11.1.1 A mass m is connected to a spring k and released from rest with the spring stretched a distance d from its static equilibrium position. It then oscillates back and forth repeatedly crossing the equilibrium. How much time passes from release until the mass moves through the equilibrium position for the second time? Neglect gravity and friction. Answer in terms of some or all of m, k, and d.

Answer:
Time span $= 3\pi\sqrt{m/k}/2$

1

10.1.1

From equation of motion

$$-K(x+x_0)+mg = m\ddot{x} \longrightarrow (1)$$

at equilibrium, the upward force is equal to the downward force

$$-Kx_0+mg=0$$

$$mg = Kx_0$$

substitute in (1)

$$m\ddot{x} = -Kx$$

$$\ddot{x}+\left(\sqrt{\frac{K}{m}}\right)^2 x = 0$$

on solving the differential equation

$$x = C.\cos\left(\sqrt{\frac{K}{m}}\cdot t+\phi\right)$$

2

At, $t=0$, $\dot{x}_0=0$, $x_0=d$

$$x = C\cdot\cos\left(\sqrt{\frac{K}{m}}t+\phi\right)$$

$$\dot{x} = -C\cdot\sqrt{\frac{K}{m}}\sin\left(\sqrt{\frac{K}{m}}t+\phi\right)$$

$$x_0 = C\cdot\cos\phi$$

$$\dot{x}_0 = 0 = -C\cdot\sqrt{\frac{K}{m}}\cdot\sin\phi$$

this means that

$$\sin\phi = 0$$

hence $\phi = 0$

$$C = d$$

$$\therefore \quad x = d\cos\left(\sqrt{\frac{K}{m}}t\right)$$

The time required by the mass to cross the equilibrium position for the second time is 3/4 the period of oscillation

$$t = \frac{3}{4}T = \frac{\frac{3}{4}\times 2\pi}{\sqrt{\frac{K}{m}}}$$

$$t = \frac{3\pi}{2}\sqrt{\frac{m}{K}}$$

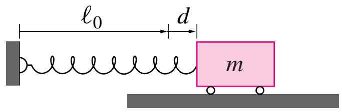
11.1.2 A spring k with rest length ℓ_0 is attached to a mass m which slides frictionlessly on a horizontal ground as shown. At time $t = 0$ the mass is released from rest with the spring stretched a distance d. Measure the mass position x relative to the wall.

a) What is the acceleration of the mass just after release?

b) Find a differential equation which describes the horizontal motion x of the mass.

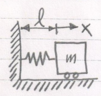
c) What is the position of the mass at an arbitrary time t?

d) What is the speed of the mass when it passes through $x = \ell_0$ (the position where the spring is relaxed)?

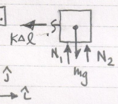


Problem 11.2

\# 3.76 TAM 203 HW 5A WEISENFELD



mass m, spring constant k
relaxed length ℓ, at $t=0$, $x=d$

A. 

FBD at $t=0^+$ (a split second after being released, $\underline{a} = ?$

$\Sigma \underline{F} = m\underline{a}$ $\quad k\Delta\ell = kx$

$-kx\hat{\imath} + N_1\hat{\jmath} + N_2\hat{\jmath} - mg\hat{\jmath} = m\underline{a}$ dot by $\hat{\imath}$

at $t=0^+$ $x=d$ $\quad \boxed{\underline{a} = -\frac{kd}{m}\hat{\imath}}$

B. differential equation for $x(t)$ dot LMB by $\hat{\imath}$

$-kx = m\ddot{x}$ where $\underline{a} = \ddot{x}\hat{\imath}$

$\boxed{\ddot{x} + \frac{k}{m}x = 0}$

C. solution to differential equation gives position at arbitrary time t, general form of solution

$x(t) = A\cos\omega t + B\sin\omega t$, $\omega = \sqrt{\frac{k}{m}}$

$\boxed{x(t) = d\cos\left(\sqrt{\frac{k}{m}}\,t\right)}$ from initial conditions $x(0) = A = d$, $\dot{x}(0) = B = 0$

D. $v(t) = \dot{x}(t) = -d\sqrt{\frac{k}{m}}\sin\left(\sqrt{\frac{k}{m}}\,t\right)$

since $x=0$ is equilibrium position, mass has no potential energy and all kinetic energy when $x=0$, so maximum of $v(t)$

$\boxed{v(x=0) = d\sqrt{\frac{k}{m}}}$

E. when does $E_k = E_p$? $\quad E_k = \frac{1}{2}m[\dot{x}(t)]^2$

Plugging in $x(t)$ & $\dot{x}(t)$: $\quad E_p = \frac{1}{2}k[x(t)]^2$

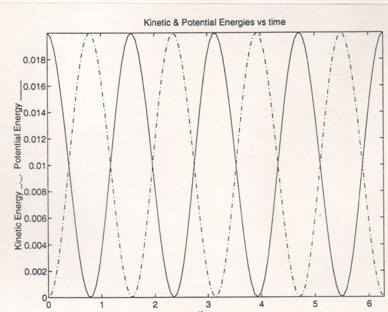
$\frac{1}{2}m\left[\frac{d^2k}{m}\sin^2\left(\sqrt{\frac{k}{m}}\,t\right)\right] = \frac{1}{2}k\left[d^2\cos^2\left(\sqrt{\frac{k}{m}}\,t\right)\right]$

