10.5 Collisions in 1D

Sometimes things interact in a sudden manner, like two cars in a head-on crash or a dropped cell-phone hitting the floor. Some sudden interactions are intentional, for example in sports the banging of racquets, bats, clubs, sticks, hands and legs with balls, pucks and bodies. And in machines there are sometimes intentionally sudden interactions like the clicking of a ratchet and the flip of an electric light switch. More esoteric 'sudden' interactions include those between subatomic particles in an accelerator and near passes of satellites with planets.

When two solids bump into each other a nearly discontinuous change in their velocities and/or angular velocities is needed to keep the bodies from interpenetrating. This sudden change in velocity demands large interaction. In the case of subatomic particles near nuclei and satellites near planets there might be no contact, but nonetheless there are large forces when the interaction distances get small. Estimating the effects of these large yet short-lived forces is the central problem in collision mechanics.

Two objects are said to *collide* when some interaction force or moment between them becomes so large that other forces acting on the bodies become negligible. For example, in a car collision the force of interaction at the bumpers may be many times the weight of the car or the reaction forces acting on the wheels. And so short acting that, although velocities change, positions change negligibly during the collision.

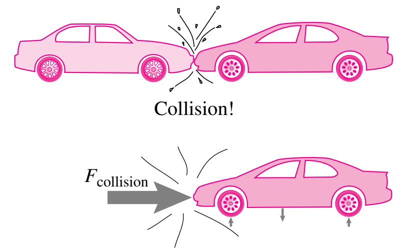


Figure 10.50: Here cars are shown colliding. A free-body diagram of the right car shows the collision force and should not show other forces which are negligibly small. Here they are shown as negligibly small forces to give the idea that they may be much smaller than the collision force. The wheel reaction forces are neglected because of the spring compliance of the suspension and tires.

Collisional free-body diagrams The analysis of collisions is a little different than the analysis of smooth motions, but still depends on free-body diagrams (See fig. 10.50). Knowing which forces to include and which to ignore in a collisional free-body-diagram is a subtle issue. Some rules of thumb:

- ignore forces from gravity, springs, and at places where contact is broken in the collision, and

- include forces at places where new contact is made, or where contact is maintained.

The elementary analysis of rigid body collisions is based on these ideas:

I. Collision forces are big, so non-collisional forces are neglected in collisional free-body diagrams.
II. Collision forces are of short duration, so the position and orientation of the colliding bodies do not change during the collision.

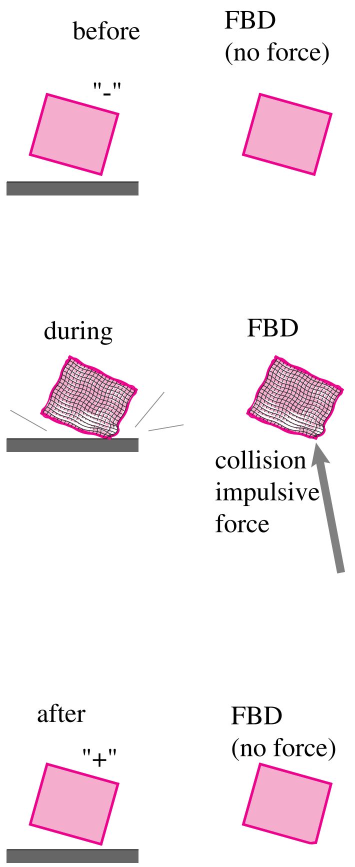


Figure 10.51: Just before a collision is called "-", and just after is called "+". The only forces that show on a collisional free-body diagram are those that are large and part of the impact. Either a force or an impulse may be shown. This figure exaggerates the difference between the before (-) and after (+) states. In analysis we assume that there is no change in the body's position or orientation from just before to just after the collision. The only net changes caused by the collision are the body's velocity and rotation rate.

① Nominally means "in name". That is, what one calls "contacting points" are not points at all, but regions of complex interaction.

What happens during a collision

During a collision between what would generally be called "rigid" bodies *things get wild*. There are huge contact forces and stresses in the regions near the nominally① contacting points, there could be plastic deformation, fracture, and frictional slip. Elastic waves may travel all over the body, reflect and scatter this way and that. Altogether the contact interaction during the collision is the result of very complex deformations (see fig. 10.51).

Deformations (the lack of rigidity) give rise to the forces between colliding bodies. So what could the phrase "rigid-object collisions" mean? It is an oxymoron. Trying to understand the collision forces in detail, and how they are related to deformations, is way beyond this book. Actually, there is no unified theory of collisions so you can't read about it in any book. Loosely one might imagine that during part of the collision material is being squeezed, this is called the *compression* phase and later on it expands back in a *restitution* phase. But the realities of collisions are not necessarily so simple; the forces and deformations can vary in complex ways.

Soon after the collision, however, the vibrations often die out, each object may have negligible permanent change in shape, and the object returns to motions that are well described by rigid-object kinematics. To find out the net effect of the collision forces we use this one key idea:

> III. The laws of mechanics apply *during* collisions even though rigid-object kinematics does not.

