c\,dx/dt = c\dot{x}$ is the dashpot force assuming a linear dashpot.

The system is a one degree of freedom system because a single coordinate x is sufficient to describe the complete motion of the system. The equation of motion for this system is

$$m\ddot{x} = -F_d - F_s \quad \text{where} \quad \ddot{x} = d^2x/dt^2. \tag{11.1}$$

Assuming a linear spring and a linear dashpot this expression becomes

$$m\ddot{x} + c\dot{x} + kx = 0. \tag{11.2}$$

We have taken care with the signs of the various terms. Make sure you can confidently derive equation 11.2 without introducing sign errors. The analytical solution of the damped-oscillator equation is in box 11.3. Some qualitative features of the damped solutions are shown in fig. 11.2

For given k and m we can think of the damping c as adjustable. A system which has small damping (small c) is *under-damped* and does not come to equilibrium quickly because oscillations last for a long time. A system which has a lot of damping (big c) is *over-damped* and does not come to equilibrium quickly because the dashpot doesn't leak fast enough. A system which is in between, *critically-damped* comes to equilibrium most quickly. The purpose of damping is often to purge motion after a disturbance. If the only design variable available for adjustment is the damping, then the quickest purge is accomplished with critical damping, $c = \sqrt{(4km)}$. In practice, any damping value close to critical is often used, more or less depending on whether a little oscillation is tolerable or not①.

①Stereotypically, the suspension of an overloaded old-fashioned luxury car is underdamped, imagine it bouncing along after a bump. And the suspension of a tight sports car is overdamped.

Summary of equations for the unforced harmonic oscillator

- $\ddot{x} + \frac{k}{m}x = 0$, mass-spring equation
- $\ddot{x} + \lambda^2 x = 0$, harmonic oscillator equation
- $x(t) = A\cos(\lambda t) + B\sin(\lambda t)$, general solution to harmonic oscillator equation
- $x(t) = R\cos(\lambda t - \phi)$, amplitude-phase version of solution to harmonic oscillator solution, $R = \sqrt{A^2 + B^2}, \phi = \tan^{-1}(\frac{B}{A})$ (See box on page 521).
- $\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0$, mass-spring-dashpot equation (see equations 11.3-11.6 for solutions)

Solution of the damped-oscillator equations

Here are some mathematical details you can use for reference. These details are of much lower status than those in box 3.1 on page 203. Only some vibrations experts remember the formulas below in detail.

Even if we make the common assumptions that m, c, and k are all positive, the whole nature of the solution of (11.2) depends on the values of those constants. The three types of solutions are categorized as follows:

- *Under-damped*: $c^2 < 4mk$. In this case the damping is small and oscillations persist forever, though their amplitude diminishes exponentially in time. The general solution for this case is:

  $$x(t) = e^{(-\frac{c}{2m})t}[A\cos(\lambda_d t) + B\sin(\lambda_d t)], \tag{11.3}$$

  where λ_d is the damped natural frequency and is given by

  $$\lambda_d = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2}. \tag{11.4}$$

- *Critically damped*: $c^2 = 4mk$. In this case the damping is at a critical level that separates the cases of under-damped oscillations from the simply decaying motion of the over-damped case. The general solution is:

  $$x(t) = Ae^{(-\frac{c}{2m})t} + Bte^{(-\frac{c}{2m})t}. \tag{11.5}$$

- *Over-damped*: $c^2 > 4mk$. Here there are no oscillations, just a simple return to equilibrium with at most one crossing through the equilibrium position on the way to equilibrium. The general solution in the over-damped case is:

$$x(t) = Ae^{\left(-\frac{c}{2m}+\sqrt{(\frac{c}{2m})^2-\frac{k}{m}}\right)t} + Be^{\left(-\frac{c}{2m}-\sqrt{(\frac{c}{2m})^2-\frac{k}{m}}\right)t}. \qquad (11.6)$$