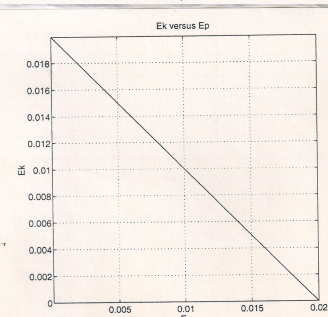
$\sqrt{\frac{k}{m}}\,t = \frac{\pi}{4}$ when $\sin^2\left(\sqrt{\frac{k}{m}}\,t\right) = \cos^2\left(\sqrt{\frac{k}{m}}\,t\right)$ $\quad \boxed{t = \frac{\pi}{4}\sqrt{\frac{m}{k}}}$

F. phase difference between $\sin^2(t)$ & $\cos^2(t)$ = $\boxed{\pi}$

\# 3.76

```
% mass in kg
m = 1;

% spring constant in N/m
k = 4;

% initial displacement in m
d = 0.1;

% natural frequency in 1/sec
omega = sqrt(k/m);

% enough time for 2 periods
t=linspace(0,2*2*pi/omega);

% compute displacement
x = d*cos(omega*t);

% compute velocity
v = -d*omega*sin(omega*t);

% compute kinetic energy
Ek = m*(v.*v)/2;

% compute potential energy
Ep = k*(x.*x)/2;

% plot Ek and Ep versus time
plot(t,Ek,'-.',t,Ep,'-');
ylabel('Kinetic Energy _._.  Potential Energy ____');
xlabel('time, sec');
title('Kinetic & Potential Energies vs time');
axis([min(t) max(t) min(Ek) max(Ek)]);

%plot Ek versus Ep
plot(Ep,Ek);
axis([min(Ep) max(Ep) min(Ek) max(Ek)]);
axis('square');
ylabel('Ek');
xlabel('Ep');
title('Ek versus Ep');
grid;
```

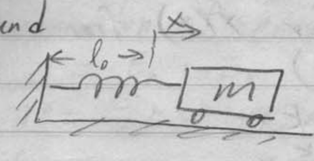




Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

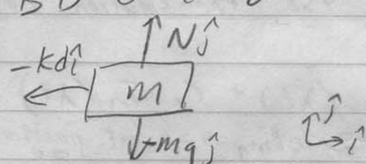
9.49

A spring with rest length l_0 attached to mass m slides frictionless on a horizontal ground



At $t=0$ mass is released with $v_0 = 0$ and spring stretch a distance d.

FBD @ $t=0$



$N\hat{\jmath} = mg\hat{\jmath}$

a) Find a of mass just after release

$\{ m\ddot{x}\hat{\imath} = -kd\hat{\imath} \}$

$\{\}\cdot\hat{\imath}$: $m\ddot{x} = -kd$

$$\boxed{\ddot{x} = \frac{-kd}{m}}$$

b) Find a differential eqn that describes horizontal motion of mass

$\{ m\ddot{x}\hat{\imath} = -kx\hat{\imath} \}$

$\{\}\cdot\hat{\imath}$: $\boxed{m\ddot{x} = -kx}$

9.49 continued

c) Find position of mass at arbitrary time t.

- we will solve for $x(t)$ from the eqn we find in (b)

$$m\ddot{x} = -kx$$

$$\ddot{x} = -\frac{k}{m}x$$

$$\ddot{x} + \frac{k}{m}x = 0$$

let $\lambda^2 = \frac{k}{m}$

$$\ddot{x} + \lambda^2 x = 0$$

according to Page 438, the solution to the above eqn is

$$x = C_1\cos(\lambda t) + C_2\sin(\lambda t)$$

apply initial conditions

At $t=0$, $x=d$ → letting $x=0$ at position spring relaxes

$$x(0) = C_1\cos(\lambda\cdot 0) + C_2\cancel{\sin(\lambda\cdot 0)}^{\,0}$$

$$d = C_1$$

At $t=0$, $v=0$

$$\dot{x} = -\lambda C_1\sin(\lambda t) + \lambda C_2\cos(\lambda t)$$

$$\dot{x}(0) = -\lambda C_1\cancel{\sin(\lambda\cdot 0)}^{\,0} + \lambda C_2\cos(\lambda\cdot 0)$$

$$0 = \lambda C_2$$

$$C_2 = 0$$

$$\boxed{x = d\cos\left(\sqrt{\tfrac{k}{m}}\,t\right)}$$

9.49 continued

d) Find speed of mass when it passes through the position where spring is relaxed

conservation of energy

$E_{total} = E_p + E_k$

$\frac{1}{2} k d^2 = \frac{1}{2} k x^2 + \frac{1}{2} m v^2$

for $x = 0$

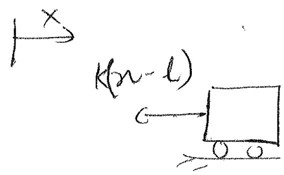
$\cancel{\frac{1}{2}} k d^2 = \cancel{\frac{1}{2}} m v^2$

$\rightarrow \boxed{v = \pm\sqrt{\frac{k}{m}}\, d}$

9.49 — additional note

If you assume x to be from the wall as stated in the problem, then

FBD



$$\sum F_{ext} = \dot{L}$$

$$-k\cdot(x-\ell_0) = m\ddot{x}$$

Part b:

$$\boxed{m\ddot{x} + kx = k\ell_0}$$

Part c:

$$\boxed{x = \ell_0 + d\cos\left(\sqrt{\frac{k}{m}}\, t\right)}$$

Part a, d: will have the same answer as above!