While the motions during a collision may be wildly complex, the general linear and angular momentum balance laws are still applicable. Rather than applying these laws to understand the details *during* a collision, we use them to summarize the overall result of the collision.

That is, in rigid-object collision analysis we do not pay attention to how the forces vary in time, or to the detailed trajectories, velocities or accelerations of any material points. Rather, we focus on the net change in the velocities of the colliding bodies that the collision forces cause. Thus, instead of using the differential-equation form of the linear momentum balance, angular-momentum balance and energy equations (Ia, IIa, and IIIa from the inside front cover) we use the time integrated forms (Ib, IIb, and IIIb). All that we note about a collisional force is its net impulse

$$\vec{\boldsymbol{P}}_{\text{coll}} = \int_{\text{collision time}} \vec{\boldsymbol{F}}_{\text{coll}}\, dt$$

in terms of which we have, for one object experiencing this impulse at point C

$$\vec{\boldsymbol{P}}_{\text{coll}} = \Delta\vec{\boldsymbol{L}}, \tag{10.53}$$

$$\vec{\boldsymbol{r}}_{C/0} \times \vec{\boldsymbol{P}}_{\text{coll}} = \Delta\vec{\boldsymbol{H}}_{/O}, \quad \text{and} \tag{10.54}$$

$$\text{Collisional dissipation} = \Delta E_{K}. \tag{10.55}$$

Most often the first two of these, the impulse-momentum equations are used to find the motion after collision. The energy equation is just a check to make sure that the collisional dissipation is positive (otherwise the collision would be an energy source).

Extra assumptions are needed

The momentum balance equations, with the assumptions already discussed, are never enough in themselves to determine the outcome of a collision. The extra assumptions come in various forms. To minimize the algebra we discuss the issues first with one-dimensional collisions.

One dimensional collisions

Here we only consider collisions in the context of one-dimensional mechanics: all motion is constrained to one direction of motion by forces which we ignore. Only momentum and forces in, say, the $\hat{\boldsymbol{\imath}}$ direction are included.

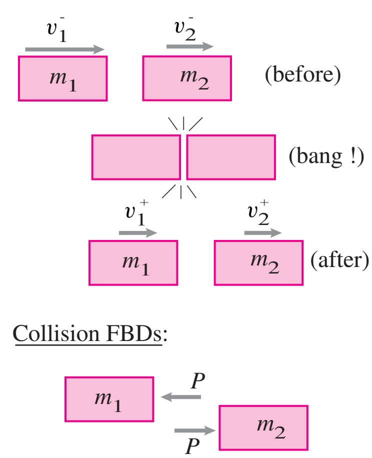


Figure 10.52: Before the collisions the masses have velocities to the right of v_1^- and v_2^-. After the collision the velocities are v_1^+ and v_2^+. During the collision the impulse P acts to the right on mass 2 and to the left on mass 1.

Example: 1-D collisions

Consider two masses which collide along their common line of motion. All velocities and momenta are positive if to the right and P is the impulse *on* mass 2 from mass 1. The relevant impulse-momentum relations are

$$\begin{aligned} &\text{For mass 1} & -P &= m_1(v_1^+ - v_1^-), \\ &\text{For mass 2} & P &= m_2(v_2^+ - v_2^-), \text{and} \\ &\text{For the system} & 0 &= (m_1 v_1^+ + m_2 v_2^+) - (m_1 v_1^- + m_2 v_2^-). \end{aligned}$$

The third equation comes from a free-body diagram of the system (*i.e.,* conservation of momentum) or by adding the first two equations. In any case, given the masses and initial velocities we have only two independent equations and we have three unknowns: v_1^+, v_2^+ and P. Momentum balance is not enough to determine the outcome of a collision.

To "close" (make solvable) the set of equations one needs to make extra assumptions.

Sticking collisions

The simplest assumption is that the masses *stick* together after the collision so

$$v_1^+ = v_2^+.$$

Such a collision is sometimes called a *perfectly plastic*, a *perfectly inelastic*, or a *dead* collision. Algebraic manipulations of the momentum equations and the "sticking" constitutive law give

$$
\begin{aligned}
v_1^+ = v_2^+ &= (m_1 v_1^- + m_2 v_2^-)/m_{\text{tot}} \qquad (\text{where } m_{\text{tot}} = m_1 + m_2) \quad \text{and} \\
P &= (v_1^- - v_2^-) m_{\text{coll}} \qquad (\text{where } m_{\text{coll}} = \frac{m_1 m_2}{m_1 + m_2}).
\end{aligned}
$$

The *collisional mass* or *contact mass* m_{coll}

$$
m_{\text{coll}} \quad = \quad \frac{m_1 m_2}{m_1 + m_2} \quad = \quad \frac{1}{\frac{1}{m_1} + \frac{1}{m_2}}
$$

is not the mass of anything. It is just a quantity that shows up repeatedly in collision calculations and theory. It is the reciprocal of the sum of the reciprocals of the two masses. If one mass is much bigger than the other, the contact mass is $m_{\text{coll}} \approx$ the smaller of the two masses. It is the proportionality constant relating the interaction force and the relative acceleration of the particles during the collision

$$
m_{\text{coll}}(a_2 - a_1) = F \qquad (\text{with } F \text{ being the force of body 1 on body 2})
$$

and is thus related to the effective mass of box 14.1 on page 690.