Measurement of damping. In the under-damped case, the damping constant c can be found by measuring the rate of decay of unforced oscillations using the 'logarithmic decrement'. The logarithmic decrement is the natural logarithm of the ratio of the amplitude of any two successive peaks. The larger the damping, the greater the rate of decay and the bigger the decrement:

$$\text{logarithmic decrement} \equiv D = \ln(\frac{x_n}{x_{n+1}}) \qquad (11.7)$$

where x_n and x_{n+1} are the heights of two successive peaks in fig. 11.4 (also seen on the 2nd and 3rd figures in fig. 11.2 on page 2). Because of the exponential envelope (bounding curve), $x_n = (\text{const.})e^{-(\frac{c}{2m})t_n}$ and $x_{n+1} = (\text{const.})e^{-(\frac{c}{2m})(t_n+T)}$.

$$D = \ln[(e^{-(\frac{c}{2m})t_n})/(e^{-(\frac{c}{2m})(t_n+T)})]$$

Simplifying this expression, we get that

$$D = \frac{cT}{2m}$$

where T is the period of oscillation. Thus, measuring the logarithmic decrement D and the period of oscillation T determines c as

$$c = \frac{2mD}{T}.$$

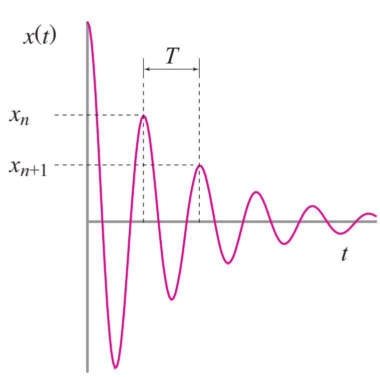


Figure 11.4: Measuring damping using 'logarithmic decrements' (see text).

Solution using complex variables. To find the solutions above it is easiest to use complex variables. Consider this general constant coefficient 2nd order linear equation:

$$A\ddot{x} + B\dot{x} + Cx = 0. \qquad (11.8)$$

We want to find the most general function $x(t)$ that satisfies this equation. We do this by making the *guess* that

$$x(t) = e^{\alpha t}.$$

Plugging this guess into eqn. (11.8) and cancelling the common non-zero factor $e^{\alpha t}$ from each term gives

$$A\alpha^2 + B\alpha + C = 0.$$

For simplicity let's assume this quadratic has two independent roots α_1 and α_2,

$$\alpha_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}.$$

For each root we have a solution to eqn. (11.8). And the solutions can be multiplied by a constant. And the solutions can be added. Thus the general solution to eqn. (11.8) is:

$$x(t) = C_1 e^{\alpha_1 t} + C_2 e^{\alpha_2 t}. \tag{11.9}$$

This seems easy enough. But the α's might be complex. And the C's too. To get a real solution we have to take the real part using the Euler equation $e^{i\lambda t} = \cos\lambda t + i\sin\lambda t)$. By this means, with lots of algebra, we could see that the solution 11.6 actually includes the solutions 11.3 and 11.5 as special cases. This algebra is carried out for the simplest case, no damping, in box 10.3 on page 527.

Numerical solution to the damped oscillator equations

As for the numerical solution of the harmonic oscillator we define $v = \dot{x}$. Thus

$$m\ddot{x} + c\dot{x} + kx = 0 \quad \Rightarrow \quad m\dot{v} + cv + kx = 0.$$

Combining the definition of v with the differential equation we get the set of two coupled first order equations

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= -\frac{k}{m}x - \frac{c}{m}v \end{aligned} \tag{11.10}$$

We can think of this as

$$\dot{z} = f(z)$$

where z is the list of two numbers $z(1) = x$ and $z(2) = v$ so

$$\frac{d}{dt}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\frac{k}{m}z_1 - \frac{c}{m}z_2 \end{bmatrix}.$$

which is standard form for numerical integration.

Energy? Note that for the damped oscillator we cannot use energy conservation to check the solution because energy is constantly lost. We could, however, make a plot of the total energy and make sure that it is an ever decreasing (monotonically decreasing) function of time.

