SAMPLE 11.1 **Springs in series versus springs in parallel:** Two massless springs with spring constants k_1 and k_2 are attached to mass A *in parallel* (although they look superficially as if they are in series) as shown in Fig. 11.5. An identical pair of springs is attached to mass B *in series*. Taking $m_A = m_B = m$, find and compare the natural frequencies of the two systems. Ignore gravity.

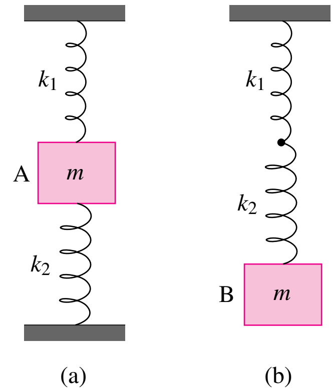

Figure 11.5

Solution Let us pull each mass downwards by a small vertical distance y and then release. Measuring y to be positive downwards, we can derive the equations of motion for each mass by writing the balance of linear momentum for each as follows.

- **Mass A:** The free-body diagram of mass A is shown in Fig. 11.6. As the mass is displaced downwards by y, spring 1 gets stretched by y whereas spring 2 gets compressed by y. Therefore, the forces applied by the two springs, $k_1 y$ and $k_2 y$, are in the same direction. The linear momentum balance of mass A in the vertical direction gives:

$$\begin{aligned} \sum F &= m a_y \\ \text{or} \qquad -k_1 y - k_2 y &= m\ddot{y} \\ \text{or} \qquad \ddot{y} + \left(\frac{k_1 + k_2}{m}\right) y &= 0. \end{aligned}$$

Let the natural frequency of this system be ω_p. Comparing with the standard simple harmonic equation $\ddot{x} + \lambda^2 x = 0$ (see box 3.1 on page 203), we get the natural frequency (λ) of the system:

$$\omega_p = \sqrt{\frac{k_1 + k_2}{m}} \tag{11.11}$$

$$\boxed{\omega_p = \sqrt{\frac{k_1+k_2}{m}}}$$

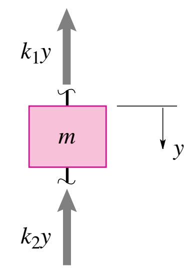

Figure 11.6: Free-body diagram of the mass.

- **Mass B:** The free-body diagram of mass B and the two springs is shown in Fig. 11.7. In this case both springs stretch as the mass is displaced downwards. Let the net stretch in spring 1 be y_1 and in spring 2 be y_2. y_1 and y_2 are unknown, of course, but we know that

$$y_1 + y_2 = y \tag{11.12}$$

Now, using the free-body diagram of spring 2 and then writing linear momentum balance we get,

$$k_2 y_2 - k_1 y_1 = \underbrace{m}_{0} a = 0$$

$$y_1 = \frac{k_2}{k_1} y_2 \tag{11.13}$$

Solving (11.12) and (11.13) we get

$$y_2 = \frac{k_1}{k_1 + k_2} y.$$

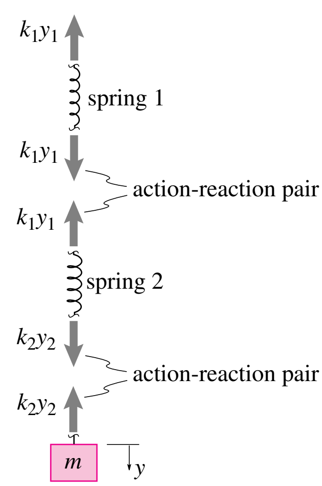

Figure 11.7: Free-body diagrams

Now, linear momentum balance of mass B in the vertical direction gives:

$$\begin{aligned} -k_2 y_2 = m a_y &= m\ddot{y} \\ \text{or} \qquad m\ddot{y} + k_2 \overbrace{\frac{k_1}{k_1+k_2} y}^{y_2} &= 0 \\ \text{or} \qquad \ddot{y} + \frac{k_1 k_2}{m(k_1+k_2)} y &= 0. \end{aligned} \tag{11.14}$$