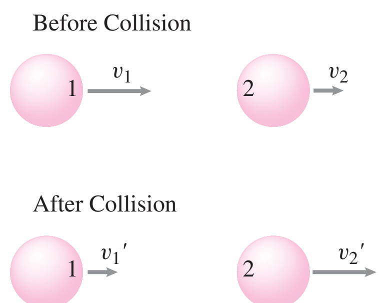


Figure 10.53: The rate of approach of two about-to-collide points is $v_1 - v_2$. The rate of separation after collision is $v_2' - v_1'$. The simplest collision law says $v_2' - v_1' = e(v_1 - v_2)$, where e is the coefficient of restitution.

More general 1-D collisions

The momentum equations can be re-arranged to better get at the essence of the situation which is that

- In the collision the system's center-of-mass velocity is unchanged, and
- The effect of the collision is to change the difference between the two mass velocities.

So we define the center-of-mass velocity v_{cm} and the velocity difference v_{rel} as

$$
v_{\text{cm}} \equiv (m_1 v_1 + m_2 v_2)/m_{\text{tot}} \quad \text{and} \quad v_{\text{rel}} \equiv v_2 - v_1.
$$

Note that before a collision the masses are approaching each other so $v_1^- > v_2^-$ and $v_{\text{rel}}^- < 0$. A little more algebra shows that for any P,

$$
\begin{aligned}
v_2^+ &= v_{\text{cm}} + \frac{m_1}{m_1 + m_2} v_{\text{rel}}^+, \\
v_1^+ &= v_{\text{cm}} - \frac{m_2}{m_1 + m_2} v_{\text{rel}}^+, \quad \text{and} \\
P &= (v_{\text{rel}}^+ - v_{\text{rel}}^-) m_{\text{coll}}
\end{aligned}
$$

That is, P acts on v_{rel} as if v_{rel} were the velocity of an object with mass m_{coll}. If $P = 0$ the equations above are a long winded way of saying that nothing happened, $v_1^+ = v_1^-$ and $v_2^+ = v_2^-$, and the masses pass right through each other.

If $P = -v_{\text{rel}}^- m_{\text{coll}}$ there is a sticking collision.

Elastic collisions

Application of the above formulas will show that if

$$P = -2v_{\text{rel}}^{-} m_{\text{coll}}$$

then the kinetic energy of the system after the collision is the same as the kinetic energy before. That is

$$\begin{aligned} {E_{\text{K}}}^{+} &= {E_{\text{K}}}^{-} \\ \frac{m_1 {v_1^{+}}^2 + m_2 {v_2^{+}}^2}{2} &= \frac{m_1 {v_1^{-}}^2 + m_2 {v_2^{-}}^2}{2}. \end{aligned}$$

Also, $v_{\text{rel}}^{+} = -v_{\text{rel}}^{-}$, the relative velocity maintains its magnitude and reverses its sign.

The coefficient of restitution

We have that as P ranges from $-v_{\text{rel}}^{-} m_{\text{coll}}$ to $-2v_{\text{rel}}^{-} m_{\text{coll}}$, the collision ranges from sticking to an energy conserving reversal of relative velocities. The *coefficient of restitution e* is introduced as a way of interpolating between these cases. The most commonly used collision law can be summarized with this simple equation,

The speed with which colliding points are separating after the collision.

The speed at which colliding points are approaching before collision.

$$\overbrace{(v_b' - v_a')} = \underbrace{e}\,\overbrace{(v_a - v_b)}, \tag{10.56}$$

The coefficient of restitution, assumed to be a constant for given materials.

Or, more simply expressed, the collision law can be defined by either of the following two equations

$$\begin{aligned} v_{\text{rel}}^{+} &= -e v_{\text{rel}}^{-} \quad \text{or} \\ P &= -(1+e) v_{\text{rel}}^{-} m_{\text{coll}}. \end{aligned}$$

If $e = 0$ we have a sticking collision. If $e = 1$ we have an energy conserving elastic collision. If e is between 0 and 1 the collision is somewhere between as dead and as alive as it can be[2]. In words the collision equation is: *the rate of separation is proportional to the rate of approach.* The coefficient e is called Newton's (see box 10.5) or Poisson's coefficient of

[2] A common mistake is to take e as a material property. It is not. e generally depends on the shapes and sizes of the contacting objects also (see box 10.4 on page 7).

restitution. Somewhat of a miracle is that a given pair of objects seems to have a coefficient of restitution that is roughly independent of the velocities. This is the result of a conspiracy by all kinds of deformation mechanisms that we don't really understand. But that e is a constant for a given pair of bodies is only an approximation that has roughly the same status (accuracy) as, say, the friction coefficient. Much lower status than the momentum balance equations.
③