SAMPLE 11.1 Springs in series versus springs in parallel: Two massless springs with spring constants k_1 and k_2 are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 11.5. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

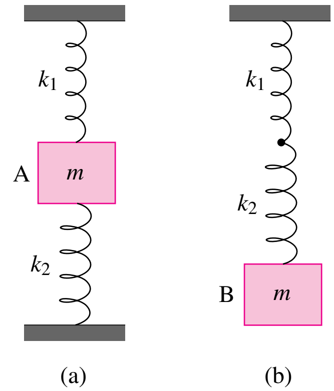


Figure 11.5

Solution Let us pull each mass downwards by a small vertical distance y and then release. Measuring y to be positive downwards, we can derive the equations of motion for each mass by writing the balance of linear momentum for each as follows.

- **Mass A:** The free-body diagram of mass A is shown in Fig. 11.6. As the mass is displaced downwards by y, spring 1 gets stretched by y whereas spring 2 gets compressed by y. Therefore, the forces applied by the two springs, k_1y and k_2y, are in the same direction. The linear momentum balance of mass A in the vertical direction gives:

$$
\begin{aligned}
\sum F &= ma_y \\
\text{or} \qquad -k_1y - k_2y &= m\ddot{y} \\
\text{or} \qquad \ddot{y} + \left(\frac{k_1+k_2}{m}\right)y &= 0.
\end{aligned}
$$

Let the natural frequency of this system be ω_p. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ (see box 3.1 on page 203), we get the natural frequency (λ) of the system:

$$\omega_p = \sqrt{\frac{k_1+k_2}{m}} \tag{11.11}$$

$$\boxed{\omega_p = \sqrt{\frac{k_1+k_2}{m}}}$$

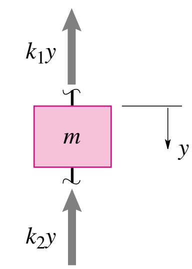


Figure 11.6: Free-body diagram of the mass.

- **Mass B:** The free-body diagram of mass B and the two springs is shown in Fig. 11.7. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{11.12}$$

Now, using the free-body diagram of spring 2 and then writing linear momentum balance we get,

$$
\begin{aligned}
k_2y_2 - k_1y_1 &= \underbrace{m}_{0} \quad a = 0 \\
y_1 &= \frac{k_2}{k_1}y_2 \qquad (11.13)
\end{aligned}
$$

Solving (11.12) and (11.13) we get

$$y_2 = \frac{k_1}{k_1+k_2}y.$$

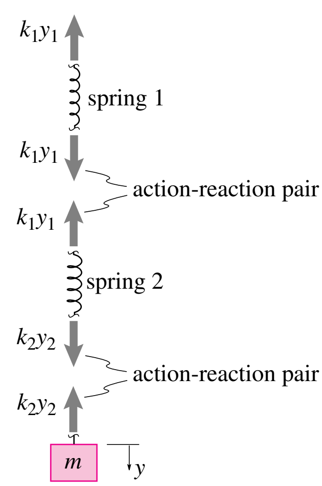


Figure 11.7: Free-body diagrams

Now, linear momentum balance of mass B in the vertical direction gives:

$$\begin{aligned} -k_2 y_2 = m a_y &= m\ddot{y} \\ \text{or} \qquad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1+k_2} y}^{y_2} &= 0 \\ \text{or} \qquad \ddot{y} + \frac{k_1 k_2}{m(k_1+k_2)} y &= 0. \end{aligned} \tag{11.14}$$

Let the natural frequency of this system be denoted by ω_s. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}. \tag{11.15}$$

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}$$

From (11.11) and (11.15)

$$\frac{\omega_p}{\omega_s} = \frac{k_1+k_2}{\sqrt{k_1 k_2}}.$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.,* the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the restoring forces add and in the other they don't. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B. The ratio of the two natural frequencies, ω_p/ω_s as a function of k_1/k_2 is plotted in fig. 11.8. As we can see, ω_p is always higher than ω_s for all k_1/k_2 ratios and its minimum value is twice that of ω_s when $k_1 = k_2$.