Let the natural frequency of this system be denoted by ω_s. Then, comparing with the standard simple harmonic equation as in the previous case, we get

$$\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}. \tag{11.15}$$

$$\boxed{\omega_s = \sqrt{\frac{k_1 k_2}{m(k_1+k_2)}}}$$

From (11.11) and (11.15)

$$\frac{\omega_p}{\omega_s} = \frac{k_1+k_2}{\sqrt{k_1 k_2}}.$$

Let $k_1 = k_2 = k$. Then, $\omega_p/\omega_s = 2$, *i.e.,* the natural frequency of the system with two identical springs in parallel is twice as much as that of the system with the same springs in series. Intuitively, the restoring force applied by two springs in parallel will be more than the force applied by identical springs in series. In one case the restoring forces add and in the other they don't. Therefore, we do expect mass A to oscillate at a faster rate (higher natural frequency) than mass B. The ratio of the two natural frequencies, ω_p/ω_s as a function of k_1/k_2 is plotted in fig. 11.8. As we can see, ω_p is always higher than ω_s for all k_1/k_2 ratios and its minimum value is twice that of ω_s when $k_1 = k_2$.

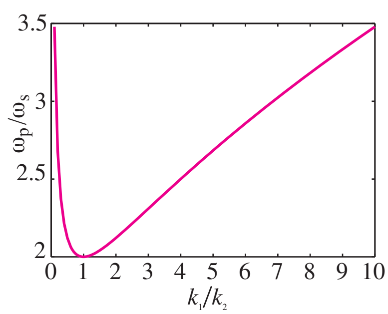

Figure 11.8: The ratio ω_p/ω_s of the two natural frequencies as a function of the spring stiffness ratio k_1/k_2.

Comments:

1. Although the springs attached to mass A do not visually seem to be in parallel, from a mechanics point of view they are parallel (see fig. 11.9). You can easily check this result by putting the two springs visually in parallel and then deriving the equation of mass A. You will get the same equations. For springs in parallel, each spring has the *same displacement* but *different forces*. For springs in series, each has *different displacements* (if $k_1 \neq k_2$) but the *same force*.

2. When many springs are connected to a mass in series or in parallel, sometimes we talk about their *effective* spring constant, *i.e.,* the spring constant of a single imaginary spring which could be used to replace all the springs attached in parallel or in series. Let the effective spring constant for springs in parallel and in series be represented by k_{pe} and k_{se} respectively. By comparing eqns. (11.11) and (11.15) with the expression for natural frequency of a simple spring mass system, we see that

$$k_{pe} = k_1 + k_2 \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2}.$$

These expressions can be easily extended for any arbitrary number of springs, say, N springs:

$$k_{pe} = k_1 + k_2 + \ldots + k_N \quad \text{and} \quad \frac{1}{k_{se}} = \frac{1}{k_1} + \frac{1}{k_2} + \ldots + \frac{1}{k_N}.$$

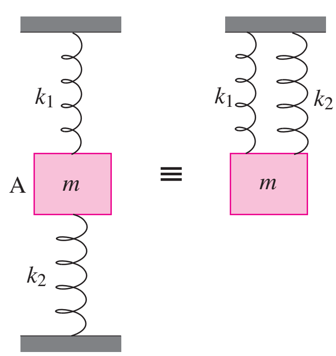

Figure 11.9: The two arrangements of springs are equivalent; they are connected in parallel.

SAMPLE 11.2 Figure 11.10 shows two responses obtained from experiments on two spring-mass systems. For each system

1. Find the natural frequency.
2. Find the initial conditions.

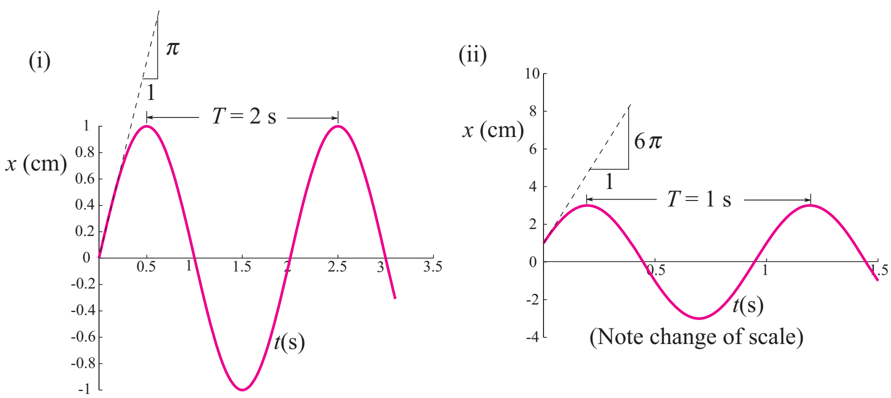

Figure 11.10:

Solution

1. **Natural frequency:** By definition, the natural frequency f is the number of cycles the system completes in one second. From the given responses we see that:

   Case(i): the system completes $\frac{1}{2}$ a cycle in 1 s.

   $$\Rightarrow \qquad f = \frac{1}{2}\ \text{Hz}.$$

   Case(ii): the system completes 1 cycle in 1 s.

   $$\Rightarrow \qquad f = 1\ \text{Hz}.$$

   It is usually hard to measure the fraction of a cycle occurring in a short time. It is easier to first find the time period, *i.e.,* the time taken to complete 1 cycle. ① Then the natural frequency can be found by the formula $f = \frac{1}{T}$. From the given responses, we find the time period by estimating the time between two successive peaks (or troughs): From Figure 11.10 we find that for

   ① To estimate the frequency of some repeated motion in an experiment, it is best to measure the time for a large number of cycles, say 5, 10 or 20, and then divide that time by the total number of cycles to get an average value for the time period of oscillation.

   Case (i):

   $$f = \frac{1}{T} = \frac{1}{2\,\text{s}} = \frac{1}{2}\ \text{Hz},$$

   Case (ii):

   $$f = \frac{1}{T} = \frac{1}{1\,\text{s}} = 1\ \text{Hz}$$

   $$\boxed{\text{case (i) } f = \tfrac{1}{2}\ \text{Hz.} \quad \text{case (ii) } f = 1\ \text{Hz.}}$$

2. **Initial conditions:** Now we are to find the displacement and velocity at $t = 0$ s for each case. Displacement is easy because we are given the displacement plot, so we just read the value at $t = 0$ from the plots:

   Case (i):

   $$x(0) = 0.$$

   Case (ii):

   $$x(0) = 1\,\text{cm}.$$

The velocity (actually the speed) is the time-derivative of the displacement. Therefore, we get the initial velocity from the slope of the displacement curve at $t = 0$.

Case (i):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{\pi\,\text{cm}}{1\,\text{s}} = 3.14\,\text{cm/s}.$$

Case (ii):

$$\dot{x}(0) = \frac{dx}{dt}(t=0) = \frac{6\pi\,\text{cm}}{1\,\text{s}} = 18.85\,\text{cm/s}.$$

Thus the initial conditions are

Case (i):	$x(0) = 0$	$\dot{x}(0) = 3.14\,\text{cm/s}$
Case (ii):	$x(0) = 1\,\text{cm}$	$\dot{x}(0) = 18.85\,\text{cm/s}$

Comments: Estimating the speed from the initial slope of the displacement curve at $t = 0$ is not a very good method because it is hard to draw an accurate tangent to the curve at $t = 0$. A slightly different line but still seemingly tangential to the curve at $t = 0$ can lead to significant error in the estimated value. A better method, perhaps, is to use the known values of displacement at different points and use the energy method to calculate the initial speed. We show sample calculations for the first system:

Case(i): We know that $x(0) = 0$. Therefore the entire energy at $t = 0$ is the kinetic energy $= \frac{1}{2}mv_0^2$. At $t = 0.5\,\text{s}$ we note that the displacement is maximum, *i.e.,* the speed is zero. Therefore, the entire energy is potential energy $= \frac{1}{2}kx^2$, where $x = x(t = 0.5\,\text{s}) = 1\,\text{cm}$.

Now, from the conservation of energy:

$$\begin{aligned}
\frac{1}{2}mv_0^2 &= \frac{1}{2}k(x_{t=0.5\,\text{s}})^2 \\
\Rightarrow \quad v_0 &= \sqrt{\frac{k}{m}}\cdot(x_{t=0.5\,\text{s}}) \\
&= \underbrace{\sqrt{\frac{k}{m}}}_{\lambda}\cdot(1\,\text{cm}) \\
&= 2\pi f\cdot(1\,\text{cm}) \\
&= 2\pi\cdot\frac{1}{2}\,\text{Hz}\cdot 1\,\text{cm} \\
&= 3.14\,\text{cm/s}.
\end{aligned}$$

Similar calculations can be done for the second system.