③**What did Newton say about collisions?** Newton swung spheres at the ends of string, banged them into each other, and measured their bounce. He took account of air friction. At the point of this quote he has already discussed momentum conservation. Here is his statement of, and justification for, what we now call "Newton's law of collisions" (eqn. (10.56)):

"In bodies imperfectly elastic the velocity of the return is to be diminished together with the elastic force; because that force (except when the parts of bodies are bruised by their impact, or suffer some such extension as happens under the strokes of a hammer) is (as far as I can perceive) certain and determined, and makes bodies to return one from the other with a relative velocity, which is in a given ratio to that relative velocity with which they met. This I tried in balls of wool, made up tightly, and strongly compressed. For, first, by letting go the pendula's bodies, and measuring their reflection, I determined the quantity of their elastic force; and then, according to this force, estimated the reflections that ought to happen in other cases of impact. And with this computation other experiments made afterwards did accordingly agree; the balls always receding one from the other with a relative velocity, which was to the relative velocity to which they met, as about 5 to 9. Balls of steel returned with almost the same velocity; those of cork with a velocity something less; but in balls of glass the proportion was as about 15 to 16. " (Newton's Principia Motte's translation revised, by Florian Cajori, Univ. of CA press, page 25, 1947)

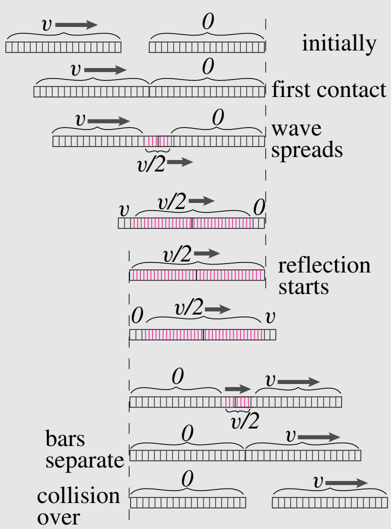
10.4 The axial collision of elastic rods: the unusual disappearance of vibrations

This box is an aside for the curious. No key skills are covered.

One can try to understand the stresses and deformations *during* a collision. This generally leads to the solution of partial differential equations. But those equations depend on material behavior that is usually not well-modeled. So even carefully generated numerical solutions may be far from reality.

To get a sense of the complexity we consider an ideal simple system, one that was somewhat controversial amongst the great 19th century scientists Cauchy, Poisson and Saint-Venant (so said E.J. Routh in 1905).

Two identical linear elastic rods. One uniform linear elastic rod with length ℓ is stationary. An identical rod approaches it with speed v from the left.



The system has no damping so no matter how the rods shake and vibrate, their elastic potential energy plus kinetic energy is constant.

Using the wave equation (see last paragraph) we can find the motions illustrated above. The pictures exaggerate the compression in the bar (For most materials the compression wouldn't be visible).

First, the undeformed left rods make contact, and a compressional sound wave starts spreading to the left and right. Between the wave fronts, in both rods, the material is compressed material moving at speed $v/2$ to the right. To the right of the right wave front the material is still. To the left of the left-moving wave front the material continues to move at v. When the wave fronts meet, the ends of their respective bars, the bars are compressed and all material is going to the right at $v/2$. Then both wave-fronts reflect off the ends of the bars and head back towards the contact point. To the left of the now right-moving wave front (on the left bar) the material is stationary and uncompressed. To the right of the left-moving wave front (on the right bar) the material is uncompressed but moving to the right at speed v . Finally, the waves meet in the center and the bars separate. The right bar is now uniformly moving to the right at speed v and the left bar is stationary.

The result of this collision is that all of the momentum of the left bar is transferred to the right bar. The separation velocity is equal in magnitude to the approach velocity. The coefficient of restitution e is 1, and the kinetic energy of the system is the same after the collision as it was before.

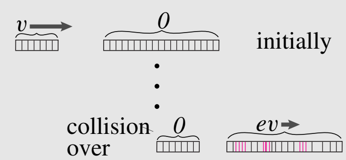
Note that the collision itself was quick. The wave-fronts move at the speed of sound, typically about 1000 m/s for metals. So for 1 meter metal rods the collision takes a few thousandths of a second. But during that few thousandths of a second, the initial energy was partitioned into elastic strain energy and kinetic energy in different time-changing regions of the bar.

Despite all the complicated details, we predict a totally 'elastic' collision. This may seem natural for collisions of elastic objects.

An elastic rod hits a rigid wall If you drop a 3 foot wooden dowel straight down on a thick concrete or stone floor it bounces quite well. Why? A wave analysis like that described above shows that a wave traveling from the first contact at the floor travels up to the top and reflects back to the bottom, leaving the rod moving uniformly up after the collisions just as fast as it was moving down before. Of course a wooden dowel is not perfectly described by the simple wave theory. And the ground is not perfectly rigid. So a real dowel's collision is not perfectly elastic.

But again we find that if we assume an elastic material we predict an elastic collision. Maybe no surprise. But the previous two examples are completely misleading! These are maybe *the only* examples where a detailed elastic theory predicts an elastic collision. More commonly it's more like the next example.