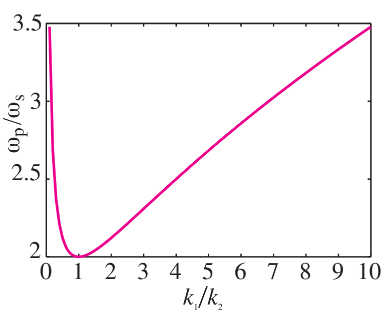


Figure 11.8: The ratio ω_p/ω_s of the two natural frequencies as a function of the spring stiffness ratio k_1/k_2.

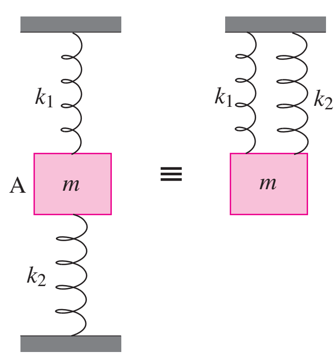


Figure 11.9: The two arrangements of springs are equivalent; they are connected in parallel.

Comments:

1. Although the springs attached to mass A do not visually seem to be in parallel, from a mechanics point of view they are parallel (see fig. 11.9). You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* (if $k_1 \neq k_2$) but the *same force*.
2. When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.,* the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by k_{pe} and k_{se} respectively. By comparing eqns. (11.11) and (11.15) with the expression for natural frequency of a simple spring mass system, we see that

$$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

These expressions can be easily extended for any arbitrary number of springs, say, N springs:

$$k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.$$

SAMPLE 11.2 Figure 11.10 shows two responses obtained from experiments on two spring-mass systems. For each system

1. Find the natural frequency.
2. Find the initial conditions.

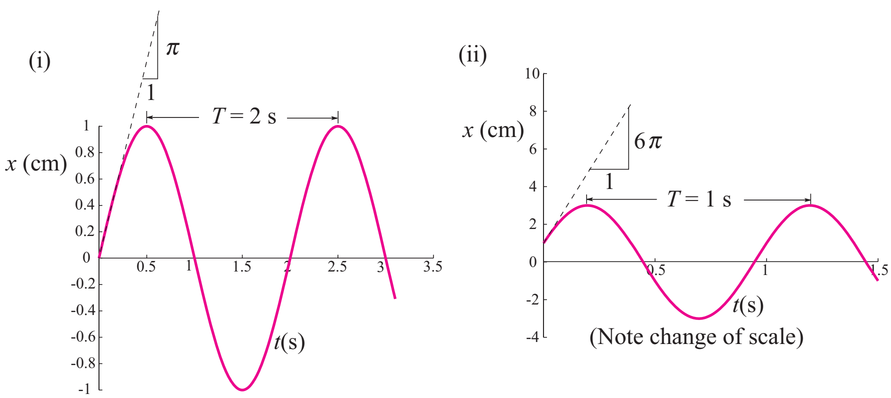


Figure 11.10:

Solution

1. **Natural frequency:** By definition, the natural frequency f is the number of cycles the system completes in one second. From the given responses we see that:

   Case(i): the system completes $\frac{1}{2}$ a cycle in 1 s.

   $$\Rightarrow \qquad f = \frac{1}{2}\ \text{Hz}.$$

   Case(ii): the system completes 1 cycle in 1 s.

   $$\Rightarrow \qquad f = 1\ \text{Hz}.$$

   It is usually hard to measure the fraction of a cycle occurring in a short time. It is easier to first find the time period, *i.e.,* the time taken to complete 1 cycle. (1) Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 11.10 we find that for

   (1) To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

   Case (i):

   $$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2}\ \text{Hz},$$

   Case (ii):

   $$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1\ \text{Hz}$$

   $$\boxed{\text{case (i) } f = \tfrac{1}{2}\ \text{Hz.} \quad \text{case (ii) } f = 1\ \text{Hz.}}$$

2. **Initial conditions:** Now we are to find the displacement and velocity at $t = 0\,$s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:

   Case (i):

   $$x(0) = 0.$$

   Case (ii):

   $$x(0) = 1\,\text{cm}.$$

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

Case (i):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{\pi\,\text{cm}}{1\,\text{s}} = 3.14\,\text{cm/s}.$$

Case (ii):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{6\pi\,\text{cm}}{1\,\text{s}} = 18.85\,\text{cm/s}.$$

Thus the initial conditions are

Case (i):	$x(0) = 0$	$\dot{x}(0) = 3.14\,\text{cm/s}$
Case (ii):	$x(0) = 1\,\text{cm}$	$\dot{x}(0) = 18.85\,\text{cm/s}$

Comments: Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

Case(i): We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5\,\text{s}$ we note that the displacement is maximum, *i.e.,* the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5\,\text{s}) = 1\,\text{cm}$.

Now, from the conservation of energy:

$$\begin{aligned}
\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\,\text{s}})^2 \\
\Rightarrow \quad v_0 &= \sqrt{\frac{k}{m}}\cdot(x_{t=0.5\,\text{s}}) \\
&= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda}\cdot(1\,\text{cm}) \\
&= 2\pi f\cdot(1\,\text{cm}) \\
&= 2\pi\cdot\frac{1}{2}\,\text{Hz}\cdot 1\,\text{cm} \\
&= 3.14\,\text{cm/s}.
\end{aligned}$$

Similar calculations can be done for the second system.

SAMPLE 11.3 A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 11.11. The spring constant $k = 1000\,\mathrm{N/m}$ and a damping rate $c = 50\,\mathrm{N\cdot s/m}$. When the block is at a distance d_0 from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

1. the equation of motion of the block.
2. the position of the block at $t = 2\,\mathrm{s}$.

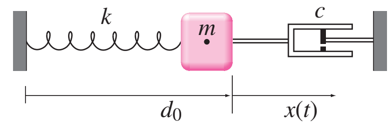


Figure 11.11: Spring-mass dashpot.

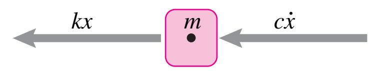


Figure 11.12: Free-body diagram of the mass at an instant t.

Solution

1. Let x be the position of the block, measured positive to the right of the static equilibrium position, at some time t. Let $\dot{x}$ be the corresponding speed. The free-body diagram of the block at the instant t is shown in Figure 11.12.

   Since the motion is only horizontal, we can write the linear momentum balance in the x-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m \underbrace{\ddot{x}}_{a_x}$$

$$\text{or} \quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \tag{11.16}$$

   which is the desired equation of motion of the block.

$$\boxed{\ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0.}$$

2. To find the position and velocity of the block at any time t we need to solve Eqn (11.16). Since the solution depends on the relative values of m, k, and c, we first compute c^2 and compare with the *critical value* $4mk$.

$$\begin{aligned} c^2 &= 2500(\mathrm{N\cdot s/m})^2 \\ \text{and} \quad 4mk &= 4\cdot 10\,\mathrm{kg}\cdot 1000\,\mathrm{N/m} = 4000(\mathrm{N\cdot s/m})^2. \\ \Rightarrow \quad c^2 &< 4mk. \end{aligned}$$

   Therefore, the system is underdamped and we may write the general solution as (see box 11.3 on page 563)

$$x(t) = e^{-\frac{c}{2m}t}\,[A\cos\lambda_D t + B\sin\lambda_D t] \tag{11.17}$$

   where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\mathrm{rad/s}.$$

   Substituting the initial conditions $x(0) = 0.5\,\mathrm{m}$ and $\dot{x}(0) = 0\,\mathrm{m/s}$ in Eqn (11.17) (we need to differentiate Eqn (11.17) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} x(0) &= 0.5\,\mathrm{m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m}\cdot A + \lambda_D \cdot B \\ \Rightarrow \quad B &= \frac{A\,c}{2m\lambda_D} = \frac{(0.5\,\mathrm{m})\cdot(50\,\mathrm{N\cdot s/m})}{2\cdot(10\,\mathrm{kg})\cdot(9.682\,\mathrm{rad/s})} = 0.13\,\mathrm{m}. \end{aligned}$$

   Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\mathrm{s}})t}\,[0.50\cos(9.68\,\mathrm{rad/s}\;t) + 0.13\sin(9.68\,\mathrm{rad/s}\;t)]\ \mathrm{m}.$$

   Substituting $t = 2\,\mathrm{s}$ in the above expression we get $x(2\,\mathrm{s}) = 0.003\,\mathrm{m}$.

$$\boxed{x(2\,\mathrm{s}) = 0.003\,\mathrm{m}.}$$

SAMPLE 11.4 A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\text{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\text{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

1. Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the nth displacement peak.
2. What percent of energy available at the first peak is lost after 5 cycles?

Solution

1. We are given that

$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$

The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.*,

$$\lambda_D t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{2\lambda_D}.$$

At this instant,

$$\begin{aligned} x(t) &= Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} \\ &= Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} \\ &= (0.3\,\text{m})\cdot e^{-0.0314} \\ &= 0.29\,\text{m}. \end{aligned}$$

Let x_n and x_{n+1} be the values of the displacement at the nth and the $(n+1)$th peak, respectively. Since x_n and x_{n+1} are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{11.18}$$

Let t_n be the time at which the nth peak displacement x_n is attained, *i.e.*,

$$x_n = Ae^{-\xi\lambda t_n} \tag{11.19}$$

Since x_{n+1} is the next peak displacement, it must occur at $t = t_n + T_D$ where T_D is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{11.20}$$

From Eqns (11.18), (11.19), and (11.20)

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} \\ &= e^{-2\xi\lambda T_D}. \end{aligned}$$

$$\boxed{\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.}$$

2. Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$\begin{aligned} E_{n+1} &= E_n\, e^{-2\xi\lambda\cdot\frac{2\pi}{\lambda\sqrt{1-\xi^2}}} \\ &= e^{-4\pi\frac{\xi}{\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \\ \Rightarrow \qquad E_{n+1} &= e^{-4\pi\xi}\, E_n. \end{aligned}$$

Applying this equation recursively for $n = n-1, n-2, \dots, 1, 0,$, we get

$$\begin{aligned} E_n &= e^{-4\pi\xi}\cdot E_{n-1} \\ &= e^{-4\pi\xi}\cdot(e^{-4\pi\xi}\cdot E_{n-2}) \\ &= (e^{-4\pi\xi})^3\cdot E_{n-3} \\ &\vdots \\ &= (e^{-4\pi\xi})^n\cdot E_0. \end{aligned}$$

Now we use this equation to find the percentage of energy of the first peak ($n=0$) lost after 5 cycles ($n=5$):

$$\begin{aligned} \Delta E_5 &= \frac{E_0 - E_5}{E_0} \times 100 \\ &= \left(1 - e^{-4\pi\xi\cdot 5}\right) \times 100 \\ &= 71.5\%. \end{aligned}$$

$$\boxed{\Delta E_5 = 71.5\%.}$$

SAMPLE 11.5 A SDOF spring-mass model from given data: The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.
Data:

peak no. n	0	1	2	3	4	5	6
time (s)	0.0000	0.6279	1.2558	1.8837	2.5116	3.1395	3.7674
peak x (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443

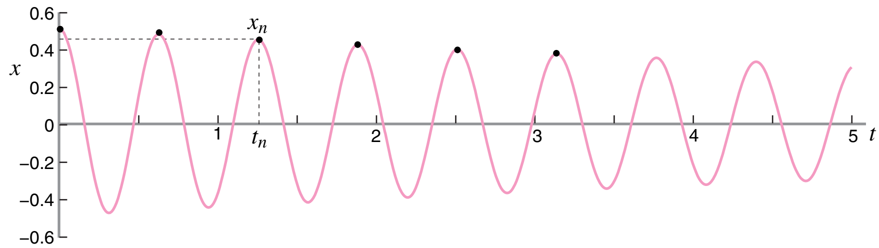


Figure 11.13: Oscillation data from the simulation of a mechanical system

Solution Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period T_D from which we can determine the damped circular frequency λ_D. From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots = 0.6279\,\mathrm{s}$$