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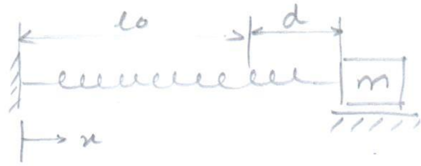
10.1.2



a) Acceleration just after release

$$= -\frac{kd}{m} \quad \text{towards wall}$$

b) $-k(x-l_0) = m\ddot{x}$

Initial conditions at $t=0$

$\dot{x} = 0$ at $t=0$

$x = l+d$ at $t=0$

c) position at arbitrary time

$$-k(x-l_0) = m\ddot{x}$$

put $y = x - l_0$

$\ddot{y} = \ddot{x}$

$m\ddot{y} = -ky$

then the solution of the differential equation

$$y = C\cos\left(\sqrt{\frac{k}{m}} + \phi\right)$$

from the initial conditions

$$y = d\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

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$$x - l_0 = d\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

$$x = l_0 + d\cos\left(\sqrt{\frac{k}{m}}\,t\right)$$

d)

$$\dot{x} = -d\sqrt{\frac{k}{m}}\,\sin\left(\sqrt{\frac{k}{m}}\cdot t\right) \longrightarrow (1)$$

When the mass is at the point where $x = l_0$ it has performed $\frac{1}{4}$ of the oscillation hence the time t is given by

$$t = \frac{\frac{2\pi}{4}}{\sqrt{\frac{k}{m}}} = \frac{\frac{\pi}{2}}{\sqrt{\frac{k}{m}}}$$

on substubing the value in (1)

we get.

$$\dot{x}_{l_0} = -d\sqrt{\frac{k}{m}}$$

11.1.3 Reconsider the spring-mass system from problem 11.1.2.

a) Find the potential and kinetic energy of the spring mass system as functions of time.

b) Assigning numerical values to the various variables, use a computer to make a plot of the potential and kinetic energy as a function of time for several periods of oscillation. Are the potential and kinetic energy ever equal at the same time? If so, at what position $x(t)$?

c) Make a plot of kinetic energy versus potential energy. What is the phase relationship between the kinetic and potential energy?

5

10.1.3

a) Potential energy $= \frac{1}{2} k (x - l_0)^2$

Kinetic energy $= \frac{1}{2} m \dot{x}^2$

Where $(x - l_0) = d \cos\left(\sqrt{\frac{k}{m}}\, t\right)$

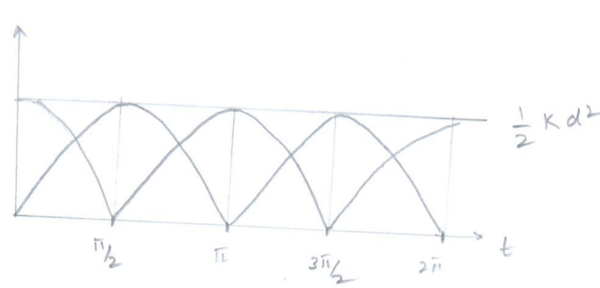
$$\dot{x} = -d \sqrt{\frac{k}{m}} \sin\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$V = \frac{1}{2} k d^2 \cos^2\left(\sqrt{\frac{k}{m}}\, t\right)$$

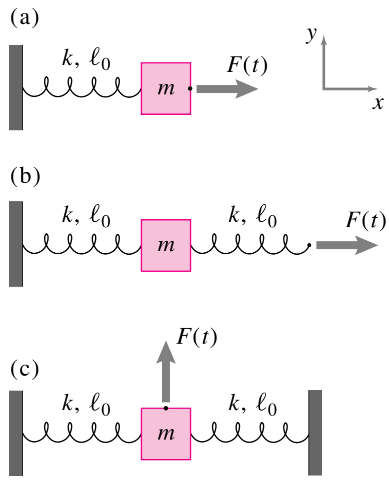
$$K_i = \frac{1}{2} d^2 k \sin^2\left(\sqrt{\frac{k}{m}}\, t\right)$$

$$\boxed{V + K = \frac{1}{2} k d^2}$$

c)



11.1.4 For the three spring-mass systems shown in the figure, find the equation of motion of the mass in each case. All springs are massless and are shown in their relaxed states. Ignore gravity. (In problem (c) assume vertical motion.)



Problem 11.4

Answer:
(a) $m\ddot{x}+kx = F(t)$, (b) $m\ddot{x}+kx = F(t)$, and (c) $m\ddot{y} + 2ky - 2k\ell_0\frac{y}{\sqrt{\ell_0^2+y^2}} = F(t)$

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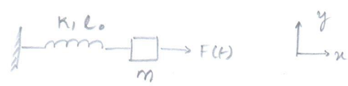
10.1.4

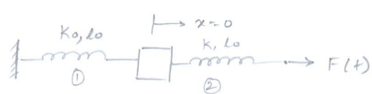
a)



$m\ddot{x} = -Kx + F(t)$

$Kx + m\ddot{x} = F(t)$

b)

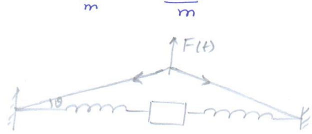


$-Kx_1 + K(x_2 - x_1) = m\ddot{x}_1$

$K(x_2 - x_1) - F(t) = \cancel{m_s \ddot{x}_2}\ 0$

$\ddot{x} + \frac{K}{m}x = \frac{F(t)}{m}$

c)



$-2K(l_0/\cos\theta - l_0)\sin\theta + F(t) = m l_0 \frac{d^2}{dt^2}(\sin\theta)$

$-2K\left(\frac{l_0}{\cos\theta} - l_0\right)\sin\theta + F(t) = m l_0\left[-\sin\theta\,\dot{\theta}^2 + \cos\theta\cdot\ddot{\theta}\right]$

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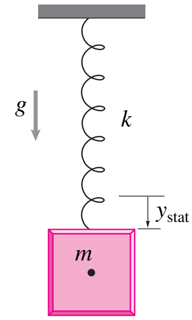
if $\theta \to 0$ then $\sin\theta = \theta$

$\cos\theta = 1$

$$F(t) = ml_0(\ddot{\theta} - \theta\dot{\theta}^2)$$

11.1.5 A mass-spring oscillator hangs vertically under gravity. The mass rests in static equilibrium by stretching the spring by an amount $y_{static} = 0.025\,\mathrm{m}$. Take your favorite value of g and find the natural frequency of the oscillator. How much time does the oscillator take to complete one oscillation?