Rods of different length If the rods have length ℓ_1 and $\ell_2 > \ell_1$ then the collision works out differently.



When the reflection from the left end of the left rod comes back to the contact point, the rods separate. The left rod is stationary but the right rod has waves moving up and back. The average speed of the right rod is $(\ell_1/\ell_2)v$ so the effective coefficient of restitution is $e = \ell_1/\ell_2 < 1$. Later, after the vibrations have died out, the energy of the system will be less than initially. Or, even if the waves don't die out, the kinetic energy that can be accounted for in rigid-body mechanics is lost to remnant vibrations. Thus a totally elastic system leads to *inelastic* collisions. It is wrong to think that the restitution constant e depends on material; it also depends on the shapes and sizes of the objects. The amount of vibrational energy left after separation depends on shape and size.

For experts only: the wave equation In one-dimensional linear elasticity the displacement u to the right, of a point at location x on one or the other rod follows this partial differential equation:

$$\frac{\partial^2 u}{\partial t^2} = \frac{E}{\rho}\frac{\partial^2 u}{\partial x^2}.$$

That is, the collision mechanics in detail is the finding of $u(x,t)$ that solves the wave equation above with the given initial conditions (one bar is moving the other isn't) and the boundary conditions (the ends of the bars have no stresses but where they are in contact they can have equal compressive stresses). The solution is most easily found by constructing right and left going waves that add to meet the initial conditions and boundary conditions (Routh).

SAMPLE 10.19 **Collision without energy loss:** A block of mass $m_1 = 2\,\mathrm{kg}$ moves with speed $v_1 = 0.5\,\mathrm{m/s}$ along the x-axis on a frictionless level ground behind another block of mass $m_2 = 10\,\mathrm{kg}$ moving at a speed $v_2 = 0.2\,\mathrm{m/s}$ in the same direction. The first block collides with the second block. Given that there is no loss of energy in this collision, find the speeds of the two blocks immediately after the collision.

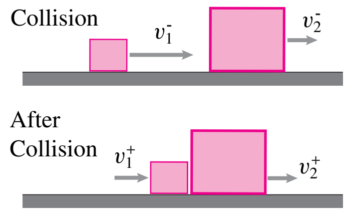


Figure 10.54

Solution We are given the speeds of two blocks (of known masses) just before the collision. It is also given that there is no loss of energy in the collision. We have to find the speed of the two masses immediately after collision.

We know that the linear momentum of the system consisting of the two blocks is conserved during the collision. Thus, if v_1^- and v_2^- are the speeds of the two masses just before the collision and v_1^+ and v_2^+ are their respective speeds immediately after the collision, then we have

$$m_1 v_1^+ + m_2 v_2^+ = m_1 v_1^- + m_2 v_2^- \tag{10.57}$$

Since there is no loss of energy in the collision, the energy of the system is conserved. Thus, $E^- = E^+$, or

$$\frac{1}{2}m_1(v_1^+)^2 + \frac{1}{2}m_2(v_2^+)^2 = \frac{1}{2}m_1(v_1^-)^2 + \frac{1}{2}m_2(v_2^-)^2. \tag{10.58}$$

Thus, we have two equations (eqn. (10.57) and eqn. (10.58)) in two unknowns, v_1^+ and v_2^+, and hence we can solve for them. It is now only a question in algebra. From eqn. (10.58), we have

$$\begin{aligned} m_1\left[(v_1^+)^2 - (v_1^-)^2\right] &= m_2\left[(v_2^-)^2 - (v_2^+)^2\right] \\ \Rightarrow \quad m_1(v_1^+ + v_1^-)(v_1^+ - v_1^-) &= m_2(v_2^- + v_2^+)(v_2^- - v_2^+) \end{aligned} \tag{10.59}$$

But, from eqn. (10.57), $m_1(v_1^+ - v_1^-) = m_2(v_2^- - v_2^+)$. Hence,eqn. (10.59) simplifies to

$$\begin{aligned} v_1^+ + v_1^- &= v_2^+ + v_2^- \\ \Rightarrow \quad v_1^+ - v_2^+ &= v_2^- - v_1^-. \end{aligned} \tag{10.60}$$

Multiplying the above equation by m_1 and subtracting from eqn. (10.57), we get

$$\begin{aligned} (m_1 + m_2)v_2^+ &= 2m_1 v_1^- + v_2^-(m_2 - m_1) \\ \Rightarrow \quad v_2^+ &= \frac{2m_1}{m_1 + m_2}v_1^- + \frac{m_2 - m_1}{m_1 + m_2}v_2^-. \end{aligned}$$

Now substituting the given values, $m_1 = 2\,\mathrm{kg}$, $m_2 = 10\,\mathrm{kg}$, $v_1^- = 0.5\,\mathrm{m/s}$ and $v_2^- = 0.2\,\mathrm{m/s}$ above, we get $v_2^+ = 0.3\,\mathrm{m/s}$. Further, substituting the values of v_2^+ in eqn. (10.60), we get $v_1^+ = 0$, *i.e.,* the first mass comes to a halt!

$$\boxed{v_1^+ = 0 \text{ and } v_2^+ = 0.3\,\mathrm{m/s}}$$

Comments: : Note that rather than using energy conservation equation directly as we did above, we could have used the given energy information t o set $e = 1$ (perfectly elastic collision) in eqn. (10.56) to get $v_2^+ - v_1^+ = -v_2^- + v_1^-$ (rather than deriving it as we did above). We can then solve this equation along with eqn. (10.57) to solve for v_1^+ and v_2^+.