Therefore,

$$\begin{aligned} T_D &= 0.6279\,\mathrm{s}. \\ \Rightarrow \quad \lambda_D &= \frac{2\pi}{T_D} = 10\,\mathrm{rad/s}. \end{aligned} \tag{11.21}$$

Now we make a table for the logarithmic decrement of the peak displacements:

peak disp. x_n (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443
$\frac{x_n}{x_{n+1}}$	1.0658	1.0648	1.0647	1.0645	1,0637	1.0627	
$\ln\left(\frac{x_n}{x_{n+1}}\right)$	0.0637	0.0628	0.0627	0.0624	0.0618	0.0608	

Thus, we get several values of the logarithmic decrement $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ ②

② Theoretically, all of these values should be the same, but it is rarely the case in practice. When x_n's are measured from an experimental setup, the values of D may vary even more.

We take the average value of D:

$$D = \bar{D} = 0.0624. \tag{11.22}$$

Let the equivalent single degree of freedom model have mass m, spring stiffness k, and damping rate c. Then

$$\lambda_D = \lambda\sqrt{1-\xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (11.21),

$$\frac{k}{m} = \lambda^2 = 100(\text{ rad/s})^2, \tag{11.23}$$

and, since $D = \frac{cT_D}{2m}$, from Eqn (11.22) we get

$$\begin{aligned} c &= \frac{2mD}{T_D} \\ &= \frac{2m(0.0624)}{0.6279\,\text{s}} \\ &= (0.1988\frac{1}{\text{s}})m. \end{aligned} \tag{11.24}$$

Equations (11.23) and (11.24) have three unknowns: k, m, and c. We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass $m = 5\,\text{kg}$. Then

$$\begin{aligned} k &= (100\frac{1}{\text{s}^2})\cdot(5\,\text{kg}) \\ &= 500\,\text{N/ m}, \end{aligned}$$

and

$$\begin{aligned} c &= (0.1988\frac{1}{\text{s}})\cdot(5\,\text{kg}) \\ &= 0.99\,\text{N}\cdot\text{s/ m}. \end{aligned}$$

$$\boxed{\begin{aligned} m &= 5\,\text{kg}, \\ k &= 500\,\text{N/ m}, \\ c &= 0.99\,\text{N}\cdot\text{s/ m}. \end{aligned}}$$

Of course, we could choose many other sets of values for m, k, and c which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine k and c uniquely from the given response.

11.1 Free Vibration of a SDOF System

Preparatory Problems

11.1.1 A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? Neglect gravity and friction. Answer in terms of some or all of m, k, and d. *

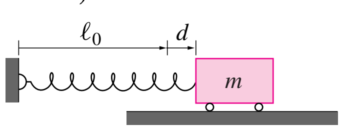
11.1.2 A spring k with rest length ℓ_0 is attached to a mass m which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released from rest with the spring stretched a distance d. Measure the mass position x relative to the wall.

a) What is the acceleration of the mass just after release?

b) Find a differential equation which describes the horizontal motion x of the mass.

c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through $x = \ell_0$ (the position where the spring is relaxed)?



Problem 11.1.2

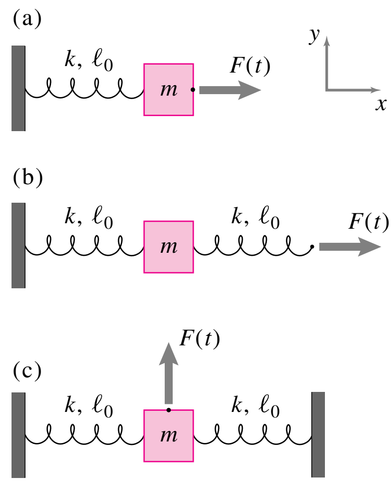
11.1.3 Reconsider the spring-mass system from problem 11.1.2.

a) Find the potential and kinetic energy of the spring mass system as functions of time.

b) Assigning numerical values to the various variables, use a computer to make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?