SAMPLE 11.3 A block of mass 10 kg is attached to a spring and a dashpot as shown in Figure 11.11. The spring constant $k = 1000\,\mathrm{N/m}$ and a damping rate $c = 50\,\mathrm{N\cdot s/m}$. When the block is at a distance d_0 from the left wall the spring is relaxed. The block is pulled to the right by 0.5 m and released. Assuming no initial velocity, find

1. the equation of motion of the block.
2. the position of the block at $t = 2\,\mathrm{s}$.

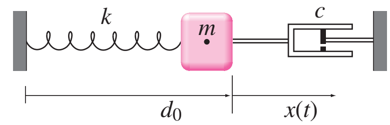

Figure 11.11: Spring-mass dashpot.

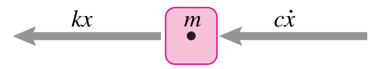

Figure 11.12: Free-body diagram of the mass at an instant t.

Solution

1. Let x be the position of the block, measured positive to the right of the static equilibrium position, at some time t. Let $\dot{x}$ be the corresponding speed. The free-body diagram of the block at the instant t is shown in Figure 11.12.

   Since the motion is only horizontal, we can write the linear momentum balance in the x-direction ($\sum F_x = m\,a_x$):

$$\underbrace{-kx - c\dot{x}}_{\sum F_x} = m\underbrace{\ddot{x}}_{a_x}$$

$$\text{or}\quad \ddot{x} + \frac{c}{m}\dot{x} + \frac{k}{m}x = 0 \qquad (11.16)$$

   which is the desired equation of motion of the block.

$$\boxed{\ddot{x} + \tfrac{c}{m}\dot{x} + \tfrac{k}{m}x = 0.}$$

2. To find the position and velocity of the block at any time t we need to solve Eqn (11.16). Since the solution depends on the relative values of m, k, and c, we first compute c^2 and compare with the *critical value* $4mk$.

$$\begin{aligned} c^2 &= 2500(\mathrm{N\cdot s/m})^2 \\ \text{and}\quad 4mk &= 4\cdot 10\,\mathrm{kg}\cdot 1000\,\mathrm{N/m} = 4000(\mathrm{N\cdot s/m})^2. \\ \Rightarrow\quad c^2 &< 4mk. \end{aligned}$$

   Therefore, the system is underdamped and we may write the general solution as (see box 11.3 on page 563)

$$x(t) = e^{-\frac{c}{2m}t}\left[A\cos\lambda_D t + B\sin\lambda_D t\right] \qquad (11.17)$$

   where

$$\lambda_D = \sqrt{\frac{k}{m} - \left(\frac{c}{2m}\right)^2} = 9.682\,\mathrm{rad/s}.$$

   Substituting the initial conditions $x(0) = 0.5\,\mathrm{m}$ and $\dot{x}(0) = 0\,\mathrm{m/s}$ in Eqn (11.17) (we need to differentiate Eqn (11.17) first to substitute $\dot{x}(0)$), we get

$$\begin{aligned} x(0) &= 0.5\,\mathrm{m} = A. \\ \dot{x}(0) &= 0 = -\frac{c}{2m}\cdot A + \lambda_D\cdot B \\ \Rightarrow\quad B &= \frac{A\,c}{2m\lambda_D} = \frac{(0.5\,\mathrm{m})\cdot(50\,\mathrm{N\cdot s/m})}{2\cdot(10\,\mathrm{kg})\cdot(9.682\,\mathrm{rad/s})} = 0.13\,\mathrm{m}. \end{aligned}$$

   Thus, the solution is

$$x(t) = e^{(-2.5\frac{1}{\mathrm{s}})t}\left[0.50\cos(9.68\,\mathrm{rad/s}\;t) + 0.13\sin(9.68\,\mathrm{rad/s}\;t)\right]\,\mathrm{m}.$$

   Substituting $t = 2\,\mathrm{s}$ in the above expression we get $x(2\,\mathrm{s}) = 0.003\,\mathrm{m}$.

$$\boxed{x(2\,\mathrm{s}) = 0.003\,\mathrm{m}.}$$

SAMPLE 11.4 A structure, modeled as a single degree of freedom system, exhibits characteristics of an underdamped system under free oscillations. The response of the structure to some initial condition is determined to be $x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t)$ where $A = 0.3\,\mathrm{m}$, $\xi \equiv$ damping ratio $= 0.02$, $\lambda \equiv$ undamped circular frequency $= 1\,\mathrm{rad/s}$, and $\lambda_D \equiv$ damped circular frequency $= \lambda\sqrt{1-\xi^2} \approx \lambda$.