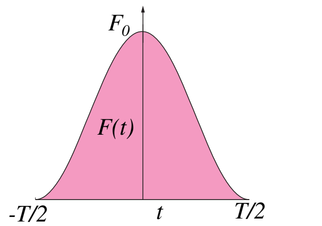


Figure 10.55

SAMPLE 10.20 Estimating peak force in a collision: A metal ball of mass $m = 0.5\,\text{kg}$ strikes a stationary surface S_1 with velocity $\vec{\boldsymbol{v}} = 10\,\text{m/s}\hat{\boldsymbol{\imath}}$ and rebounds with velocity $\vec{\boldsymbol{v}} = -9\,\text{m/s}\hat{\boldsymbol{\imath}}$. In a different experiment the same ball strikes another stationary surface S_2 with the same initial velocity and has the same rebound velocity. The contact times during the two experiments were different: 0.1 s and 0.001 s respectively. Assuming that the collisional force between the ball and the two surfaces can be modeled as $F(t) = \frac{F_0}{2}(1 + \cos\frac{2\pi t}{T})$ (see fig. 10.55) where $-T/2 \le t \le T/2$ and T is the contact time, find the peak force F_0 in each case.

Solution Let the collisional impulse acting on the ball be $\vec{\boldsymbol{P}}$ (see fig. 10.56) given by

$$\vec{\boldsymbol{P}} = \int_{-T/2}^{T/2} \vec{\boldsymbol{F}}(t)dt.$$

From impulse-momentum relationship, we have

$$\vec{\boldsymbol{P}} = \Delta\vec{\boldsymbol{L}} = m\Delta\vec{\boldsymbol{v}}.$$

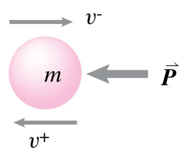


Figure 10.56: Free-body diagram of the ball during collision with collisional impulse $\vec{\boldsymbol{P}}$ acting on it.

Since in the case of each surface, $\Delta\vec{\boldsymbol{v}}$ is the same ($\vec{\boldsymbol{v}}^+ - \vec{\boldsymbol{v}}^- = -19\,\text{m/s}\hat{\boldsymbol{\imath}}$), the change in linear momentum $\Delta L = m\Delta\vec{\boldsymbol{v}}$ is also the same. Hence, the impulse acting on the ball in each case has to be the same. Now, let $\vec{\boldsymbol{P}}_1$ and $\vec{\boldsymbol{P}}_2$ be the impulses acting on the ball during the collision with surface S_1 and S_2 respectively. Then,

$$\begin{aligned}
\vec{\boldsymbol{P}}_1 &= -\int_{-T_1/2}^{T_1/2} \vec{\boldsymbol{F}}_1(t)dt\,\hat{\boldsymbol{\imath}} \\
&= -\int_{-T_1/2}^{T_1/2} \frac{(F_0)_1}{2}\left(1 + \cos\frac{2\pi t}{T_1}\right)dt\hat{\boldsymbol{\imath}} \\
&= -\frac{(F_0)_1 T_1}{2}\,\hat{\boldsymbol{\imath}}
\end{aligned}$$

Similarly,

$$\vec{\boldsymbol{P}}_2 = -\frac{(F_0)_2 T_2}{2}\,\hat{\boldsymbol{\imath}}.$$

Now, setting $\vec{\boldsymbol{P}}_1 = \Delta L$, we get

$$\begin{aligned}
-\frac{(F_0)_1 T_1}{2}\hat{\boldsymbol{\imath}} &= -m\Delta v\,\hat{\boldsymbol{\imath}} \\
\Rightarrow \quad (F_0)_1 &= \frac{2m\Delta v}{T_1} \\
&= \frac{2 \cdot 0.5\,\text{kg} \cdot 19\,\text{m/s}}{0.1\,\text{s}} \\
&= 190\,\text{N}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
(F_0)_2 &= \frac{2m\Delta v}{T_2} \\
&= \frac{2 \cdot 0.5\,\text{kg} \cdot 19\,\text{m/s}}{0.001\,\text{s}} \\
&= 19000\,\text{N}.
\end{aligned}$$

Clearly, the peak force is inversely proportional to the collision time. In fact, it is easy to see that for the given model of the impulsive force, the peak force $F_0 = \frac{2m\Delta v}{T}$. Thus if the change in momentum is constant, then the peak force varies as $1/T$.