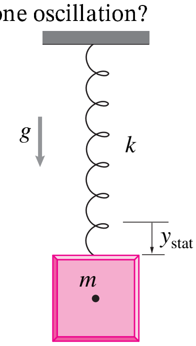
c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

11.1.4 For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.) *



Problem 11.1.4

11.1.5 A mass-spring oscillator hangs vertically under gravity. The mass rests in static equilibrium by stretching the spring by an amount $y_{\text{static}} = 0.025\,\text{m}$. Take your favorite value of g and find the natural frequency of the oscillator. How much time does the oscillator take to complete one oscillation?



Problem 11.1.5

11.1.6 A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient b to its right and a spring with constant k and rest length ℓ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance x from its position where the spring is unstretched. There is gravity.

a) Draw a free-body diagram of the mass at the instant of interest.

b) Derive the equation of motion of the mass. *

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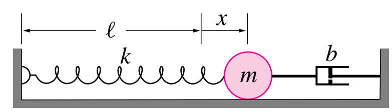

Problem 11.1.6

11.1.7 A mass-spring-dashpot system has $m = 1\,\text{kg}$, $k = 10\,\text{kN/m}$, and $c = 5\,\text{kg/s}$. Find the natural frequency, damping ratio, and the damped frequency of the system. Specify whether the system is underdamped, critically damped or overdamped. *

11.1.8 The natural frequency, λ_n, of a SDOF system is 150 rad/s. Find the minimum damping (ξ) that the system must have for the resonant frequency to occur below 100 rad/s? *

More-Involved Problems

11.1.9 The equation of motion of an unforced mass-spring-dashpot system is, $m\ddot{x} + c\dot{x} + kx = 0$, as discussed in the text. For a system with $m = 0.4\,\text{kg}, c = 10\,\text{kg/s}$, and $k = 5\,\text{N/m}$,

a) Find whether the system is underdamped, critically damped, or overdamped. *

b) Sketch a typical solution of the system.

c) Make an accurate plot of the response of the system (displacement vs time) for the initial conditions $x(0) = 0.1\,\text{m}$ and $\dot{x}(0) = 0$.

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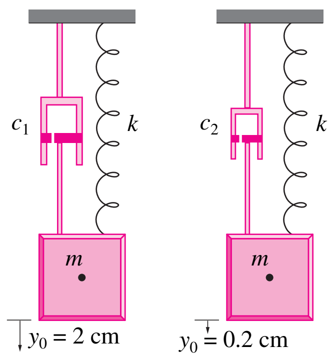
11.1.10 You are given to design a SDOF damped oscillator that should show no oscillations at all when disturbed from the equilibrium (*i.e.,* it should return to equilibrium without overshooting on the other side). You are given a spring with stiffness $k = 500\,\mathrm{N/m}$, a hydraulic damper with $c = 10\,\mathrm{kg/s}$, and you have a choice of masses from $m = 1\,\mathrm{kg}$ to $m = 10\,\mathrm{kg}$ in increments of half kg. Find the appropriate mass.

11.1.11 Two SDOF oscillators with the same k and m but different c's are hung from the ceiling as shown in the figure. The one on the left is pulled down 2 cm and let go. The other is pulled down by 0.2 cm and let go. Which oscillator undergoes more oscillations before reaching the steady state. Find the steady state displacement of each mass.



Problem 11.1.11

11.1.12 Experiments conducted on free oscillations of a damped oscillator reveal that the amplitude of oscillations drops to 25% of its peak value in just 3 periods of oscillations. The period of oscillation is measured to be 0.6 s and the mass of the system is known to be 1.2 kg. Find the damping coefficient and the spring stiffness of the system. *

11.1.13 You are required to design a mass-spring-dashpot system that, if disturbed, returns to its equilibrium position the quickest. You are given a mass, $m = 1\,\mathrm{kg}$, and a damper with $c = 10\,\mathrm{kg/s}$. What should be the stiffness of the spring? Your solution needs to include your definition of "'quickest". *