Problem 11.5

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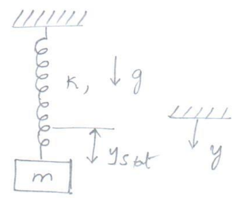
10.1.5



$$m\ddot{y} = mg - Ky$$

let $x = y - y_{stat}$

$$m\ddot{x} = mg - K(x + y_{stat})$$

$$m\ddot{x} + Kx = 0$$

$$\omega = \sqrt{\frac{K}{m}}$$

since from equilibrium condition

$$mg = K y_{stat}$$

Period of oscillation

$$T = \frac{2\pi}{\omega}$$

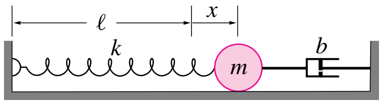
$$T = 2\pi\sqrt{\frac{m}{K}}$$

11.1.6 A mass moves on a frictionless surface. It is connected to a dashpot with damping coefficient b to its right and a spring with constant k and rest length ℓ to its left. At the instant of interest, the mass is moving to the right and the spring is stretched a distance x from its position where the spring is unstretched. There is gravity.

a) Draw a free-body diagram of the mass at the instant of interest.

b) Derive the equation of motion of the mass.

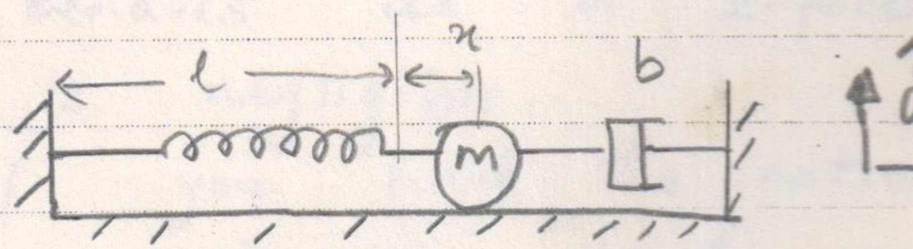
.



Problem 11.6

Answer:
LHS of Linear Momentum Balance:
$\sum \vec{F} = -(kx + b\dot{x})\hat{\imath} + (N - mg)\hat{\jmath}$.

(2.8)



(a) FBD of the mass:

F_s = spring force = (kx)

F_D = force in dashpot = (

$\underline{W}$ = weight = $(mg)(-\hat{j}$

$\underline{N}$ = normal reaction =

(b) LHS of LMB $\sum \underline{F} =$

11.1.7 A mass-spring-dashpot system has $m = 1\,\text{kg}$, $k = 10\,\text{kN/m}$, and $c = 5\,\text{kg/s}$. Find the natural frequency, damping ratio, and the damped frequency of the system. Specify whether the system is underdamped, critically damped or overdamped.

Answer:
$\omega_n = 100\,\text{rad/s}$, $\xi = 0.025$, System is underdamped.

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10.1.7

$$m\ddot{x} + b\dot{x} + Kx = 0 \longrightarrow (1)$$

The solution is of the form

$$x_c = Be^{st}$$

substitute in (1)

$$(ms^2 + bs + K)\,Be^{st} = 0$$

$$ms^2 + bs + K = 0$$

the roots are

$$S_{1,2} = \frac{1}{2m}\left(-b \pm \sqrt{b^2 - 4mK}\right)$$

since there are two roots the complementary solution is

$$x_c = B_1 e^{S_1 t} + B_2 e^{S_2 t}$$

Now we rewrite the equation as

$$\frac{K}{m} = \omega_n^2 \quad , \quad \frac{b}{m} = 2\xi\omega_n$$

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$$S_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2 - 1}\ \omega_n$$

Now the Natural frequency

$$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{10^4}{1}} = 100 \text{ rad/sec}$$

Damping ratio

$$\xi = \frac{b}{2\sqrt{km}}$$

$$\xi = \frac{5}{\cancel{2}\sqrt{4\times10^4}} = 2.5\times10^{-2}$$

$$\xi < 1$$

The system is underdamped

Damped frequency,

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$\omega_d = 99.96 \text{ rad/sec}$$

11.1.8 The natural frequency, λ_n, of a SDOF system is 150 rad/s. Find the minimum damping (ξ) that the system must have for the resonant frequency to occur below 100 rad/s?

Answer:
$\xi \geq 0.74$

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10·1·8

$\omega = 150 \text{ rad/s}$

$\omega_d = \omega\sqrt{1-\xi^2}$

$100 \geq 150\sqrt{1-\xi^2}$

$0.6666 \geq \sqrt{1-\xi^2}$

$0.4444 \geq 1-\xi^2$

$-0.5556 \geq -\xi^2$

$0.5556 \leq \xi^2$

$\xi \geq 0.7453$

Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*

11.1.9 The equation of motion of an unforced mass-spring-dashpot system is, $m\ddot{x} + c\dot{x} + kx = 0$, as discussed in the text. For a system with $m = 0.4\,\mathrm{kg}, c = 10\,\mathrm{kg/\,s}$, and $k = 5\,\mathrm{N/m}$,

a) Find whether the system is underdamped, critically damped, or overdamped.

b) Sketch a typical solution of the system.

c) Make an accurate plot of the response of the system (displacement vs time) for the initial conditions $x(0) = 0.1\,\mathrm{m}$ and $\dot{x}(0) = 0$.

.