1. Find an expression for the ratio of energies of the system at the $(n+1)$th displacement peak and the nth displacement peak.
2. What percent of energy available at the first peak is lost after 5 cycles?

Solution

1. We are given that

$$x(t) = Ae^{-\xi\lambda t}\sin(\lambda_D t).$$

The structure attains its first displacement peak when $\sin\lambda_D t$ is maximum, *i.e.*,

$$\lambda_D t = \frac{\pi}{2} \quad \Rightarrow \quad t = \frac{\pi}{2\lambda_D}.$$

At this instant,

$$\begin{aligned} x(t) &= Ae^{-\xi\cdot\lambda\cdot\frac{\pi}{2\lambda_D}} \\ &= Ae^{-\frac{\pi}{2}\cdot\frac{\xi}{\sqrt{1-\xi^2}}} \\ &= (0.3\,\mathrm{m})\cdot e^{-0.0314} \\ &= 0.29\,\mathrm{m}. \end{aligned}$$

Let x_n and x_{n+1} be the values of the displacement at the nth and the $(n+1)$th peak, respectively. Since x_n and x_{n+1} are peak displacements, the respective velocities are zero at these points. Therefore, the energy of the system at these peaks is given by the potential energy stored in the spring. That is

$$E_n = \frac{1}{2}kx_n^2 \quad \text{and} \quad E_{n+1} = \frac{1}{2}kx_{n+1}^2. \tag{11.18}$$

Let t_n be the time at which the nth peak displacement x_n is attained, *i.e.*,

$$x_n = Ae^{-\xi\lambda t_n} \tag{11.19}$$

Since x_{n+1} is the next peak displacement, it must occur at $t = t_n + T_D$ where T_D is the time period of damped oscillations. Thus

$$x_{n+1} = Ae^{-\xi\lambda(t_n+T_D)} \tag{11.20}$$

From Eqns (11.18), (11.19), and (11.20)

$$\begin{aligned} \frac{E_{n+1}}{E_n} &= \frac{\frac{1}{2}k(Ae^{-\xi\lambda(t_n+T_D)})^2}{\frac{1}{2}k(Ae^{-\xi\lambda t_n})^2} \\ &= e^{-2\xi\lambda T_D}. \end{aligned}$$

$$\frac{E_{n+1}}{E_n} = e^{-2\xi\lambda T_D}.$$

2. Noting that $T_D = \frac{2\pi}{\lambda_D}$ and $\lambda_D = \lambda\sqrt{1-\xi^2}$, we get

$$\begin{aligned} E_{n+1} &= E_n\, e^{-2\xi\lambda\cdot\frac{2\pi}{\lambda\sqrt{1-\xi^2}}} \\ &= e^{-4\pi\frac{\xi}{\sqrt{1-\xi^2}}} \approx e^{-4\pi\xi} \\ \Rightarrow \qquad E_{n+1} &= e^{-4\pi\xi}\, E_n. \end{aligned}$$

Applying this equation recursively for $n = n-1, n-2, \ldots, 1, 0,$, we get

$$\begin{aligned} E_n &= e^{-4\pi\xi}\cdot E_{n-1} \\ &= e^{-4\pi\xi}\cdot(e^{-4\pi\xi}\cdot E_{n-2}) \\ &= (e^{-4\pi\xi})^3\cdot E_{n-3} \\ &\vdots \\ &= (e^{-4\pi\xi})^n\cdot E_0. \end{aligned}$$

Now we use this equation to find the percentage of energy of the first peak ($n = 0$) lost after 5 cycles ($n = 5$):

$$\begin{aligned} \Delta E_5 &= \frac{E_0 - E_5}{E_0} \times 100 \\ &= \left(1 - e^{-4\pi\xi\cdot 5}\right) \times 100 \\ &= 71.5\%. \end{aligned}$$

$$\Delta E_5 = 71.5\%.$$

SAMPLE 11.5 A SDOF spring-mass model from given data: The following table is obtained for successive peaks of displacement from the simulation of free vibration of a mechanical system. Make a single degree of freedom mass-spring-dashpot model of the system choosing appropriate values for mass, spring stiffness, and damping rate.
Data:

peak no. n	0	1	2	3	4	5	6
time (s)	0.0000	0.6279	1.2558	1.8837	2.5116	3.1395	3.7674
peak x (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443