$(F_0)_1 = 190\,\text{N}$ and $(F_0)_2 = 19000\,\text{N}$

SAMPLE 10.21 A two-ball multiple collision experiment: A tennis ball of approximate mass $m_1 = 60\,\mathrm{gm}$ and a basketball of approximate mass $m_2 = 600\,\mathrm{gm}$ are used in a fun collision experiment. The two balls are held in air, one on top of the other with a tiny gap between them, at a height h from the ground as shown in the figure. The two balls are released simultaneously from rest. The coefficient of restitution between the tennis ball and the basketball is $e_1 = 0.6$ and that between the basketball and the floor is $e_2 = 0.9$. Assume that the collision between the two balls takes place immediately after the basketball rebounds from the floor. Find the height of the tennis ball flight in terms of h as a result of the collision.

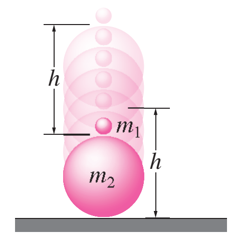


Figure 10.57

Solution We need to track two separate collisions here — one between the basketball and the floor, and second, between the tennis ball and the basketball. We can find the relevant vertical velocities before and after the collisions to determine the velocity of the tennis ball's flight which we can use to find the height of the flight. We will assume upward velocities to be positive.

Collision-1: Just before the basketball hits the floor, let its vertical velocity be v_2^- and let the tennis ball's speed at the same instant be v_1^-. Since both balls undergo free fall from height h before attaining these speeds, we have $v_1^- = v_2^- = -\sqrt{2gh}$. Now let v_2^+ be the speed of the basketball immediately after the collision with the ground (see fig. 10.58). Then, $v_2^+ = -e_2 v_2^- = e_2\sqrt{2gh}$.

Collision-2: We assume that the second collision, the collision between the tennis ball and the basketball, takes place immediately after the first collision. Hence, the velocity of the tennis ball just before the collision with the basketball can be assumed to be $v_1^- = -\sqrt{2gh}$. The second collision is shown in fig. 10.59. The after impact velocities of the two balls are v_1^+ and v_2^{++}. Now, from collision law, we have

$$v_1^+ - v_2^{++} = -e_1(v_1^- - v_2^+) = -e_1(-\sqrt{2gh} - e_2\sqrt{2gh}) = \sqrt{2gh}e_1(1+e_2). \qquad (10.61)$$

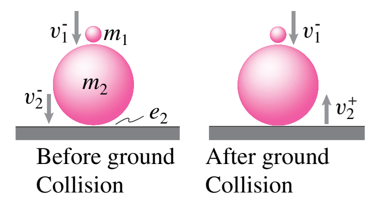


Figure 10.58: Collision of the basketball with the ground.

The conservation of linear momentum for the two-ball system gives

$$\begin{aligned} m_1 v_1^+ + m_2 v_2^{++} &= m_1 v_1^- + m_2 v_2^+) \\ \Rightarrow \quad v_1^+ + \frac{m_2}{m_1} v_2^{++} &= v_1^- + \frac{m_2}{m_1} v_2^+ \end{aligned}$$

Taking $M = m_2/m_1$, and substituting the values of v_1^- and v_2^+, we get

$$v_1^+ + M v_2^{++} = \sqrt{2gh}(Me_2 - 1). \qquad (10.62)$$

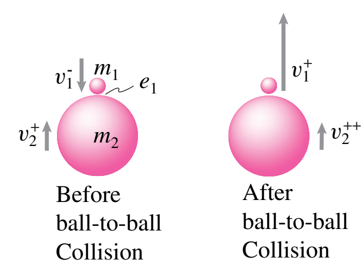


Figure 10.59: Collision of the tennis ball with the basketball.

Now solving eqn. (10.61) and eqn. (10.62) simultaneously, we get

$$v_1^+ = \frac{\sqrt{2gh}}{1+M}\left[M(e_1 + e_2 + e_1 e_2) - 1\right].$$

This is the velocity with which the tennis ball takes off on its vertical flight. Let the height of this flight be h_f. Then, from constant acceleration motion formula, we get $(v_1^+)^2 = 2gh_f$, or $h_f = (v_1^+)^2/2g$. Thus, from the derived expression for v_1^+ above, we get

$$h_f = \frac{h}{(1+M)^2}\left[M(e_1 + e_2 + e_1 e_2) - 1\right]^2.$$

Substituting $M = m_2/m_1 = 10$, $e_1 = 0.6$, and $e_2 = 0.9$ above, we get $h_f = 3.11h$. Thus the tennis ball flies off to three times its original height.

$$\boxed{h_f = 3.11h}$$

Note: From the expression obtained for v_1^+, we see that if M is very large then $v_1^+ = \sqrt{2gh}(e_1 + e_2 + e_1 e_2)$ and $h_f = (e_1 + e_2 + e_1 e_2)^2 h$.