Answer:
Overdamped

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10.1.9

$$m\ddot{x} + c\dot{x} + Kx = 0$$

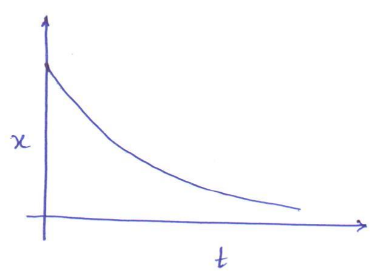
$$S_{1,2} = -\xi\omega_n \pm \sqrt{\xi^2 - 1}\cdot\omega_n$$

a) $$\xi = \frac{c}{2\sqrt{Km}} = \frac{10}{2\sqrt{0.4\times 5}} = 3.535$$

$$\xi > 1.$$

The system is overdamped.

b)



c) The equation of motion is given by.

$$x_c = A e^{-\xi\omega_n t}\cdot \sin(\omega_d t + \psi)$$

$$\dot{x}_c = A\left[-\xi\omega_n e^{-\xi\omega_n t}\sin(\omega_d t + \psi) + e^{-\xi\omega_n t}\cdot\omega_d\cdot\cos(\omega_d t + \psi)\right.$$

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c) The equation of motion is given by.

$$x = B_1 e^{s_1 t} + B_2 e^{s_2 t} \longrightarrow (1)$$

$$\dot{x} = B_1 s_1 e^{s_1 t} + B_2 s_2 e^{s_2 t} \longrightarrow (2)$$

$$s_1 = -\xi\omega_n + \sqrt{\xi^2 - 1}\cdot\omega_n$$

$$s_1 = -\xi\omega_n - \sqrt{\xi^2 - 1}\ \omega_n$$

Now, put $\xi = 3.535$

then,

$$s_1 = -3.535\,\omega_n + 3.39\,\omega_n = -0.144\,\omega_n$$

$$s_2 = -3.535\,\omega_n - 3.39\,\omega_n = -6.925\,\omega_n$$

Now, $\omega_n = \sqrt{\frac{K}{m}} = \sqrt{\frac{5}{0.4}} = 3.535$ rad/s

then,

$$s_1 = -0.5$$

$$s_2 = -24.5$$

From equation (1) and (2) at $t=0$

$$x_0 = B_1 + B_2$$

$$\dot{x}_0 = B_1 s_1 + B_2 s_2$$

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From the initial condition we know that

$x_0 = 0.1\ m$

$\dot{x}_0 = 0$

then

$B_1 S_1 = -B_2 S_2$

$B_1 = -B_2 \dfrac{S_2}{S_1}$

$B_1 = -49\ B_2$

$B_1 + B_2 = 0.1$

$-49 B_2 + B_2 = 0.1.$

$-48 B_2 = 0.1$

$B_2 = -2.08 \times 10^{-3}$

$B_1 = 0.102$

We get,

$x = 0.102\ e^{-0.5t} - 2.08 \times 10^{-3}\ e^{-24.5t}$

MARIGOT FACKENTHAL
MKF47

PROBLEM 11.1.9

GIVEN: EOM of unforced mass-spring-damper system: $m\ddot{x} + c\dot{x} + kx = 0$

where $m = 0.4\,kg$ $\quad c = 10\,kg/s$ $\quad k = 5\,N/m$

FIND:

a) underdamped, critically damped, or overdamped?

b) sketch of typical solution

c) accurate plot of x vs. t
using I.C.'s $x(0) = 0.1\,m$
$\dot{x}(0) = 0$

SOLUTION:

a) what is the damping behavior?
let's write the characteristic equation for this system:

$$mr^2 + cr + k = 0$$

$\hookrightarrow$ roots: $\dfrac{-c \pm \sqrt{c^2 - 4mk}}{2m}$

recall that: repeated roots $\rightarrow$ critical
real, separate roots $\rightarrow$ overdamped
complex roots $\rightarrow$ underdamped

(see section 11.1 "Damped Vibrations" in book for more review)

PG 1

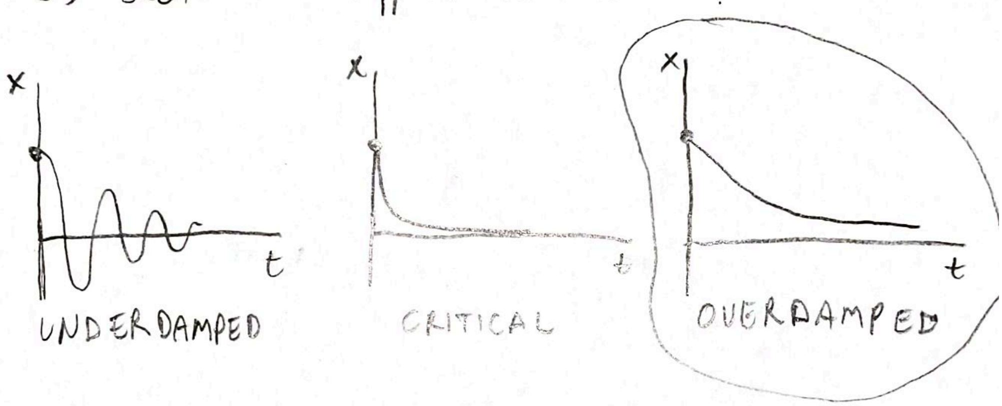
what do we have in our case?
plug in parameters:

$m = 0.4\,\text{kg}$ $\quad c = 10\,\text{kg/s}$ $\quad k = 5\,\text{N/m}$

roots: $$\frac{-10\,\text{kg/s} \pm \sqrt{(10\,\text{kg/s})^2 - 4(0.4\,\text{kg})(5\,\text{N/m})}}{2(0.4\,\text{kg})}$$

$$= \underbrace{-0.5104\,\tfrac{1}{s},\ -24.49\,\tfrac{1}{s}}_{\text{real, separate roots}} \rightarrow \boxed{\text{OVERDAMPED}}$$

b) sketch of typical solution...



still approaches setpoint, but not quickly.

c) accurate plot of x vs. t

↳ cont.