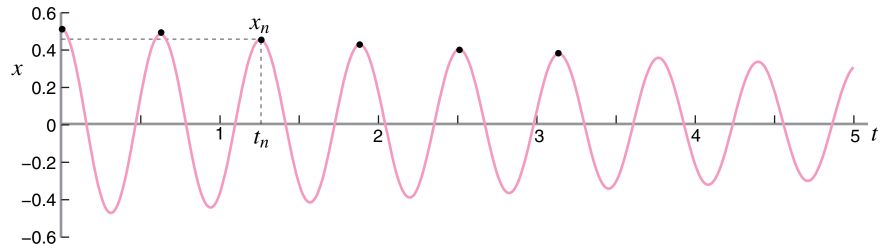

Figure 11.13: Oscillation data from the simulation of a mechanical system

Solution Since the data provided is for successive peak displacements, the time between any two successive peaks represents the period of oscillations. It is also clear that the system is underdamped because the successive peaks are decreasing. We can use the logarithmic decrement method to determine the damping in the system.

First, we find the time period T_D from which we can determine the damped circular frequency λ_D. From the given data we find that

$$t_2 - t_1 = t_3 - t_2 = t_4 - t_3 = \cdots = 0.6279\,\mathrm{s}$$

Therefore,

$$\begin{aligned} T_D &= 0.6279\,\mathrm{s}. \\ \Rightarrow \quad \lambda_D &= \frac{2\pi}{T_D} = 10\,\mathrm{rad/s}. \end{aligned} \tag{11.21}$$

Now we make a table for the logarithmic decrement of the peak displacements:

peak disp. x_n (m)	0.5006	0.4697	0.4411	0.4143	0.3892	0.3659	0.3443
$\frac{x_n}{x_{n+1}}$	1.0658	1.0648	1.0647	1.0645	1,0637	1.0627	
$\ln\left(\frac{x_n}{x_{n+1}}\right)$	0.0637	0.0628	0.0627	0.0624	0.0618	0.0608	

Thus, we get several values of the logarithmic decrement $D = \ln\left(\frac{x_n}{x_{n+1}}\right)$ ②

② Theoretically, all of these values should be the same, but it is rarely the case in practice. When x_n's are measured from an experimental setup, the values of D may vary even more.

We take the average value of D:

$$D = \bar{D} = 0.0624. \tag{11.22}$$

Let the equivalent single degree of freedom model have mass m, spring stiffness k, and damping rate c. Then

$$\lambda_D = \lambda\sqrt{1-\xi^2} \approx \lambda = \sqrt{\frac{k}{m}}.$$

Thus, from Eqn (11.21),

$$\frac{k}{m} = \lambda^2 = 100(\text{ rad/s})^2, \tag{11.23}$$

and, since $D = \frac{cT_D}{2m}$, from Eqn (11.22) we get

$$\begin{aligned} c &= \frac{2mD}{T_D} \\ &= \frac{2m(0.0624)}{0.6279\,\text{s}} \\ &= (0.1988\frac{1}{\text{s}})m. \end{aligned} \tag{11.24}$$

Equations (11.23) and (11.24) have three unknowns: k, m, and c. We cannot determine all three uniquely from the given information. So, let us pick an arbitrary mass $m = 5\,\text{kg}$. Then

$$\begin{aligned} k &= (100\frac{1}{\text{s}^2})\cdot(5\,\text{kg}) \\ &= 500\,\text{N/ m}, \end{aligned}$$

and

$$\begin{aligned} c &= (0.1988\frac{1}{\text{s}})\cdot(5\,\text{kg}) \\ &= 0.99\,\text{N}\cdot\text{s/ m}. \end{aligned}$$

$$\begin{aligned} m &= 5\,\text{kg}, \\ k &= 500\,\text{N/ m}, \\ c &= 0.99\,\text{N}\cdot\text{s/ m}. \end{aligned}$$

Of course, we could choose many other sets of values for m, k, and c which would match the given response. In practice, there is usually a little more information available about the system, such as the mass of the system. In that case, we can determine k and c uniquely from the given response.