10.5 1D Collisions

Preparatory Problems

10.5.1 Before a collision two particles, $m_A = 1\,\text{kg}$ and $m_B = 2\,\text{kg}$, have velocities of $v_A^- = 10\,\text{m/s}$ and $v_B^- = 5\,\text{m/s}$. After the collision the velocity of A is $v_A^+ = 6\,\text{m/s}$.

a) What is the momentum of A before the collision?

b) What is the momentum of B before the collision?

c) What is the system momentum before the collision?

d) What is the momentum of A after the collision?

e) What is the system momentum after the collision?

f) What is the momentum of B after the collision?

g) What is the impulse that A applies to B during the collision?

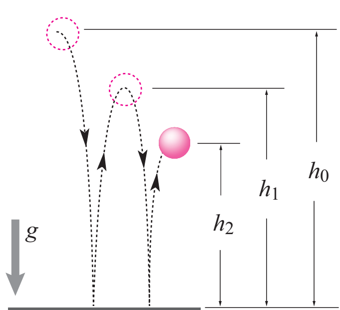
h) What is the impulse that B applies to A during the collision?

i) What is the kinetic energy of the system before the collision?

j) What is the kinetic energy of the system after the collision?

k) What is the coefficient of restitution?

10.5.2 A ball is dropped from a height of $h_0 = 10\,\text{m}$ onto a hard stationary surface. After the first bounce, it reaches a height of $h_1 = 6.4\,\text{m}$. What is the coefficient of restitution between the ball and ground? What is the height of the second bounce, h_2?



Problem 10.5.2

10.5.3 A 20 gram, 500 m/s bullet embeds in an initially-stationary 50 kg rigid block. What is the coefficient of restitution? What is the velocity of the block after this collision? *

10.5.4 A ball of mass m is dropped vertically from a height h. The only force acting on the ball in its flight is gravity. The ball strikes the ground with speed v^- and after collision it rebounds vertically with reduced speed v^+ directly proportional to the incoming speed, $v^+ = ev^-$, where $0 < e < 1$. What is the maximum height the ball reaches after one bounce, in terms of h, e, and g. *

10.5.5 Set up the following equations in matrix form and solve for v_A and v_B, if $v_0 = 2.6\,\text{m/s}$, $e = 0.8$, $m_A = 2\,\text{kg}$, and $m_B = 500\text{g}$:

$$m_A v_0 = m_A v_A + m_B v_B$$
$$-e v_0 = v_A - v_B.$$

More-Involved Problems

10.5.6 Before a collision two particles, $m_A = 7\,\text{kg}$ and $m_B = 9\,\text{kg}$, have velocities of $v_A^- = 6\,\text{m/s}$ and $v_B^- = 2\,\text{m/s}$. The coefficient of restitution is $e = .5$. Find the impulse of mass A on mass B and the velocities of the two masses after the collision.

10.5.7 Two frictionless masses $m_A = 2\,\text{kg}$ and $m_B = 5\,\text{kg}$ travel on straight collinear paths with speeds $V_A = 5\,\text{m/s}$ and $V_B = 1\,\text{m/s}$, respectively. The masses collide since $V_A > V_B$. Find the amount of energy lost in the collision. The coefficient of restitution is $e = 0.5$.

Problem 10.5.7

10.5.8 A ball of mass m is dropped from height h onto the solid hard ground where its coefficient of restitution is $e < 1$. The gravitational constant is g.

a) How many times does the ball bounce before it comes to a stop?

b) How long does it take from first release until it comes to a stop?

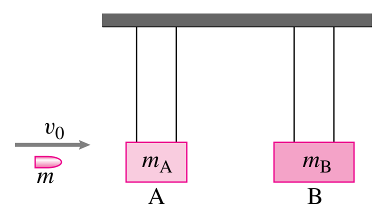
c) What is the total distance the ball travels before coming to a stop (add up and down distances)?

10.5.9 A bullet of mass m with initial speed v_0 is fired in the horizontal direction through block A of mass m_A and becomes embedded in block B of mass m_B. Each block is suspended by thin wires. The bullet causes A and B to start moving with speed of v_A and v_B respectively. Determine

a) the initial speed v_0 of the bullet in terms of v_A and v_B, *

b) the velocity of the bullet as it travels from block A to block B,and *

c) the energy loss due to friction as the bullet (1) moves through block A and (2) penetrates block B. *



Problem 10.5.9

10.5.10 A basketball with mass m_b is dropped from height h onto the hard solid ground on which it has coefficient of restitution e_b. Just on top of the basketball, falling with it and then bouncing against it after the basketball hits the ground, is a small rubber ball with mass m_r that has a coefficient of restitution e_r with the basketball.

a) In terms of some or all of m_b, m_r, h, g, e_b and e_r how high does the rubber ball bounce (measure height relative to the collision point)?

b) Assuming the coefficients of restitution are less than or equal to one, for given h, what mass and restitution parameters maximize the height of the bounce of the rubber ball and what is that height?

10.5.11 Show that it is necessary that $|e| \leq 1$ for the net kinetic energy (sum of the two kinetic energies of the colliding particles) to not increase.

10.5.12 According to the problem above, unless energy is created in the collision (as in an explosion), $-1 \leq e \leq 1$. Show that, for given masses and given initial velocities, that the loss of system kinetic energy is maximized by $e = 0$.