PG 2

we now have a solution for $x(t)$:

$$\boxed{x(t) = 0.1022e^{(-0.51\frac{1}{s})t} - 0.0022e^{(-24\frac{1}{s})t}}$$

we now plot this in MATLAB.

→ see FIGURE 1 on PG 8.

② NUMERICAL SOLUTION

let's go ALL THE WAY BACK to the ODE:

$$m\ddot{x} + c\dot{x} + kx = 0$$

We want to be able to give this equation to a numerical integrator, so let's put it into <u>first-order</u> or <u>"state space"</u> form:

$$\dot{x} = v$$

$$\dot{v} = (-cv - kx)\frac{1}{m}$$

once our numerical integrator returns our array of position estimations, we plot!

→ see FIGURE 2 on PG 8.

we now have a solution for $x(t)$:

$$\boxed{x(t) = 0.1022e^{(-0.51\bar{1})t} - 0.0022e^{(-24\frac{1}{3})t}}$$

we now plot this in MATLAB.

$\longrightarrow$ see FIGURE 1 on PG 8.

② NUMERICAL SOLUTION

let's go ALL THE WAY BACK to the ODE:

$$m\ddot{x} + c\dot{x} + kx = 0$$

* We want to be able to give this equation to a numerical integrator, so let's put it into first-order or "state space" form:

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Once our numerical integrator returns our array of position estimations, we plot!

$\longrightarrow$ see FIGURE 2 on PG 8.

my functions:

* numerical integrator: myEulerSolver

equations of motion, state space: eqn

PG 4

```
% p11_1_9.m

% DYNAMICS HW 3
% PROBLEM 12
% RUINA-PRATAP 11.1.9

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% ALL UNITS ARE SI

clear; clf; close all; clc;
set(0,'DefaultAxesFontSize', 10)


% SETUP
 ********************************************************************

% parameters [CHANGE THESE]
p.m = 0.4;
p.c = 10;
p.k = 5;

% timing
t0 = 0;  tf = 15; n = 5000;
tarray = linspace(t0,tf,n)';

% initial conditions [CHANGE THESE]
rx0 = 0.1;
vx0 = 0;
z0 = [rx0; vx0];  % use semicolons to make it a 4x1
% also, make sure the entries are in the standard order (ex. x1, x2,
 v1, v2)


% ANALYTICAL SOLUTION
 ******************************************************

% analytical solution coefficients
% calculated in written work
C1 = 0.1022;
C2 = -0.0022;
r1 = -0.51;
r2 = -24;

rxAna = C1*exp(r1*tarray) + C2*exp(r1*tarray);


% NUMERICAL SOLUTION
 *******************************************************

[tarray, zarray] = myEulerSolver(@eqn, tarray, z0, p);
```
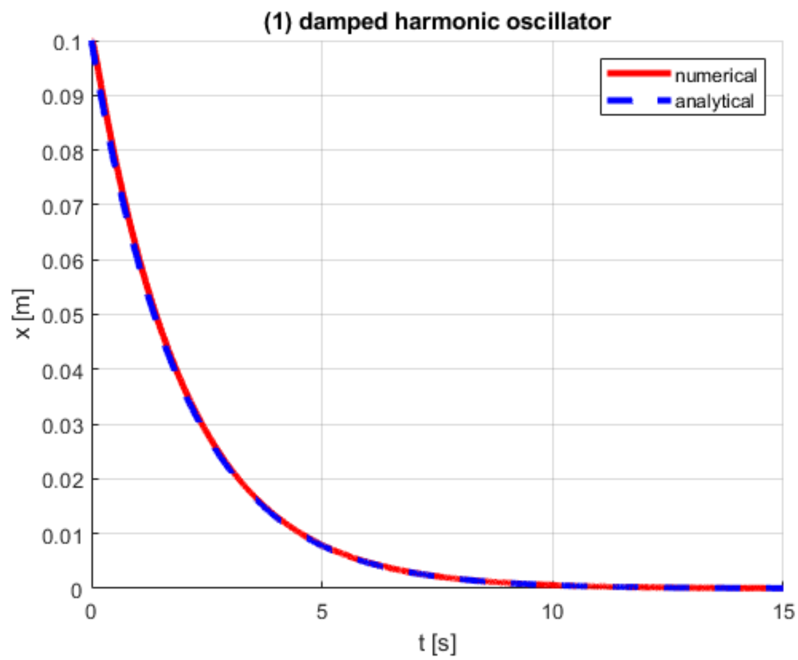
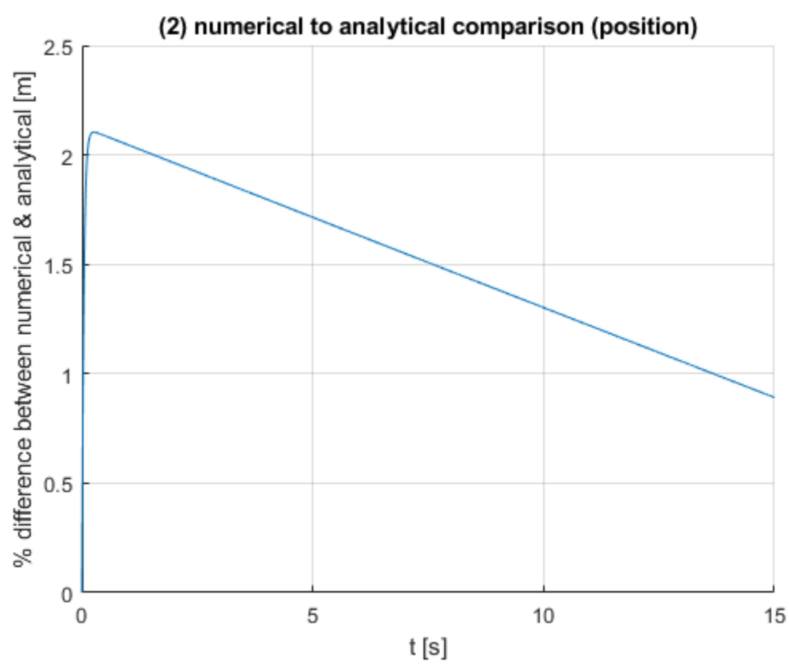

```
% outputs
rxNum = zarray(:,1);
vxNum = zarray(:,2);


% PLOTTING
 ****************************************************************

% ---------------
% POSITION V. TIME
figure(1)
hold on
grid on

% plot numerical solution
plot(tarray, rxNum, '-r', 'LineWidth', 3)

% plot analytical solution
plot(tarray, rxAna, '--b', 'LineWidth', 3)

% figure settings
xlabel('t [s]')
ylabel('x [m]')
title('(1) damped harmonic oscillator')
legend('numerical', 'analytical')
hold off

% ---------------
% NUMERICAL TO ANALYTICAL COMPARISON FOR POSITION V. TIME
figure(2)
hold on
grid on

% plot difference between analytical and numerical solutions
plot(tarray, 100*(rxNum-rxAna)./rxAna)

% figure settings
xlabel('t [s]')
ylabel('% difference between numerical & analytical [m]')
title('(2) numerical to analytical comparison (position)')
hold off




% FUNCTIONS
 ***************************************************************

% equations of motion
function dz = eqn(t,z,p) % time, state, struct of matricies

    % unpack parameters ***********************************
```

```
    m = p.m;
    c = p.c;
    k = p.k;

    % unpack z vector *************************************
    % positions
    rx = z(1);
    % velocities
    vx = z(2);

    % fill dz vector **************************************
    % first entries become velocities
    dz(1) = vx;
    % next entries become accelerations
    % CHANGE THESE LINES ACCORDING TO EOM
    dz(2) = (1/m) * (-c*vx - k*rx);

    % transpose dz from row to column vector **************
    dz = dz';

end


% Solver using Euler's method (Stolen from Teaya Yang)
function  [tarray, zarray] = myEulerSolver(eqn, tarray, z0, p)
% Generating zarray matrix
ntimes = length(tarray);
zarray = zeros(ntimes, length(z0));
zarray(1,:) = z0';
h = tarray(2) - tarray(1); % step size


    % Applying Euler's method at each step
    for i  = 1:ntimes-1
        znow    = zarray(i,:)';   tnow = tarray(i);  % set present
 values
        zdot = eqn(tnow,znow,p);   % evaluate RSH (right hand side) at
 present time

        % Euler's method
        znew = znow + zdot * h;

        % Append to zarray
        zarray(i+1,:) = znew'; % Add a row to the array of solutions.
                                  % Recall, z is a column, zarray is in rows.
    end % loop

end  % function
```


Disclaimer: *We have lost track of the many people who wrote these solutions. They are not official and have not been reviewed by the authors for correctness.*





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11.1.10 You are given to design a SDOF damped oscillator that should show no oscillations at all when disturbed from the equilibrium (*i.e.,* it should return to equilibrium without overshooting on the other side). You are given a spring with stiffness $k = 500\,\mathrm{N/m}$, a hydraulic damper with $c = 10\,\mathrm{kg/s}$, and you have a choice of masses from $m = 1\,\mathrm{kg}$ to $m = 10\,\mathrm{kg}$ in increments of half kg. Find the appropriate mass.

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10.1.10

$K = 500\ N/m$

$C = 10\ Kg/s$

Range of mass = 1 kg to 10 kg in the increment of half kg.

Suppose that the system is critically damped then

$\xi = 1$.

we have.

$$\xi = \frac{c}{\sqrt{4mK}}$$

$$\sqrt{m} = \frac{c}{\xi\sqrt{4K}}$$

$$m = \frac{c^2}{\xi^2 \cdot 4 \cdot K}$$

$$m = \frac{c^2}{4 \cdot K} = \frac{10 \times 10}{4 \times 500}$$

$$m = 0.05\ Kg$$

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Now, let us consider the mass given in the problem

$m = 1$ kg.

$$\xi = \frac{c}{\sqrt{4mk}}$$

$$\xi = \frac{10}{\sqrt{4 \times 1 \times 500}}$$

$$\xi = 0.22$$

Now take, $m = 10$ kg.

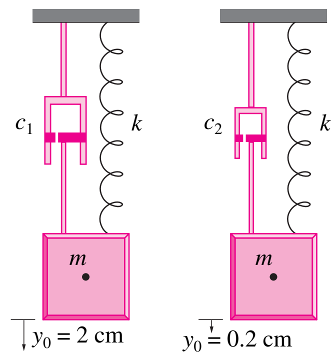
$$\xi = \frac{c}{\sqrt{4mk}}$$

$$\xi = \frac{10}{\sqrt{4 \times 10 \times 500}}$$

$$\xi = 0.0707$$

within the mass range of 1 kg to 10 kg the system is underdamped and it oscillates.

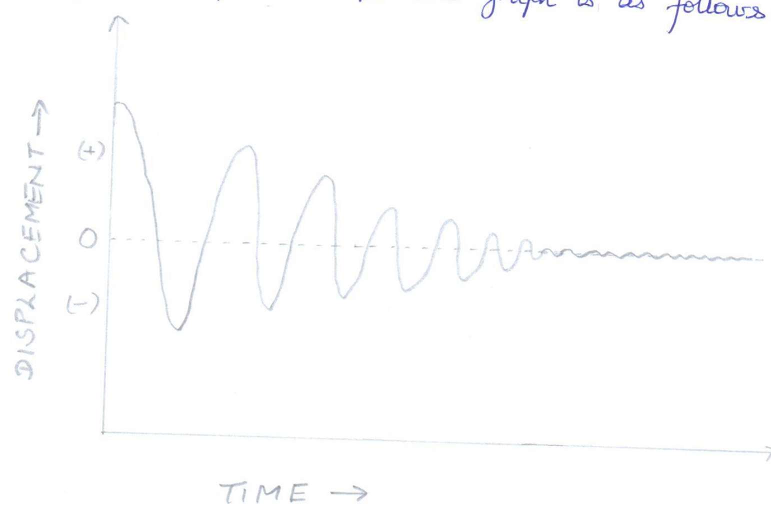
11.1.11 Two SDOF oscillators with the same k and m but different c's are hung from the ceiling as shown in the figure. The one on the left is pulled down 2 cm and let go. The other is pulled down by 0.2 cm and let go. Which oscillator undergoes more oscillations before reaching the steady state. Find the steady state displacement of each mass.



Problem 11.11

10.1.11 Irrespective of k and initial displacement, the SDOF oscillator with lower 'c' will undergo more number of oscillations, before reaching the steady state.

For damped oscillations without continuous external forcing, the displacement v/s time graph is as follows:



So, as can be seen, displacement decays with progress of time. The steady state displacement for both SDOF systems, therefore, would be zero.

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11.1.12 Experiments conducted on free oscillations of a damped oscillator reveal that the amplitude of oscillations drops to 25% of its peak value in just 3 periods of oscillations. The period of oscillation is measured to be 0.6 s and the mass of the system is known to be 1.2 kg. Find the damping coefficient and the spring stiffness of the system.

Answer:
$k = 132\,\mathrm{N/m}$, $c = 1.76\,\mathrm{kg/s}$.

10.1.12

$T_d = 0.6\ s$

$m = 1.2\ Kg$

Aplitude drops to 25% of the peak value in 3 periods of oscillation.

Logarithmic Decrement

$$X_i = X e^{-\xi \omega_n t_i}$$

next cycle occurs one period later at $t_i + T_d$

$$X_{i+1} = X e^{-\xi \omega_n (t_i + T_d)}$$

ratio of successive amplitudes

$$\frac{X_i}{X_{i+1}} = e^{\xi \omega_n T_d} = \text{constant}.$$

now,

$$\frac{X_0}{X_3} = \frac{X_0}{X_1} \cdot \frac{X_1}{X_2} \cdot \frac{X_2}{X_3} = e^{3\xi \omega_n T_d}$$

$$T_d = \frac{2\pi}{\omega_d} = \frac{2\pi}{\omega_n \sqrt{1-\xi^2}}$$

$$\ln\left(\frac{X_0}{X_3}\right) = 3\xi \omega_n T_d$$

$$\ln\left(\frac{X_0}{X_3}\right) = \frac{3 \times 2\pi \xi}{\sqrt{1-\xi^2}}$$

$$\ln\left(\frac{x_0}{0.25\,x_0}\right) = \frac{3\times 2\pi\,\xi}{\sqrt{1-\xi^2}}$$

$$\left(\frac{1.38}{3\times 2\pi}\right)^2 = \frac{\xi^2}{\left(\sqrt{1-\xi^2}\right)^2}$$

$$5.4\times 10^{-3} = \frac{\xi^2}{1-\xi^2}$$

$$5.4\times 10^{-3} = 1.0054\,\xi^2$$

$$5.37\times 10^{-3} = \xi^2$$

$$\xi = 0.07$$

Now,

$$\tau_d = \frac{2\pi}{\omega_d}$$

$$\omega_d = \frac{2\pi}{\tau_d} = \frac{2\pi}{0.6} = 10.47 \text{ rad/s}$$

$$\omega_d = \omega_n\sqrt{1-\xi^2}$$

$$10.47 = \omega_n \times 0.997$$

$$\omega_n = 10.49 = \sqrt{\frac{k}{m}}$$

$$110.04 = \frac{k}{m}$$

$$110.04\times m = k$$

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$K = 132.04\ N/m$

$$\xi = \frac{c}{\sqrt{4Km}}$$

$$\xi\sqrt{4Km} = c$$

$$c = 0.07\left(\sqrt{4 \times 1.2 \times 132.04}\right)$$

$$c = 1.76\ kg/s$$

11.1.13 You are required to design a mass-spring-dashpot system that, if disturbed, returns to its equilibrium position the quickest. You are given a mass, $m = 1\,\mathrm{kg}$, and a damper with $c = 10\,\mathrm{kg/s}$. What should be the stiffness of the spring? Your solution needs to include your definition of "'quickest".

Answer:
$k = 25\,\mathrm{N/m}$

10.1.13 $\quad m = 1\,kg \quad C = 10\,kg/sec. \quad k = ?$

For the system to return to equilibrium position in the quickest amount of time possible, it has to be critically damped.

$$\left(\frac{C_c}{2m}\right)^2 - \frac{k}{m} = 0$$

$$\Rightarrow \frac{C_c^2}{4m^2} = \frac{k}{m} \qquad \Rightarrow k = \frac{C_c^2}{4m} = \frac{10\times10}{4\times1} = 25\,N/m$$